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Varieties of small Kodaira dimension whose cotangent bundles are semiample

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We work in the category of complex projective algebraic varieties, and study the fundamental structures of nonsingular varieties of Kodaira dimension 0 and 1 whose cotangent bundles are semiample. Our results are summarized as follows.

A nonsingular variety $X$ is called a para-abelian variety if it admits a finite unramified Galois covering $A \rightarrow X$ with an abelian variety $A$. It is clear that a para-abelian variety $X$ is attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 0$. Conversely, we obtain the following:

**Theorem I.** Let $X$ be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 0$. Then $X$ is a para-abelian variety.

To simplify our statement of the next result, we introduce a special type of variety.

**Definition.** Let $V = F \times C$ be the product of a para-abelian variety $F$ and a nonsingular curve $C$ of genus $g$, and let $X = Y/G$ be the quotient of $V$ by a finite group $G$ which acts effectively both on $V$ and on $C$ so that:

1. $\phi \circ \sigma = \sigma \circ \phi$ for every $\sigma \in G$ and for the projection $\phi : V \rightarrow C$;
2. If $\sigma \in G$ has a fixed point $v \in V$, then $\sigma(v') = v'$ for every point $v' \in \phi^{-1}(\phi(v))$.

For each point $c \in C$ put $G_c = \{ \sigma \in G | \sigma(v') = v' \text{ for every point } v' \in \phi^{-1}(\phi(c)) \}$, and set

$$R = \sum_{c \in C} (|G_c| - 1),$$

where $|G_c|$ is the order of the subgroup $G_c$. Then, in case $R < 2g - 2$, we call $X$ a variety of type $Q_+$. We shall show that a variety $X$ of type $Q_+$ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Such a variety $X$ may seem too typical for the converse to be verified. Nevertheless we obtain the following:

**Theorem II.** Let $X$ be a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$. Then $X$ is a variety of type $Q_+$. 
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**Notation and Terminology**

- $\mathcal{L} \otimes^m$ the $m$th tensor power of a line bundle $\mathcal{L}$
- $S^m \mathcal{E}$ the $m$th symmetric tensor power of a vector bundle $\mathcal{E}$
- $\det \mathcal{E}$ the determinant bundle of a vector bundle $\mathcal{E}$
- $\mathcal{E}^*$ the dual bundle of a vector bundle $\mathcal{E}$
- $\mathbf{P}(\mathcal{E})$ the projective space bundle $\text{Proj}(\oplus_{m \geq 0} S^m \mathcal{E})$ associated to a vector bundle $\mathcal{E}$
- $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ the tautological line bundle of $\mathbf{P}(\mathcal{E})$
- $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ the $m$th tensor power of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$
- $c_1(\mathcal{E})$ the first Chern class of a vector bundle $\mathcal{E}$
- $\mathcal{O}_X$ the structure sheaf of a variety $X$
- $\mathcal{T}_X$ the tangent sheaf of a variety $X$
- $\Omega^1_X$ the sheaf of regular 1-forms on a variety $X$ (the cotangent bundle of a variety $X$)
- $\omega_X$ the canonical sheaf of a variety $X$
- $\Omega_{X/Y}$ the sheaf of relative differentials of a variety $X$ over a variety $Y$

A **vector bundle** means a locally free sheaf of finite rank. A line bundle is said to be **spanned** if it is generated by its global sections. A vector bundle $\mathcal{E}$ is defined to be **semiaample** if for some positive integer $m$ the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$ is spanned. We say that a surjective homomorphism $h$ of vector bundles is **splitting** if the short exact sequence derived from $h$ splits.

Given a line bundle $\mathcal{L}$ on a nonsingular variety $X$, we let $N(\mathcal{L})$ be the set of all positive integers $m$ such that $H^0(X, \mathcal{L} \otimes^m) \neq 0$, and for each $m \in N(\mathcal{L})$ let $\Phi_m : X \to \mathbf{P}(H^0(X, \mathcal{L} \otimes^m))$ be the canonical rational map. Then we put

$$\kappa(\mathcal{L}, X) = \begin{cases} \max \{ \dim \Phi_m(X) \mid m \in N(\mathcal{L}) \} & \text{if } N(\mathcal{L}) \neq \emptyset, \\ -\infty & \text{if } N(\mathcal{L}) = \emptyset. \end{cases}$$

This is the $\mathcal{L}$-dimension of $X$ introduced by Iitaka [5]. For the canonical sheaf $\omega_X$ of $X$, we put $\kappa(X) = \kappa(\omega_X, X)$ and call it the **Kodaira dimension** of $X$.

