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TAKASHI ICHIKAWA

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The universal periods of curves and the Schottky problem

TAKASHI ICHIKAWA
Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840, Japan

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Introduction

The aim of this paper is to give a new approach to the Schottky problem which means, in this paper, to characterize Siegel modular forms vanishing on the jacobian locus. Our characterization uses the formal power series expressing periods of algebraic curves and deduces new relations between the Fourier coefficients.

In [10] and [7], Schottky and Manin-Drinfeld obtained the infinite product expression of the multiplicative periods of Schottky uniformized Riemann surfaces and Mumford curves respectively. First, we show the infinite products can be expressed as certain formal power series, which we call the *universal power series for multiplicative periods of algebraic curves*, in terms of the *Koebe coordinates* which are the fixed points and the ratios of generators of the corresponding Schottky groups. The existence of the universal periods is suggested by the result of Gerritzen [4] which determined a fundamental domain in the Schottky space over a complete valuation field. Our result also gives an algorithm to compute the universal periods. By using our result, one can deduce Mumford's result on the periods of degenerate Riemann surfaces ([9], IIIb, §5) and the asymptotic behavior of the periods of Riemann surfaces ([3], Corollary 3.8).

Next, we study the Schottky problem by using the universal periods. Let f be a Siegel modular form of degree g over $\mathbf{C}$ given by ($Z = (z_{ij}) \in$ the Siegel upper half space of degree g)

$$f(Z) = \sum_{T} a(T) \exp(2\pi\sqrt{-1} \cdot Tr(TZ)).$$

Since $\exp(2\pi\sqrt{-1} \cdot z_{ij})$ are the multiplicative periods of the Riemann surface with period matrix Z, $f|_{q_{ij}=p_{ij}}$ ($q_{ij} := \exp(2\pi\sqrt{-1} \cdot z_{ij})$, p_{ij}: the universal periods) is the expansion of the form (on the moduli space of curves of genus g) induced from f in terms of the Koebe coordinates. Then as an analogy of a part of the q-

expansion principle, we have

$$f \text{ vanishes on the jacobian locus} \Leftrightarrow f|_{q_{ij}=p_{ij}} = 0.$$

This follows from the retrosection theorem ([6]) which says that every Riemann surface can be Schottky uniformized. As an immediate consequence of this result, we have

THEOREM (cf. Corollary 3.3). *Let*

$$f(Z) = \sum_T a(T) \exp(2\pi\sqrt{-1} \cdot Tr(TZ))$$

be any Siegel modular form of degree g over $\mathbf{C}$ *vanishing on the jacobian locus, and put* $S = \min\{Tr(T) \mid a(T) \neq 0\}$. *Then for any set* $\{s_1, \ldots, s_g\}$ *of non-negative g-integers satisfying* $\Sigma_{i=1}^g s_i = S$,

$$\sum_{t_{ii}=s_i} a(T) \prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)}\right)^{2t_{ji}} = 0 \quad (T = (t_{ij})),$$

where $x_{\pm 1}, \ldots, x_{\pm g}$ *are variables.*

It is shown by Faltings [2] that Siegel modular forms over any field can be defined and have Fourier expansions in that field. In this paper, we prove our results for Siegel modular forms over any field by using the theory of Mumford curves ([8]) and the irreducibility of the moduli space of curves of given genus ([1]).

1. Mumford curves

In this section, we review some results in [4], [7], and [8].

1.1. Let K be a complete discrete valuation field with valuation $\|$. Let $PGL_2(K)$ act on $\mathbf{P}^1(K)$ by the Möbius transformations, i.e.,

$$g(z) = \frac{az+b}{cz+d}\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod K^\times \in PGL_2(K),\ z \in \mathbf{P}^1(K)\right).$$

Let Γ be a Schottky group over K, i.e., a finitely generated discrete subgroup of $PGL_2(K)$ consisting of hyperbolic elements. Then it is known that Γ is a free group (cf. [5]). Let g be the rank of Γ, and $\{\gamma_i \mid i = 1, \ldots, g\}$ be a set of free generators of Γ. Since γ_i is hyperbolic, this can be expressed as

$$\gamma_i = \begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta_i \end{pmatrix}\begin{pmatrix} \alpha_i & \alpha_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod K^\times,$$

where $\alpha_i, \alpha_{-i} \in \mathbf{P}^1(K)$ and $\beta_i \in K^\times$ with $|\beta_i| < 1$. Then $\alpha_{\pm i}$ are the fixed points of γ_i and β_i is one of the ratios of the eigenvalues of γ_i. Let

