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Effective divisors of higher rank on a curve and the Siegel formula

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Section 0

In this paper we introduce a notion of effective divisors of higher rank on a curve and use this to give a self contained proof of the so called Siegel formula within the framework of elementary algebraic geometry. As in the case of ordinary divisors one can partition the set of effective r-divisors into linear equivalence classes. One has the obvious equality between the cardinality of the total set and the sum of the cardinalities of the equivalence classes. The Siegel formula appears as a straightforward consequence of this equality.

The Siegel formula was first proved in [6] and [4] and played there a key role in computing the Betti numbers of the moduli variety of stable bundles on a curve (cf. Appendix). The original proof consisted in showing that the Siegel formula is equivalent to the Tamagawa number of the special linear group over a function field being 1. This latter fact had already been established in [8] using Fourier analysis on adelizations of algebraic groups (cf. Appendix).

In section 1 we give the definition of positive r-divisor and sketch the proof of the Siegel formula. This will be completed in section 2.

Finally, for the convenience of the reader, in the Appendix we give some information on the original proof of the Siegel formula and how it is used to compute the Betti numbers of the moduli variety of stable bundles.

Section 1

Let X be an absolutely irreducible, smooth projective curve of genus g over a field F. Let $\mathcal{O}$ be the structure sheaf of X and let K be the field of rational functions on X. By a point on X we shall always mean a closed point.

We define a divisor of rank r (or an r-divisor) D to be an $\mathcal{O}$-submodule of rank r of the constant sheaf K^r. This amounts to giving a family $(D_p)_{p\in X}$ where D_p is an $\mathcal{O}_p$-submodule of rank r of the constant sheaf K^r, and for all but a finite number of points we have $D_p = \mathcal{O}_p^r$. When $r = 1$ this definition is equivalent to the usual one. One associates to $(D_p)_{p\in X}$ the divisor $\Sigma n_p p$ with $n_p = -\min\{\mathrm{ord}_p f \mid f \in D_p\}$. If $D = (D_p)_{p\in X}$ is any r-divisor one can define its dual, $\check{D}$, and its determinant, $\det D$. We set $\check{D} = (\check{D}_p)_{p\in X}$ where $\check{D}_p = \{v \in K^r \mid v \cdot u \in \mathcal{O}_p \, \forall u \in D_p\}$, $v \cdot u$ being the standard scalar product on the K-vector space K^r, and $\det D = (\wedge^r D_p)_{p\in X}$ where $\wedge^r K^r$ is identified with K via the determinant.

If E is a rank r vector bundle, its stalk at the generic point being isomorphic to K^r, then E is isomorphic to some r-divisor.

We shall say that an r-divisor is effective, or positive, if it contains $\mathcal{O}^r$ and that it is negative if it is contained in $\mathcal{O}^r$. Positive and negative divisors correspond to each other bijectively by duality.

There is a one to one correspondence between positive r-divisors and torsion quotients of $\mathcal{O}^r$. To a r-divisor D corresponds the quotient of $\mathcal{O}^r$ by the dual of D. This allows us to use Grothendieck's theory of Quot schemes ([5]). Let $\mathrm{Quot}^n_{\mathcal{O}^r}$ be the functor which associates to a scheme T the set of quotients

$$\mathcal{O}^r_{X\times T} \to Q \to 0,$$

with Q flat over $\mathcal{O}_T$ and such that $\forall t \in T$, Q_t is a length n quotient of $\mathcal{O}^r$. According to Grothendieck's theory this functor is represented by a projective scheme which we shall denote by $X^{(r,n)}$. In particular, the set of points of $X^{(r,n)}$ rational over F can be identified with the set of positive r-divisors of degree n. It is easy to see that $X^{(r,n)}$ is an irreducible smooth projective variety of dimension rn.