A **fibration** is a dominating morphism of normal varieties with connected fibres. A **fibre bundle** is an analytically locally trivial fibration.

**1. Semiaample vector bundles**

In this section, we study some fundamental properties of semiaample vector bundles. We use frequently the following lemmata:
LEMMA 1 (Fujita [2]). Let $f : X \rightarrow Y$ be a dominating morphism of nonsingular varieties and let $\mathcal{E}$ be a vector bundle on $Y$. Then $\mathcal{E}$ is semiample if and only if the pull-back $f^* \mathcal{E}$ is semiample.

LEMMA 2 (Fujita). Let $\mathcal{E}, \mathcal{F}$ be vector bundles on a nonsingular variety $X$. Then the direct sum $\mathcal{E} \oplus \mathcal{F}$ is semiample if and only if both $\mathcal{E}$ and $\mathcal{F}$ are semiample.

**Proof.** Put $\mathcal{G} = \mathcal{E} \oplus \mathcal{F}$. The natural surjective homomorphisms $\mathcal{G} \rightarrow \mathcal{E}$, $\mathcal{G} \rightarrow \mathcal{F}$ define embeddings $i_1 : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{G})$, $i_2 : \mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{G})$ such that $i_1^* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, $i_2^* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ respectively. Hence $\mathcal{E}$ and $\mathcal{F}$ are semiample if so is $\mathcal{G}$. Put $Y_1 = i_1(\mathbb{P}(\mathcal{E}))$, $Y_2 = i_2(\mathbb{P}(\mathcal{F}))$. Then the natural injective homomorphisms $\mathcal{E} \rightarrow \mathcal{G}$, $\mathcal{F} \rightarrow \mathcal{G}$ define morphisms $j_1 : \mathbb{P}(\mathcal{E}) \setminus Y_1 \rightarrow \mathbb{P}(\mathcal{G})$, $j_2 : \mathbb{P}(\mathcal{F}) \setminus Y_2 \rightarrow \mathbb{P}(\mathcal{G})$ such that $j_1^* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}) \setminus Y_1}$, $j_2^* \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F}) \setminus Y_2}$ respectively. We have $Y_1 \cap Y_2 = \emptyset$. Therefore, if both $\mathcal{E}$ and $\mathcal{F}$ are semiample, then so is $\mathcal{G}$. Q.E.D.

LEMMA 3 (Iitaka [5]). Let $f : X \rightarrow Y$ be a dominating morphism of nonsingular varieties and let $\mathcal{L}$ be a line bundle on $Y$. Then $\kappa(f^* \mathcal{L}, X) = \kappa(\mathcal{L}, Y)$.

LEMMA 4 (cf. Proposition 4.1 in [1]). Let $h : \mathcal{E} \rightarrow \mathcal{L}$ be a surjective homomorphism from a vector bundle $\mathcal{E}$ to a line bundle $\mathcal{L}$. If there exists a positive integer $m$ for which the derived homomorphism $S^m h : S^m \mathcal{E} \rightarrow \mathcal{L} \otimes m$ is splitting, then $h$ is splitting.

**Proof.** If $m \geq 2$, consider the derived homomorphism $S^{m-1} h : S^{m-1} \mathcal{E} \rightarrow \mathcal{L} \otimes m$. Tensoring with $\mathcal{L}$, we obtain a homomorphism $\alpha : S^{m-1} \mathcal{E} \otimes \mathcal{L} \rightarrow \mathcal{L} \otimes m$. On the other hand, the dual homomorphism $h^* : \mathcal{L}^* \rightarrow \mathcal{E}^*$ gives rise to the Koszul type exact sequence $0 \rightarrow S^m \mathcal{F} \rightarrow \mathcal{L}^* \rightarrow S^m \mathcal{F} \rightarrow 0$, where $\mathcal{F}$ is the cokernel of $h^*$. Hence we obtain a homomorphism $\beta : S^m \mathcal{E} \rightarrow S^m \mathcal{E} \otimes \mathcal{L}$, and then we have $S^m h = \alpha \circ \beta$. This implies that $\alpha$ is splitting and hence so is $S^{m-1} h$. Then by induction we can find that $h$ is splitting.

Q.E.D.

PROPOSITION 1. Let $\mathcal{E}$ be a semiample vector bundle on a nonsingular variety $X$, and let $h : \mathcal{E} \rightarrow \mathcal{O}_X$ be a nonzero homomorphism. Then $h$ is surjective and splitting.