$$[a, b; c, d] = \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}$$

denote the cross-ratio of four points. In [4], Gerritzen showed that if $(\alpha_i, \alpha_{-i}, \beta_i)_{1 \leqslant i \leqslant g} \in (\mathbf{P}^1(K) \times \mathbf{P}^1(K) \times K^\times)^g$ satisfies $\alpha_j \neq \alpha_k$ $(j \neq k)$ and $|\beta_i| < \min\{|[\alpha_j, \alpha_k, \alpha_i, \alpha_{-i}]|\}$ for any $i, j, k \in \{\pm 1, \ldots, \pm g\}$ with j, $k \neq \pm i$, then the subgroup of $PGL_2(K)$ generated by γ_i is a Schottky group with free generators γ_i. For each Schottky group Γ of rank g over K, let D_Γ be the set of points which are not limits of fixed points of elements of $\Gamma - \{1\}$ in $\mathbf{P}^1(K)$. Then in [8], Mumford proved that the quotient D_Γ/Γ is the K-analytic space associated with a unique proper and smooth curve of genus g over K with multiplicative reduction, and that any proper and smooth curve of genus g over K with multiplicative reduction can be obtained in this way. Then D_Γ/Γ is called the Mumford curve associated with Γ, and we denote it by C_Γ.

1.2. Let x_i, x_{-i}, and y_i $(i = 1, \ldots, g)$ be variables, and put $\Omega = \mathbf{Q}(x_{\pm i}, y_i)$. Let f_i be the element of $PGL_2(\Omega)$ given by

$$(1.2.1) \quad f_i = \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_i \end{pmatrix} \begin{pmatrix} x_i & x_{-i} \\ 1 & 1 \end{pmatrix}^{-1} \bmod \Omega^\times,$$

and F the subgroup of $PGL_2(\Omega)$ generated by f_i $(i = 1, \ldots, g)$. Then F is a free group with generators f_i. For $1 \leqslant i, j \leqslant g$, let $\psi_{ij}\colon F \to \Omega^\times$ be the map given by

$$\psi_{ij}(f) = \begin{cases} y_i & (\text{if } i = j \text{ and } f \in \langle f_i \rangle) \\ [x_i, x_{-i}; f(x_j), f(x_{-j})] & (\text{otherwise}). \end{cases}$$

Then it is easy to see that ψ_{ij} depends only on double coset classes $\langle f_i \rangle f \langle f_j \rangle$ $(f \in F)$. Then in [7], Manin and Drinfeld showed that for any Schottky group $\Gamma = \langle \gamma_i \rangle$ over K,

$$q_{ij} = \prod_f \psi_{ij}(f)|_{x_{\pm i} = \alpha_{\pm i}, y_i = \beta_i}$$

(f runs through all representatives of $\langle f_i \rangle \backslash F / \langle f_j \rangle$) converges in $K^\times$ and q_{ij} $(1 \leqslant i, j \leqslant g)$ are the multiplicative periods of C_Γ, i.e., the K-analytic space associated with the jacobian variety of C_Γ is given by the quotient

$$(K^\times)^g \Big/ \left(\left(\prod_j q_{ij}^{n_j} \right)_{1 \leqslant i \leqslant g} \,\middle|\, n_j \in \mathbf{Z} \right).$$

2. Periods as power series

2.1. Let the notation be as above. For each integer $k=-1,\dots,-g$, put $y_k=y_{-k}$ and $f_k=(f_{-k})^{-1}$. Then f_k satisfies (1.2.1) also. If the reduced expression of $f\in F$ is $\Pi_{p=1}^n f_{\sigma(p)}$ ($\sigma(p)\in\{\pm 1,\dots,\pm g\}$), then we put $n(f)=n$. Put

$$R=\mathbf{Z}\left[x_i,\ \prod_{j<k} 1/(x_j-x_k)\right]\quad (i,j,k\in\{\pm 1,\dots,\pm g\}).$$

Let A be the ring of formal power series over R with variables $y_1,\dots,y_g$, i.e.,

$$A=R[[y_1,\dots,y_g]],$$

and I the ideal of A generated by y_i $(i=1,\dots,g)$.

2.2. LEMMA. *For any $f\in F$ and $i\in\{\pm 1,\dots,\pm g\}$, $f(x_i)\in A$. Moreover, if the reduced expression of $f\in F-\langle f_i\rangle$ is $f_k\cdot f'$, then $f(x_i)\in x_k+I$.*

Proof. We prove this by the induction on $n(f)$. Let f be an element of $F-\langle f_i\rangle$ with reduced expression $f_k\cdot f'$ such that $f(x_i)=x_k+a$ for some $a\in I$. Then for any $j\neq -k$,

$$(f(x_i)-x_{-j})^{-1}=\{(x_k-x_{-j})(1+a/(x_k-x_{-j})\}^{-1}\in A.$$

Hence

$$(f_j\cdot f)(x_i)=\left\{x_j-\frac{(f(x_i)-x_j)x_{-j}y_j}{f(x_i)-x_{-j}}\right\}\cdot\left\{1-\frac{(f(x_i)-x_j)y_j}{f(x_i)-x_{-j}}\right\}^{-1}$$

belongs to x_j+I. Assume that $f\in\langle f_i\rangle$. Then $f(x_i)=x_i$, and hence for any $j\neq\pm i$, $(f_j\cdot f)(x_i)\in x_j+I$.