The map which associates to a positive r-divisor its determinant is functorial, hence defines a morphism

$$\det\colon X^{(r,n)} \to X^{(1,n)}$$

By composing this with the canonical morphism $X^{(1,n)} \to \mathrm{Pic}^n X$ we get a morphism $\det\colon X^{(r,n)} \to \mathrm{Pic}^n X$. For $L \in \mathrm{Pic}^n X$ we set $X^{(r,L)} = \det^{-1}(L)$. The variety $X^{(r,L)}$ parametrizes positive r-divisors having determinant isomorphic to L. In particular the set of points of $X^{(r,L)}$ rational over F can be identified with the set of positive r-divisors having determinant isomorphic to L.

There is a natural Abel-Jacobi map from the set of positive r-divisors to the set of isomorphism classes of rank r vector bundles sending a divisor to its isomorphism class as an $\mathcal{O}$-module. To a divisor, given as a quotient $\mathcal{O}^r \xrightarrow{\tau} Q \to 0$, it associates the isomorphism class of the dual of $\ker \tau$.

We shall say that two divisors are linearly equivalent if the associated isomorphism classes of vector bundles coincide. If E is a rank r vector bundle, we shall denote by $X^{(r)}_E$ the set of all positive r-divisors whose associated vector bundle is isomorphic to E.

Also, we denote by $\mathrm{Hom}_{inj}(L, N)$ the set of injective homomorphism of L in N,

L and N being $\mathcal{O}$-modules.

Clearly $X_E^{(r)}$ corresponds bijectively with $\mathrm{Hom}_{inj}(\mathcal{O}^r, E)/\mathrm{Aut}\, E$, the quotient set of $\mathrm{Hom}_{inj}(\mathcal{O}^r, E)$ by the natural action of the group of automorphisms of E.

In particular $X_E^{(r)} \neq \phi$ if and only if E is generated by its global sections over a nonempty open set.

From now on we shall restrict ourselves to the case where the base field is $\mathbb{F}_q$, a field with q elements. In this case one can consider $Z(t)$, the zeta function of X, which can be defined as the formal power series

$$Z(t) = \sum_{n \geqslant 0} c_n t^n,$$

where c_n is the number of degree n positive divisors on X.

It is well known that

$$Z(t) = \frac{P(t)}{(1-t)(1-qt)},$$

where $P(t)$ is a polynomial of degree $2g$ such that $t^{2g} q^g P(1/qt) = P(t)$. In particular, $P(1)$ is the number of isomorphism classes of line bundles of any given degree.

More generally we are led, for $r \geqslant 1$, to consider the series

$$Z^{(r)}(t) = \sum_{n \geqslant 0} c_n^{(r)} t^n$$

where $c_n^{(r)}$ is the number of degree n positive r-divisors on X.

We shall see (Proposition 1 of section 2) that

$$Z^{(r)}(t) = Z(t) \cdot Z(qt) \cdot \cdots \cdot Z(q^{r-1} t).$$

In particular this gives

$$\lim_{n \to \infty} \frac{c_n^{(r)}}{q^{rn}} = \frac{P(1)}{q-1} q^{g-1} \cdot Z(q^{-2}) \cdot \cdots \cdot Z(q^{-r}) \tag{1}$$

For L a line bundle of degree n let $c_n^{(r,L)}$ be the number of positive r-divisors whose determinant is isomorphic to L. We shall see that for $n > 2r(g-1)$ we have

$$c_n^{(r,L)} = \frac{c_n^{(r)}}{P(1)}. \tag{2}$$

On the other hand, partitioning the set of positive r-divisors having determinant isomorphic to L into linear equivalence classes, we have

$$c_n^{(r,L)} = \sum_{E \in I(r,L)} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E)|}{|\mathrm{Aut}\, E|},$$

where $I(r, L)$ is the set of isomorphism classes of vector bundles of rank r having determinant isomorphic to L, and $|A|$ is the cardinality of the set A.