**Proof.** We let $m$ be a positive integer for which the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ is spanned. The homomorphism $h$ is surjective on some open subset $U \subseteq X$. Then there exists a morphism $\rho : U \rightarrow \mathbb{P}(\mathcal{E})$ such that $\rho^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{O}_U$. Therefore the natural homomorphism $H^0(U, S^m \mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}_U \rightarrow \mathcal{O}_U$ is surjective on $U$, and hence on $X$. This implies that the derived homomorphism $S^m h : S^m \mathcal{E} \rightarrow \mathcal{O}_X$ is surjective and splitting, and hence that $h$ is surjective. Then by Lemma 4 we obtain the result.

Q.E.D.

PROPOSITION 2. Let $\mathcal{E}$ be a semiample vector bundle on a nonsingular variety $X$. Then the determinant bundle $\det \mathcal{E}$ is semiample.

**Proof.** We let $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ be the projective space bundle associated to $\mathcal{E}$, and put $r = \text{rank} \mathcal{E}$. Let $m$ be a positive integer for which the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ is
spanned. For any point \( x \in X \), choosing suitable \( r \) global sections of \( \mathcal{O}_{P(\mathcal{E})}(m) \) and taking the intersection of the divisors defined by them, we can find a nonnegative cycle \( \xi \) on \( P(\mathcal{E}) \) which represents the class \( m'c_1(\mathcal{O}_{P(\mathcal{E})}(1))^r \) and which does not meet \( \pi^{-1}(x) \). Then the projection \( \pi_*(\xi) \) is a nonnegative cycle on \( X \) which represents the class \( m'c_1(\mathcal{E}) \) (cf. [3]) and which does not contain the point \( x \). This implies that the \( m' \)th tensor power \( (\det \mathcal{E})^{\otimes m'} \) of \( \mathcal{E} \) has a global section which does not vanish at \( x \). Thus we see that the line bundle \( (\det \mathcal{E})^{\otimes m'} \) is spanned. Q.E.D.

COROLLARY 1. Let \( \mathcal{E} \) be a semiample vector bundle on a nonsingular variety \( X \). Then the vector bundle \( \mathcal{E}^* \otimes \det \mathcal{E} \) is semiample.

Proof. Let \( \pi: P(\mathcal{E}^* \otimes \det \mathcal{E}) \to X \) be the projective space bundle associated to \( \mathcal{E}^* \otimes \det \mathcal{E} \), and let \( \mathcal{N} \) be the cokernel of the natural homomorphism \( \mathcal{O}_{P(\mathcal{E}^* \otimes \det \mathcal{E})}(-1) \otimes \pi^* \det \mathcal{E} \to \pi^* \mathcal{E} \). Since \( \mathcal{E} \) is semiample, so is \( \mathcal{N} \). Hence the line bundle \( \mathcal{O}_{P(\mathcal{E}^* \otimes \det \mathcal{E})}(1) \cong \det \mathcal{N} \) is semiample. Q.E.D.

REMARK. If \( \mathcal{E} \) is a semiample vector bundle on a nonsingular variety \( X \), then it follows from Proposition 2 that \( \kappa(\det \mathcal{E}, X) \geq 0 \), where the equality holds if and only if \( c_1(\mathcal{E}) = 0 \) modulo torsion.

PROPOSITION 3. Let \( \mathcal{E} \) be a semiample vector bundle on a nonsingular variety \( X \) such that \( \kappa(\det \mathcal{E}, X) = 0 \). Then there exists a finite unramified covering \( f: \tilde{X} \to X \) such that the pull-back \( f^* \mathcal{E} \) is a trivial bundle.

Proof. Let \( \pi, r, m \) be the same as in Proof of Proposition 2, and let \( \Phi: P(\mathcal{E}) \to P(H^0(P(\mathcal{E}), \mathcal{O}_{P(\mathcal{E})}(m))) \) be the canonical morphism. Clearly \( \dim \Phi(P(\mathcal{E})) \geq r - 1 \). If \( \dim \Phi(P(\mathcal{E})) \geq r \), then we can find a positive cycle which represents the class \( m'c_1(\mathcal{E}) \). However this contradicts the fact that \( \kappa(\det \mathcal{E}, X) = 0 \). Thus we have \( \dim \Phi(P(\mathcal{E})) = r - 1 \). Therefore if we let \( W \) be an irreducible component of a smooth fibre of \( \Phi \) and let \( \lambda_1: W \to X \) be the restriction of the projection \( \pi \) to \( W \), then \( \lambda_1 \) is a finite covering. Furthermore, it follows that

\[
\mathcal{O}_{P(\mathcal{E})}(m) \otimes \mathcal{O}_W \cong \mathcal{O}_W, \quad (1.1)
\]
\[
\omega_W \cong \omega_{P(\mathcal{E})} \otimes \mathcal{O}_W. \quad (1.2)
\]

On the other hand, we have

\[
\omega_{P(\mathcal{E})} \cong \mathcal{O}_{P(\mathcal{E})}(-4) \otimes \pi^*(\omega_X \otimes \det \mathcal{E}) \quad (1.3)
\]

(cf. Proposition 8.4 in [4]). Recall that there exists a positive integer \( k \) for which \( (\det \mathcal{E})^{\otimes k} \cong \mathcal{O}_X \). Then from (1.1)--(1.3) we obtain \( \omega_W^{\otimes k} \cong \lambda_1^* \omega_X^{\otimes k} \). However the finite covering \( \lambda_1 \) induces a nonzero homomorphism \( \lambda_1^* \omega_X \to \omega_W \). Therefore this implies that \( \omega_W \cong \lambda_1^* \omega_X \), and hence that \( \lambda_1 \) is a finite unramified covering. By
virtue of (1.1), we can find a finite unramified covering \( \lambda_2: V \to W \) for which \( \lambda_2^*(\mathcal{O}_P(1) \otimes \mathcal{O}_W) \cong \mathcal{O}_V \). Put \( \lambda = \lambda_1 \circ \lambda_2: V \to X \). Then \( \lambda \) is a finite unramified covering. The universal quotient \( \pi^* \mathcal{E} \to \mathcal{O}_P(1) \) of \( \mathcal{O}(\mathcal{E}) \) induces a surjective homomorphism \( \lambda^* \mathcal{E} \to \mathcal{O}_V \) on \( V \). Then it follows from Proposition 1 that \( \lambda^* \mathcal{E} \cong \mathcal{O}_V \oplus \mathcal{F} \) with a vector bundle \( \mathcal{F} \) of rank \( r - 1 \). By Lemma 2, \( \mathcal{F} \) is semiample, and by Lemma 3, \( \kappa(\det \mathcal{F}, V) = 0 \). Hence, using induction, we obtain the result.

**Q.E.D.**

**COROLLARY 2.** Let \( \mathcal{E} \) be a semiample vector bundle on a nonsingular variety \( X \) such that \( \kappa(\det \mathcal{E}, X) = 0 \). Then the dual bundle \( \mathcal{E}^* \) is semiample, and \( \kappa(\det \mathcal{E}^*, X) = 0 \).

**Proof.** The result follows immediately from Proposition 3 and Lemma 1, 3. Q.E.D.

**COROLLARY 3.** Let \( \mathcal{E} \) be a semiample vector bundle of rank \( r \) on a nonsingular variety \( X \) such that \( \kappa(\det \mathcal{E}, X) = 0 \) and \( \dim H^0(X, \mathcal{E}) = k \). Then \( \mathcal{E} \cong \mathcal{F}_1 \oplus \mathcal{F}_2 \) with a trivial bundle \( \mathcal{F}_1 \) of rank \( k \) and a semiample vector bundle \( \mathcal{F}_2 \) of rank \( r - k \) such that \( \kappa(\det \mathcal{F}_2, X) = 0 \) and \( H^0(X, \mathcal{F}_2) = 0 \).

**Proof.** Put \( \mathcal{F}_1 = H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \cong \bigoplus_{1 \leq i \leq k} \mathcal{L}_i \) with \( \mathcal{L}_i \cong \mathcal{O}_X \) (\( i = 1, 2, \ldots, k \)). Then the natural homomorphism \( h: \mathcal{E}^* \to \mathcal{F}_1^* \) induces a nonzero homomorphism \( h_i: \mathcal{E}^* \to \mathcal{L}_i^* \) for every \( i \). By Corollary 2 the dual bundle \( \mathcal{E}^* \) is semiample. Hence, by Proposition 1 we obtain the result. Q.E.D.

**COROLLARY 4.** Let \( h: \mathcal{E} \to \mathcal{F} \) be a generically surjective homomorphism of vector bundles on a nonsingular variety \( X \). If \( \mathcal{E} \) and \( \mathcal{F} \) are semiample and if \( \kappa(\det \mathcal{F}, X) = 0 \), then \( h \) is surjective and splitting.

**Proof.** By Proposition 3 there exists a finite unramified covering \( f: \tilde{X} \to X \) such that \( f^* \mathcal{F} \) is a trivial bundle. The homomorphism \( h \) is surjective and splitting if so is the pull-back \( f^*h: f^*\mathcal{E} \to f^*\mathcal{F} \). Therefore we may assume that \( \mathcal{F} \) is a trivial bundle. Then the result follows from Proposition 1. Q.E.D.