Fix now a point $p \in X$ and for any $\mathcal{O}$-module E set $E(k) = E \otimes \mathcal{O}(kp)$. The map $E \mapsto E(k)$ induces bijections $I(r, L) \xrightarrow{\sim} I(r, L(rk))$. Using these, and the isomorphism $\mathrm{Aut}\, E \simeq \mathrm{Aut}\, E(k)$ we can write the following formula

$$c_{n+lrk}^{(r,L(rk))} = \sum_{E \in I(r,L)} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{|\mathrm{Aut}\, E|}, \tag{3}$$

where l is the degree of the point p.

Divide (3) by $q^{r\chi(E(k))}$, where $\chi(E(k))$ is the Euler-Poincaré characteristic of $E(k)$ and E is any element of $I(r, L)$; then take the limit as $k \to \infty$.

We shall show that one can interchange the limit with the sum and that

$$\lim_{k \to \infty} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}} = 1 \tag{4}$$

Therefore, taking (1) and (2) into account, we have

$$\frac{q^{(r^2-1)(g-1)}}{q-1} Z(q^{-2}) \cdot \cdots \cdot Z(q^{-r}) = \sum_{E \in I(r,L)} \frac{1}{|\mathrm{Aut}\, E|} \tag{5}$$

This is precisely the Siegel formula.

We remark that (4) is not unexpected since the "generic" element of $\mathrm{Hom}(\mathcal{O}^r, E(k))$ is injective and

$$\frac{|\mathrm{Hom}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}} = \frac{q^{rh^0(E(k))}}{q^{rh^0(E(k))}} = 1 \quad \text{for } k \gg 0.$$

We also remark that computing $c_n^{(r)}$ and $c_n^{(r,L)}$ amounts in principle to computing the zeta functions of $X^{(r,n)}$ and $X^{(r,L)}$. From these, using the Weil conjectures one can deduce the Betti numbers of these varieties (see Corollary to Proposition 1). In [2] E. Bifet, applying the theory of Bialynicki-Birula, gives a geometrical description of the varieties $X^{(r,n)}$ and $X^{(r,L)}$ valid in any characteristic.

Section 2

Let us proceed to the proof of the Siegel formula.

LEMMA 1. *Let A be a D.V.R. on $\mathbb{F}_q$, let $\mathbf{m}$ be its maximal ideal and let $d = \dim_{\mathbb{F}_q} A/\mathbf{m}$. If $c_n(r, A)$ is the number of submodules E of A^r such that $\dim_{\mathbb{F}_q} A^r/E = d \cdot n$ then we have*:

$$c_n(r, A) = \sum_{k_1 + \cdots + k_r = n} q^{d\left(\sum_{i1}^{r} k_i(i-1)\right)}$$

Proof. Every $r \times r$ matrix M with coefficients in A determines the submodule of A^r generated by the columns of M.

We say that M and M' are equivalent whenever they determine the same submodule. This happens if and only if $M' = MP$ for some invertible matrix P. By performing elementary operations on columns it is immediately seen that every rank r matrix is equivalent to a lower triangular matrix $M = (m_{ij})$ with $m_{ii} = \pi^{k_i}$, where π is a generator of $\mathbf{m}$. The integers $k_1, \ldots, k_r$ are uniquely determined, and the coefficients m_{ij} $(j < i)$ are determined modulo $\mathbf{m}^{k_i}$. If E is the submodule determined by M, then $\dim_{\mathbb{F}_q} A^r/E = d(\Sigma_{i=1}^r k_i)$. Hence we can write

$$c_n(r, A) = \sum_{\substack{k_1 + \cdots + k_r = n \\ k_i \geq 0}} q^{d\left(\sum_{i=1}^{r} k_i(i-1)\right)}$$

LEMMA 2. *If D is an effective divisor and $c(r, D)$ is the number of positive r-divisors having determinant D, then*

$$c(r, D) = \sum_{\substack{D_1 + \cdots + D_r = D \\ D_i \geq 0}} q^{\sum_{i=1}^{r} \deg D_i(i-1)}$$

Proof. $c(r, D)$ is also the number of negative r-divisors having determinant $-D$. Let $D = \Sigma_{j=1}^h n_j p_j$ and $d_j = \deg p_j = \dim_{\mathbb{F}_q} \mathcal{O}p_j/\mathbf{m}_{p_j}$.

We have

$$c(r, D) = \prod_{j=1}^{h} c_{n_j}(r, \mathcal{O}_{p_j}) = \prod_{j=1}^{h} \left(\sum_{k_{j1} + \cdots + k_{jr} = n_j} q^{d_j\left(\sum k_{ji}(i-1)\right)} \right)$$

$$= \sum_{\substack{k_{ji}, k_{j1} + \cdots + k_{jr} = n_j \\ k_{ji} \geq 0}} q^{\sum_{i=1}^{r} \left(\sum_{j=1}^{h} d_j k_{ji}\right)(i-1)}$$

Setting $D_i = \Sigma_{j=1}^h k_{ji} p_j$ the conclusion of the lemma follows.

PROPOSITION 1. *Let $c_n^{(r)}$ be the number of effective r-divisors of degree n and set $Z^{(r)}(t) = \Sigma_{n \geq 0} c_n^{(r)} t^n$. For L a line bundle of degree n, let $c_n^{(r,L)}$ be the number of effective r-divisors having determinant isomorphic to L.*

Then we have:

$$(1) \quad Z^{(r)}(t) = \prod_{j=1}^{r} Z(q^{j-1}t)$$

$$(2) \quad c_n^{(r,L)} = \frac{c_n^{(r)}}{P(1)} \quad \text{for every } n > 2r(g-1)$$

where $P(t)$ is the numerator of $Z(t)$.

Proof. From Lemma 2 it follows that

$$\begin{aligned} c_n^{(r)} &= \sum_{\substack{\deg D = n \\ D \geq 0}} \left(\sum_{D_1 + \cdots + D_r = D} q^{\sum_{i=1}^{r} \deg D_i (i-1)} \right) \\ &= \sum_{\substack{\deg(D_1 + \cdots + D_r) = n \\ D_i \geq 0}} q^{\sum_{i=1}^{r} \deg D_i (i-1)} \\ &= \sum_{j_1 + \cdots + j_r = n} c_{j_1} \cdot c_{j_2} \cdot \cdots \cdot c_{j_r} \cdot q^{\sum_{i=1}^{r} j_i (i-1)}, \end{aligned}$$

where c_j is the number of degree j effective divisors. Therefore (1) is proved.

As for (2) we can write

$$c_n^{(r,L)} = \sum_{\substack{D_1 + \cdots + D_r \in |L| \\ D_i \geq 0}} q^{\sum_{i=1}^{r} \deg D_i (i-1)},$$

where $|L|$ is the complete linear system associated to L. Fix degrees $j_1, \ldots, j_r$ for $D_1, \ldots, D_r$ with $\Sigma_{i=1}^{r} j_i = n > 2r(g-1)$. One of the j_i, say j_k, will be greater than $2g-2$. We satisfy the condition $D_1 + \cdots + D_r \in |L|$ by choosing arbitrarily D_i of degree j_i, for $i \neq k$, and letting D_k vary in the complete linear system $|L - \mathcal{O}(\Sigma_{i \neq k} D_i)|$. This linear system has degree j_k greater than $2g-2$, hence consists of $c_{j_k}/P(1)$ elements. It follows that

$$c_n^{(r,L)} = \sum_{j_1 + \cdots + j_r = n} \frac{1}{P(1)} c_{j_1} \cdots c_{j_r} q^{\sum_{i=1}^{r} j_i (i-1)} = \frac{c_n^{(r)}}{P(1)}$$

as asserted.

COROLLARY. *Let $P_u(S^{(r,n)}) = \Sigma_i b_i(X^{(r,n)}) u^i$ be the Betti-Poincaré polynomial of $X^{(r,n)}$. We have*

$$\sum_{n=0}^{\infty} P_u(X^{(r,n)}) t^n = \prod_{j=0}^{r-1} \frac{(1 + u^{2j+1} t)^{2g}}{(1 - u^{2j} t)(1 - u^{2j+2} t)}.$$

In particular, the Euler-Poincaré characteristic of $X^{(r,n)}$ *is* $(-1)^n\binom{2r(g-1)}{n}$ *and it vanishes for* $n > 2r(g-1)$.