**2. Varieties with semiample cotangent bundle**

Let \( X \) be a para-abelian variety. Then \( X \) admits a finite unramified covering \( f: A \to X \) with an abelian variety \( A \). Since \( f^*\Omega_A^1 \cong \Omega_A^1 \) is a trivial bundle, by Lemma 1 the cotangent bundle \( \Omega_A^1 \) is semiample, and \( \kappa(X) = 0 \) by Lemma 3. Conversely we have Theorem I, which follows immediately from Proposition 3.

**Proof of Theorem I.** By Proposition 3, there exists a finite unramified Galois covering \( f: A \to X \) such that \( f^*\Omega_A^1 \) is a trivial bundle. Since \( \Omega_A^1 \cong f^*\Omega_X^1 \), the covering space \( A \) is an abelian variety (cf. [6]). Q.E.D.

Before proving the second theorem, we have to study varieties of type \( Q_+ \).
PROPOSITION 4. A variety $X$ of type $Q_+$ is a nonsingular variety with semiample cotangent bundle such that $\kappa(X) = 1$.

Proof. We use the same notation as in Introduction. Put $\Gamma = \{ \sigma \in G | \sigma(v) = v \}$ for some point $v \in V$, and let $H \subseteq G$ be the subgroup generated by $\Gamma$. Then $V/H \to X$ is a finite unramified covering. Hence by Lemma 1 and 3, we may assume that $G = H$.

Let $v$ be an arbitrary point in $V$ and put $c = \varphi(v), w = \psi(v)$ where $\psi: V \to F$ is the projection. Let $s$ be a regular element of $\mathcal{O}_{C,v}$, and let $\{ t_1, t_2, \ldots, t_n \}$ be a regular system of parameters of $\mathcal{O}_{F,w}$, where $n = \dim F$. Then we can regard the set $\{ s, t_1, t_2, \ldots, t_n \}$ as a regular system of parameters of $\mathcal{O}_{V,v}$. For each $\sigma_i \in G$, the restriction of the action $\sigma_i$ to the fibre $\varphi^{-1}(c)$ is the identity. Therefore $\sigma_i$ gives rise to an automorphism $\tilde{\sigma}_i$ of the local ring $\mathcal{O}_{V,v}$ such that $\tilde{\sigma}_i(t_j) = t_j + \varepsilon_{ij} s$ with some $\varepsilon_{ij} \in \mathcal{O}_{V,v}$ for every $j$. Put $T_j = |G_1|^{-1} \sum_{\sigma \in G_0} \tilde{\sigma}_i(t_j)$. Then for every $j$ we have

$$T_j = t_j + \varepsilon_j s \quad \text{with some } \varepsilon_j \in \mathcal{O}_{V,v}, \quad (2.1)$$
$$\tilde{\sigma}_i(T_j) = T_j \quad \text{for every } i. \quad (2.2)$$

The group $G_v$ acts also on $\mathcal{O}_{C,c}$. Hence for each $i$ we have $\tilde{\sigma}_i(s) = \zeta_i s + \eta_i s^2$ with some $\zeta_i \in \mathbb{C}^*$ and some $\eta_i \in \mathcal{O}_{C,c}$. Put $S = |G_1|^{-1} \sum_{\sigma \in G_0} \zeta_i^{-1} \tilde{\sigma}_i(s)$. Then we have

$$S = s + \eta s^2 \quad \text{with some } \eta \in \mathcal{O}_{C,c}, \quad (2.3)$$
$$\tilde{\sigma}_i(S) = \zeta_i S \quad \text{for every } i. \quad (2.4)$$

Since $G_v$ acts effectively on $C$, $\zeta_{i_1} \neq \zeta_{i_2}$ if $i_1 \neq i_2$. Let $\mathcal{O}_{V,v}$ be the completion of the local ring $\mathcal{O}_{V,v}$, and let $\mathcal{L}_v$ be the subring of all invariant elements in $\mathcal{O}_{V,v}$ with respect to the action of $G_v$. Then from (2.1)–(2.4) we obtain $\mathcal{O}_{V,v} \cong C[[S, T_1, T_2, \ldots, T_n]]$ and $\mathcal{L}_v \cong C[[S^d, T_1, T_2, \ldots, T_n]]$ with $d = |G_v|$. Note that $\mathcal{L}_v$ is a regular local ring. Let $f: V \to X$ be the quotient morphism and put $x = f(v)$. Then the completion $\mathcal{O}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{L}_v$. Thus we see that the quotient space $X$ is nonsingular, and obtain the following commutative diagram with exact rows:

$$\begin{array}{cccccc}
0 & \rightarrow & \varphi^* \omega & \rightarrow & f^* \Omega^1_x & \rightarrow & \psi^* \Omega^1_v & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \varphi^* \omega_C & \rightarrow & \Omega^1_C & \rightarrow & \psi^* \Omega^1_v & \rightarrow & 0,
\end{array}$$

where $\omega = \omega_C \otimes \mathcal{O}_C(-\sum_{c \in C}(|G_1| - 1) \cdot c)$.