Proof. Using the Weil conjectures, the series $\Sigma_{n=0}^{\infty} P_u(X^{(r,n)})t^n$ can be obtained from the formula giving $Z^{(r)}(t)$ by substituting q with u^2 and ω_i with $-u$, where $\omega_1, \ldots, \omega_{2g}$ are the algebraic integers appearing in $P(t) = \Pi_{i=1}^{2g}(1-\omega_i t)$.

PROPOSITION 2. *Let*

$$\chi(r,n) = n + r(1-g) \quad \textit{and} \quad \sigma(r) = \frac{q^{(r^2-1)(g-1)}}{q-1} Z(q^{-2}) \cdots Z(q^{-r})$$

be the Siegel number. We have

$$\lim_{n\to\infty} \frac{c_n^{(r)}}{q^{r\chi(r,n)}} = P(1)\sigma(r)$$

and, consequently,

$$\lim_{n\to\infty} \frac{c_n^{(r,L)}}{q^{r\chi(r,n)}} = \sigma(r)$$

Proof. If $b(t) = \Sigma_{n\geqslant 0} b_n t^n$ is a series that converges at $t = 1$, and

$$\frac{b(t)}{1-t} = \sum_{n\geqslant 0} a_n t^n, \quad \text{then} \ \lim_{n\to\infty} a_n = b(1).$$

Consider the series

$$Z^{(r)}\left(\frac{t}{q^r}\right) = \sum_{n\geqslant 0} \frac{c_n^{(r)} t^n}{q^{rn}} = \left[\prod_{j=2}^{r} Z(q^{-j}t) \frac{P(1/qt)}{(1-1/qt)}\right] \frac{1}{1-t}.$$

The product in square brackets converges at $t = 1$, and we can write

$$\lim_{n\to\infty} \frac{c_n^{(r)}}{q^{rn}} = \prod_{j=2}^{r} Z(q^{-j}) \frac{P(1/q)}{1-1/q} = \sigma(r) \frac{P(1)}{q^{r^2(g-1)}}.$$

LEMMA 3. *Let E be any rank r vector bundle on X. We have:*

$$\lim_{k\to\infty} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}} = 1$$

Proof. Since

$$\frac{|\mathrm{Hom}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}} = 1$$

for large k, it suffices to show that

$$\lim_{k\to\infty} \frac{|\mathrm{Hom}(\mathcal{O}^r, E(k)) - \mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{lr^2k}} = 0.$$

Write $\mathcal{O}_k = \mathcal{O}/\mathcal{O}(-k)$. Then $E(k)/E$ is isomorphic to $\mathcal{O}_k^r$ and we have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{O}^r, E) \to \mathrm{Hom}(\mathcal{O}^r, E(k)) \to \mathrm{Hom}(\mathcal{O}^r, \mathcal{O}_k^r).$$

Any non-injective homomorphism $\mathcal{O}^r \to E(k)$ is sent to a morphism that factors through a surjective $\mathcal{O}^r \to \mathcal{O}_k^s$ for some $s < r$.

Thus, in order to establish the Lemma, it suffices to prove that (for any $s < r$) the number of these morphisms does not exceed $Cq^{l(r^2-1)k}$, for some constant C.

Let $\Phi_{s,k}$ be the map

$$\mathrm{Hom}_{\mathrm{surj}}(\mathcal{O}^r, \mathcal{O}_k^s) \times \mathrm{Hom}(\mathcal{O}_k^s, \mathcal{O}_k^r) \to \mathrm{Hom}(\mathcal{O}^r, \mathcal{O}_k^r)$$

given by composition. The group $G_{s,k}$ of automorphisms of $\mathcal{O}_k^s$ acts freely on the domain of $\Phi_{s,k}$ by $g\cdot(\alpha, \beta) = (g\circ\alpha, \beta\circ g^{-1})$.

Now $\Phi_{s,k}$ is clearly constant along the orbits. Moreover, if

$$\pi_{s,k}\colon \mathrm{Hom}(\mathcal{O}_k^s, \mathcal{O}_k^s) \to \mathrm{Hom}(\mathcal{O}_1^s, \mathcal{O}_1^s)$$

is the obvious surjective map, then $\pi_{s,k}^{-1}(G_{s,1}) = G_{s,k}$ by Nakayama's Lemma. Hence we have

$$|G_{s,k}| = |G_{s,1}|q^{ls^2(k-1)} \quad \text{and} \quad |\mathrm{Im}\,\Phi_{s,k}| \leqslant \frac{q^{lr(sk)}q^{ls(rk)}}{|G_{s,1}|q^{ls^2(k-1)}} \leqslant Cq^{l[r^2-(r-s)^2]k},$$

thereby proving the Lemma.

LEMMA 4. *For $r \geqslant 1$ and a line bundle L set:*

$$\tilde{\sigma}(r, L) = \sum_{E\in I(r,L)} \frac{1}{|\mathrm{Aut}\, E|}$$

We have: $\tilde{\sigma}(r, L) \leqslant \sigma(r)$. *In particular* $\tilde{\sigma}(r, L)$ *is finite.*

Proof. Recall formula (3):

$$c_{n+lrk}^{(r,L(rk))} = \sum_{E \in I(r,L)} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{|\mathrm{Aut}\, E|}.$$

We divide this expression by $q^{r\chi(E(k))}$ and take the limit as $k \to \infty$. The result follows from proposition 2 and lemma 3 because

$$\lim_{k \to \infty} \sum_{i \in I} a_{ki} \geqslant \sum_{i \in I} \lim_{k \to \infty} a_{ki}$$

whenever $a_{ki} \geqslant 0$.

COROLLARY. *For* $r \geqslant 1$ *and* $n \in \mathbf{Z}$ *set*

$$\tilde{\sigma}(r, n) = \sum_{E \in I(r,n)} \frac{1}{|\mathrm{Aut}\, E|}.$$

We have: $\tilde{\sigma}(r, n) \leqslant P(1)\sigma(r)$. *In particular* $\tilde{\sigma}(r, n)$ *is finite.*

LEMMA 5. *For every* $r \geqslant 1$ *we have*:

$$\lim_{n \to \infty} \sum_{\substack{E \in I(r,n) \\ H^1(E) \neq 0}} \frac{1}{|\mathrm{Aut}\, E|} = 0$$

Proof. By the previous corollary, given j and $\varepsilon > 0$ there exists a finite subset $I_f(r,j)$ of $I(r,j)$ such that

$$\tilde{\sigma}(r, j) - \varepsilon \leqslant \sum_{E \in I_f(r,j)} \frac{1}{|\mathrm{Aut}\, E|}.$$

If k is large enough then $H^1(E(k)) = 0$ for every $E \in I_f(r,j)$. Therefore:

$$\tilde{\sigma}(r, j) - \varepsilon \leqslant \sum_{E \in I_f(r,j)} \frac{1}{|\mathrm{Aut}\, E(k)|} \leqslant \sum_{\substack{F \in I(r,j+lrk) \\ H^1(F) = 0}} \frac{1}{|\mathrm{Aut}\, F|}.$$

Since $\tilde{\sigma}(r, j) = \tilde{\sigma}(r, j + lrk)$, this gives:

$$\sum_{\substack{F \in I(r,j+lrk) \\ H^1(F) \neq 0}} \frac{1}{|\mathrm{Aut}\, F|} < \varepsilon$$

for any large enough k. Letting j take the values $1, 2, \ldots, lr$ we obtain the lemma.