We claim that $f^* \Omega^1_x \cong \psi^* \Omega^1_v \otimes \varphi^* \omega$. Let $\{ e_1, e_2, \ldots, e_k \}$ be a basis of the vector space $H^0(F, \Omega^1_f)$. Each $\sigma \in G$ gives rise to an automorphism $\sigma^* \in H^0(V, \Omega^1_v)$. If $\sigma \in \Gamma$, then for some point $c \in C$ the restriction of the action $\sigma$ to the
fibre $\varphi^{-1}(c)$ is the identity, and therefore the image of $\sigma^*(\psi^*e_j)$ in $H^0(V, \psi^*\Omega_X^1)$ is $\psi^*e_j$. However, since $G = H$, this is true for every $\sigma \in G$. If we put $E_j = |G|^{-1} \sum_{\sigma \in G} \sigma^*(\psi^*e_j)$, then for all $\sigma \in G$ we have $\sigma^*(E_j) = E_j$. Hence every $E_j$ is a section of the vector bundle $f^*\Omega_X^1$, whose image in $H^0(V, \psi^*\Omega_X^1)$ is $\psi^*e_j$. By Corollary 3 we have $\Omega_X^1 \cong \Omega_0 \oplus \Omega$, where $\Omega_0 = \oplus_{1 \leq j \leq k} \mathcal{O}_V \cdot e_j$ and $\Omega$ is a semiample vector bundle of rank $n - k$ such that $\kappa(\det \Omega, F) = 0$ and $H^0(F, \Omega) = 0$. Put $\mathcal{F}_0 = \psi^*\Omega_0$, $\mathcal{F} = \psi^*\Omega$ and put $\mathcal{E} = \oplus_{1 \leq j \leq k} \mathcal{O}_V \cdot E_j$. Then we have $\psi^*\Omega_X^1 \cong \mathcal{F}_0 \oplus \mathcal{F}$ and $\mathcal{E} \cong \mathcal{F}_0$. Thus we obtain $f^*\Omega_X^1 \cong \mathcal{E} \oplus \mathcal{G}_1$, $\Omega_X^1 \cong \mathcal{E} \oplus \mathcal{G}_2$ with some vector bundles $\mathcal{G}_1, \mathcal{G}_2$ for which we have the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
0 & \rightarrow & \varphi^*\omega & \rightarrow & \mathcal{G}_1 & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \varphi^*\omega_C & \rightarrow & \mathcal{G}_2 & \rightarrow & \mathcal{F} & \rightarrow & 0.
\end{array}
$$

Let

$$
\begin{align*}
\delta_1 : H^0(V, \mathcal{F} \otimes \mathcal{F}^*) & \rightarrow H^1(V, \varphi^*\omega \otimes \mathcal{F}^*), \\
\delta_2 : H^0(V, \mathcal{F} \otimes \mathcal{F}^*) & \rightarrow H^1(V, \varphi^*\omega_C \otimes \mathcal{F}^*)
\end{align*}
$$

be the canonical homomorphisms and let $1_{\mathcal{F}} \in H^0(V, \mathcal{F} \otimes \mathcal{F}^*)$ be the identity of $\mathcal{F}$. Since the bottom exact sequence splits, we see that $\delta_2(1_{\mathcal{F}}) = 0$. Consider the following exact sequence:

$$
H^0(V, \mathcal{H} \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega \otimes \mathcal{F}^*) \rightarrow H^1(V, \varphi^*\omega_C \otimes \mathcal{F}^*),
$$

where $\mathcal{H} = \varphi^*\omega_C/\varphi^*\omega$. By Corollary 2 and 3, we have $H^0(F, \Omega^*) = 0$. Hence it can be easily checked that $H^0(V, \mathcal{H} \otimes \mathcal{F}^*) = 0$. Thus we find that $\delta_1(1_{\mathcal{F}}) = 0$, and hence obtain

$$
f^*\Omega_X^1 \cong \mathcal{E} \oplus \mathcal{G}_1 \cong \mathcal{E} \oplus \mathcal{F} \oplus \varphi^*\omega \cong \mathcal{F}_0 \oplus \mathcal{F} \oplus \varphi^*\omega \cong \psi^*\Omega_F \oplus \varphi^*\omega.
$$

Since $R < 2g - 2$, the line bundle $\omega$ is ample. Hence by Lemma 1 and 2, we see that the cotangent bundle $\Omega_X^1$ is semiample. Furthermore we have $\kappa(X) = \kappa(f^*\omega_X, V) = \kappa(\varphi^*\omega_F \otimes \psi^*\omega, V) = \kappa(\psi^*\omega, V) = \kappa(\omega, C) = 1$ by Lemma 3.