LEMMA 6. *For every* $r \geqslant 1$ *we have*

$$\lim_{n \to \infty} \sum_{\substack{E \in I(r,n) \\ H^1(E) \neq 0}} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E)|}{q^{r\chi(E)}|\mathrm{Aut}\, E|} = 0$$

Proof. The result follows from lemma 5 and the inequality

$$\frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E)|}{q^{r\chi(E)}} \leqslant q^{r^2 g}.$$

This is obvious if $|\mathrm{Hom}_{inj}(\mathcal{O}^r, E)| = 0$.

On the other hand the existence of an exact sequence

$$0 \to \mathcal{O}^r \to E \to Q \to 0,$$

Q being a torsion module, gives $h^1(E) \leqslant rg$ and hence $h^0(E) \leqslant \chi(E) + rg$. This implies that:

$$|\mathrm{Hom}_{inj}(\mathcal{O}^r, E)| \leqslant |\mathrm{Hom}(\mathcal{O}^r, E)| = q^{rh^0(E)} \leqslant q^{r\chi(E) + r^2 g}$$

and the inequality still holds.

PROOF OF THE SIEGEL FORMULA. We know that:

$$\sigma(r) = \lim_{k \to \infty} \sum_{E \in I(r,L)} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}|\mathrm{Aut}\, E|}$$

By the previous lemma for every $\varepsilon > 0$ we have:

$$\sigma(r) - \varepsilon \leqslant \sum_{\substack{E \in I(r,L) \\ H^1(E(k)) = 0}} \frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}|\mathrm{Aut}\, E|}$$

whenever $k \geqslant k(\varepsilon)$. But if $H^1(E(k)) = 0$, then

$$\frac{|\mathrm{Hom}_{inj}(\mathcal{O}^r, E(k))|}{q^{r\chi(E(k))}} \leqslant 1$$

and we have:

$$\sigma(r) - \varepsilon \leqslant \sum_{\substack{E \in I(r,L) \\ H^1(E(k)) = 0}} \frac{1}{|\mathrm{Aut}\, E|} \leqslant \sum_{E \in I(r,L)} \frac{1}{|\mathrm{Aut}\, E|}.$$

The Siegel formula follows from this and Lemma 4.

Appendix

Every rank r vector bundle admits a canonical filtration (the Harder-Narashimhan filtration ([6])) with semistable graded pieces.

Using this filtration and Siegel's formula it is possible by induction on the rank to isolate the contribution given by the semistable bundles to the right-hand side of Siegel's formula. Moreover if E is a semistable vector bundle whose degree is prime with respect to its rank it is well known that E is stable and $\mathrm{Aut}\, E \simeq \mathbb{F}_q^*$. Consequently it is possible to compute the number of stable bundles of rank r having determinant isomorphic to L when $\deg L$ and r are co-prime. This is equivalent to the computation of the number of rational points over $\mathbb{F}_q$ of the coarse moduli variety of stable bundles of rank r having determinant isomorphic to L. Hence one obtains the zeta function and, by the Weil conjectures [3], the Betti numbers in l-adic cohomology of the above variety.

If X is defined over $\mathbf{C}$, these numbers by the comparison theorem coincide with the Betti numbers in ordinary cohomology.

In [4] and [6] the Siegel formula is proved by showing that it is equivalent to the Tamagawa number of $\mathrm{SL}(r, K)$ being 1. A proof of this, valid for any global field, can be found in [8]. The relationship between the Siegel formula and the Tamagawa number of $\mathrm{SL}(r, K)$ is as follows. Let $\hat{\mathcal{O}}_p$ and $\hat{K}_p$ be the completions of $\mathcal{O}_p$ and K for the m_p-adic topology, where m_p is the maximal ideal of $\mathcal{O}_p$. To give an r-divisor D is equivalent to giving a family $(\hat{D}_p)_{p \in X}$ where $\hat{D}_p$ is an $\hat{\mathcal{O}}_p$-submodule of rank r of $\hat{K}_p^r$ such that $\hat{D}_p = \hat{\mathcal{O}}_p^r$ for almost all p in X. The action (on the right) of $G = \mathrm{SL}(r, K)$ on K^r extends to an action of the adèle group G_A of G on the set of all r-divisors. An orbit of G_A consist precisely of those r-divisor having the same determinant, while an orbit of G consists of those r-divisors which are isomorphic as $\mathcal{O}$-modules and have the same determinant. Fix an r-divisor $D = (\hat{D}_p)_{p \in X}$ and let $\mathscr{R}_D$ be its stabilizer under the action of G_A. We have $\mathscr{R}_D = \Pi_{p \in X} \mathscr{R}_{\hat{D}_p}$ with $\mathscr{R}_{\hat{D}_p} = \{T \in \mathrm{SL}(r, \hat{K}_p) | (T^{-1}(\hat{D}_p) = \hat{D}_p\}$, a conjugate of $\mathrm{SL}(r, \hat{\mathcal{O}}_p)$. Consider the map $\psi\colon G_A/G \to I(r, \det D)$ sending γG to the class of $D \cdot \gamma$. It is easy to see that it is surjective. The map ψ factors through $\bar{\psi}\colon \mathscr{R}_D \backslash G_A/G \to I(r, \det D)$. $\mathscr{R}_D \backslash G_A/G$ can be identified with the set of isomorphism classes of r-divisors whose determinant is equal to $\det D$. Therefore we have

$$|\bar{\psi}^{-1}(E)| = \left|\frac{\mathbb{F}_q^*}{\operatorname{Im}\det(\operatorname{Aut} E)}\right|,$$

where $\det\colon \operatorname{Aut} E \to \mathbb{F}_q^*$ is the homomorphism sending an automorphism of E to its determinant. On the other hand, if $\alpha \in \bar{\psi}^{-1}(E)$, then its isotropy group $\mathscr{R}_\alpha$ is isomorphic to $\{\varphi \in \operatorname{Aut} E \mid \det \varphi = 1\}$. This gives the following equality

$$\sum_{\alpha \in \bar{\psi}^{-1}(E)} \frac{1}{|\mathscr{R}_\alpha|} = \frac{q-1}{|\operatorname{Aut} E|} \tag{8}$$

On each local field $\hat{K}_p$ there is a unique Haar measure μ_p such that $\mu_p(\hat{\mathcal{O}}_p) = 1$. Fix a right invariant form ω of degree $r^2 - 1$ on G. This gives a form, hence a right invariant Haar measure τ_p, on each analytic variety $\mathrm{SL}(r, \hat{K}_p)$. We have

$$\tau_p(\mathrm{SL}(r, \hat{\mathcal{O}}_p)) = \frac{1}{q^{(r^2-1)d}} |\mathrm{SL}(r, \mathcal{O}_p/m_p)| = \prod_{j=2}^{r} \left(1 - \frac{1}{q^{j \cdot \deg p}}\right),$$

where $d = \deg p = \dim_{\mathbb{F}_q} \mathcal{O}_p/m_p$. The Tamagawa measure τ on G_A, which is independent of the choice of ω, is given by $\tau = q^{-(r^2-1)(g-1)} \prod_{p \in X} \tau_p$. Since

$$Z(t) = \prod_{p \in X} \frac{1}{1 - t^{\deg p}}$$

one has

$$\tau(\mathscr{R}_D)^{-1} = q^{(r^2-1)(g-1)} Z(q^{-2}) \cdots Z(q^{-r}). \tag{9}$$

The Tamagawa number of $G = \mathrm{SL}(r, K)$ is by definition $\tau(G_A/G)$. Decomposing G_A/G into orbits for the left action of $\mathscr{R}_D$ we have

$$\tau(G_A/G) = \sum_{\alpha \in \mathscr{R}_D \backslash G_A/G} \tau(\mathscr{R}_D/\mathscr{R}_\alpha) = \sum_{\alpha \in \mathscr{R}_D \backslash G_A/G} \frac{\tau(\mathscr{R}_D)}{|\mathscr{R}_\alpha|}$$

It is now clear, using (1) and (2), that the Siegel formula is equivalent to the Tamagawa number of $\mathrm{SL}(r, K)$ being one.

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