Q.E.D.

REMARK. In the above proof, it can be easily seen that the condition $R < 2g - 2$ is not only sufficient but also necessary for the quotient $X$ to be attended with semiample cotangent bundle and of Kodaira dimension $\kappa(X) = 1$. 
Proof of Theorem II. By Proposition 2, the line bundle $\omega_X$ is semiample. Hence we have a fibration $\Phi: X \to B$ with a nonsingular curve $B$ such that $\omega_X^{\otimes k} \cong \Phi^* L_0$ for some positive integer $k$ and some line bundle $L_0$ on $B$. Any smooth fibre of $\Phi$ is a para-abelian variety by Theorem I. Let $\mathcal{L}$ be the full subbundle of $\Omega^1_X$ associated to the pull-back $\Omega^* \omega_B$ of the canonical sheaf $\omega_B$. For each point $b \in B$, decompose the fibre $\Phi^{-1}(b) = \sum a_i D_i$ as a sum of irreducible components and set $D(\Phi)_b = \sum (a_i - 1) D_i$. Put $D(\Phi) = \sum_{b \in B} D(\Phi)_b$. Then we have $\mathcal{L} \cong \Phi^* \omega_B \otimes \mathcal{O}_X(D(\Phi))$ (cf. [10]). Consider the natural homomorphism $h: \mathcal{F}_X \to \mathcal{L}^*$. There exists a closed subset $Y$ of codimension 2 such that $h$ is surjective at every point in $X \setminus Y$. Tensoring with $\omega_X$, we obtain a homomorphism $h_1: \mathcal{F}_X \otimes \omega_X \to \mathcal{L}^* \otimes \omega_X$. By Corollary 1 the vector bundle $\mathcal{F}_X \otimes \omega_X$ is semiample. Hence for some positive integer $m$, the homomorphism

$$h_2: H^0(X, S^m(\mathcal{F}_X \otimes \omega_X)) \otimes_C \mathcal{O}_X \to (\mathcal{L}^* \otimes \omega_X)^{\otimes m}$$

derived from $h_1$ is surjective at every point in $X \setminus Y$. Write $\mathcal{M} = (\mathcal{L}^* \otimes \omega_X)^{\otimes m}$. Since the direct image $\Phi_* \mathcal{M}$ is a line bundle and since $H^0(B, \Phi_* \mathcal{M}) = H^0(X, \mathcal{M})$, one has $\Phi^* \Phi_* \mathcal{M} \cong \mathcal{M}$ and the zero set of each global section consists of fibres of $\Phi$. Therefore $h_2$ must be surjective at every point in $X$. This implies that $h$ is surjective, and hence that any fibre $\Phi^{-1}(b)$ is a multiple of a smooth irreducible component.

Choosing a suitable finite covering $\gamma: C \to B$ with a nonsingular curve $C$ and taking the normalization $V$ of the product $X \times_B C$, we obtain a smooth fibration $\phi: V \to C$, a finite covering $f: V \to X$,

$$
\begin{array}{ccc}
V & \xrightarrow{f} & X \\
\phi \downarrow & & \downarrow \Phi \\
C & \to & B
\end{array}
$$

and the following commutative diagram with exact rows:

$$0 \to f^* \mathcal{L} \to f^* \Omega^1_X \to \Omega^1_{V/C} \to 0$$

$$0 \to \phi^* \omega_C \to \Omega^1_V \to \Omega^1_{V/C} \to 0$$

(cf. Theorem 6.3 in [8]). From the upper exact sequence, we obtain the following one:

$$0 \to \Omega^*_{V/C} \otimes f^* \omega_X \to f^* (\mathcal{F}_X \otimes \omega_X) \to f^* (\mathcal{L}^* \otimes \omega_X) \to 0.$$
Since the homomorphism \( h_2 \) is surjective and since \((Y^* \otimes \omega_X) \otimes m\) is the pull-back of the line bundle \( \Phi_* \mathcal{M} \) on \( B \), we can find an open covering \( \{ U_i \} \) of \( B \) such that on each open subset \( \Phi^{-1}(U_i) \) the restricted homomorphism \( S^m(\mathcal{F}_X \otimes \omega_X)|_{\Phi^{-1}(U_i)} \rightarrow (L^*_p \otimes \omega_X)^{\otimes m}|_{\Phi^{-1}(U_i)} \) is splitting, and hence so is the homomorphism \( \mathcal{F}_X \otimes \omega_X|_{\Phi^{-1}(U_i)} \rightarrow \mathcal{L}_{p}^* \otimes \omega_X|_{\Phi^{-1}(U_i)} \) by Lemma 4. Then, restricted on each open subset \( \varphi^{-1}(\gamma^{-1}(U_i)) \), the exact sequence (2.6) splits, and therefore so do both of the exact sequences in (2.5). This implies that the canonical homomorphism \( \mathcal{F}_C \rightarrow R^1 \varphi_* (\Omega^*_{V/C}) \) vanishes at every point in \( C \), and hence that \( \varphi \) is a fibre bundle (cf. Theorem 5.1 in [7]).

Since the fibre of \( \varphi \) is a para-abelian variety, we see that \( \kappa(\det \omega_{V/C}, V) = 0 \) (cf. [9]). Therefore by Proposition 3, we have a finite unramified Galois covering \( \mu: \bar{V} \rightarrow V \) such that \( \mu^* \Omega_{V/C} \) is a trivial bundle. Clearly we may assume that the projection \( \bar{\varphi}: \bar{V} \rightarrow C \) is a fibration. Then, since \( \Omega_{\bar{V}/C} \) is a trivial bundle, \( \bar{\varphi} \) is a fibre bundle whose fibre \( A \) is an abelian variety. Furthermore we may assume that \( \bar{\varphi} \) has a section \( \rho: C \rightarrow \bar{V} \). Choose and fix a basis of \( H^0(\bar{V}, \Omega_{\bar{V}/C}) \). Then the basis and the section \( \rho(C) \) determine isomorphisms \( \bar{\varphi}^{-1}(c) \cong A \) for all \( c \in C \), which define an isomorphism \( \bar{V} \cong A \times C \). Each element \( \chi \) in the Galois group \( \text{Gal}(\bar{V}/V) \) gives rise to automorphisms of the fibres \( \bar{\varphi}^{-1}(c) \), and hence defines a continuous mapping \( \chi_+: C \rightarrow \text{Aut}(A) \), where \( \text{Aut}(A) \) is the group of automorphisms of \( A \). However the order of the element \( \chi \) is finite. Therefore \( \chi_+ \) must be constant. Hence we see that \( \varphi \) is a trivial fibre bundle whose fibre is a para-abelian variety.

We may assume that \( \gamma \) is a Galois covering. Then, since the variety \( V \) is the normalization of the product \( X \times_B C \), it follows that \( f \) is a finite Galois covering whose Galois group \( G = \text{Gal}(V/X) \) acts effectively both on \( V \) and on \( C \) so that \( \varphi \circ \sigma = \sigma \circ \varphi \) for every \( \sigma \in G \). Let \( F \) be an arbitrary fibre of \( \varphi \). Then by Corollary 4 the natural homomorphism \( f^* \Omega_X^k \otimes \omega_F \rightarrow \Omega_F^k \) is surjective, and hence the restriction of \( f \) to \( F \) is unramified. This implies that if \( \sigma \in G \) has a fixed point \( v \in V \), then \( \sigma(v') = v' \) for every point \( v' \in \varphi^{-1}(\varphi(v)) \). Finally from Remark to Proposition 4, we infer that the condition \( R < 2g - 2 \) holds. Thus we find that \( X \) is a variety of type \( Q_+ \).

Q.E.D.

REMARK. Let \( X \) be the same as in Definition of a variety of type \( Q_+ \), and assume that the condition \( R > 2g - 2 \) holds in place of \( R < 2g - 2 \). Then we call \( X \) a \textit{variety of type} \( Q_- \). In Proof of Proposition 4, we can easily see that a variety \( X \) of type \( Q_- \) is a nonsingular variety with semiample tangent bundle such that \( \kappa^{-1}(X) = 1 \), where \( \kappa^{-1}(X) = \kappa(\omega_X^*, X) \) is the \textit{anti-Kodaira dimension} of \( X \). Conversely, in the same manner as in Proof of Theorem II, we obtain the following:

\textbf{THEOREM II'}. \textit{Let \( X \) be a nonsingular variety with semiample tangent bundle such that \( \kappa^{-1}(X) = 1 \). Then \( X \) is a variety of type \( Q_- \).}
References