2.3. LEMMA. *For any $f\in F$ having the reduced expression $f'\cdot f_l$ with $l\neq\pm j$, $f(x_j)-f(x_{-j})\in I^{n(f)}$.*

Proof. We prove this by the induction on $n(f)$. If $n(f)=1$, then $f=f_l$ for some $l\neq\pm j$, and hence

$$f_l(x_j)-f_l(x_{-j})=\frac{(x_l-x_{-l})^2(x_j-x_{-j})y_l}{\{x_j-x_{-l}-y_l(x_j-x_l)\}\{x_{-j}-x_{-l}-y_l(x_{-j}-x_l)\}}$$

belongs to I. Assume that the reduced expression of $f\in F$ is $f_k\cdot f'\cdot f_l$ $(l\neq\pm j)$ and $f(x_j)-f(x_{-j})\in I^{n(f)}$. Then by Lemma 2.2, there exists $b\in I$ such that

$f(x_j)=x_k+b$. Therefore, for any $m\neq -k$,

$$\{f(x_j)-x_{-m}-y_m(f(x_j)-x_m)\}^{-1}=\{(x_k-x_{-m})+(b-y_m(f(x_j)-x_m))\}^{-1}$$

belongs to A. Similarly, $\{f(x_{-j})-x_{-m}-y_m(f(x_{-j})-x_m)\}^{-1}$ belongs to A. Hence

$$\begin{aligned}&(f_m\cdot f)(x_j)-(f_m\cdot f)(x_{-j})\\&=\frac{(x_m-x_{-m})^2(f(x_j)-f(x_{-j}))y_m}{\{f(x_j)-x_{-m}-y_m(f(x_j)-x_m)\}\{f(x_{-j})-x_{-m}-y_m(f(x_{-j})-x_m)\}}\end{aligned}$$

belongs to $I^{n(f)+1}$.

2.4. PROPOSITION. *For any $f\in F$ having the reduced expression $f_k\cdot f'\cdot f_l$ with $k\neq \pm i$ and $l\neq \pm j$,*

$$[x_i,\,x_{-i};\,f(x_j),\,f(x_{-j})]\in 1+I^{n(f)}.$$

Proof. By Lemma 2.2, $f(x_j)-x_k$ and $f(x_{-j})-x_k$ belongs to I. Hence $(f(x_j)-x_{-i})^{-1}$ and $(f(x_{-j})-x_i)^{-1}$ belongs to A. Therefore, by Lemma 2.3,

$$[x_i,\,x_{-i};\,f(x_j),\,f(x_{-j})]=1+\frac{(x_i-x_{-i})(f(x_j)-f(x_{-j}))}{(f(x_j)-x_{-i})(f(x_{-j})-x_i)}$$

belongs to $1+I^{n(f)}$.

2.5. COROLLARY. *For any $1\leqslant i,j\leqslant g$, the infinite product $\Pi_f\,\psi_{ij}(f)$ (f runs through all representatives of $\langle f_i\rangle\backslash F/\langle f_j\rangle$) is convergent in A.*

2.6. We call $\Pi_f\psi_{ij}(f)\in A$ the *universal power series for multiplicative periods of algebraic curves* and denote them by p_{ij}. As seen in 1.2, when $f_i\in PGL_2(\Omega)$ ($i=1,\ldots,g$) are specialized to generators of any Schottky group Γ over a complete discrete valuation field, p_{ij} are specialized to the multiplicative periods q_{ij} of C_Γ.

2.7. By the above formulas, one can compute p_{ij} explicitly. For example,

$$p_{ij}\equiv\left(1+\sum_{|k|\neq i,j}\frac{(x_i-x_{-i})(x_j-x_{-j})(x_k-x_{-k})^2}{(x_i-x_k)(x_{-i}-x_k)(x_j-x_{-k})(x_{-j}-x_{-k})}\,y_{|k|}\right)$$
$$\bmod\begin{cases}I^2 & (\text{if } i\neq j)\\ I^3 & (\text{if } i=j),\end{cases}$$

where

$$c_{ij} = \begin{cases} [x_i, x_{-i}; x_j, x_{-j}] & (\text{if } i \neq j) \\ y_i & (\text{if } i = j). \end{cases}$$

3. The Schottky problem

3.1. Let k be a field and let f be a Siegel modular form of degree $g \geqslant 2$ and weight $h > 0$ defined over k, i.e., an element of

$$\Gamma(\mathscr{X}_g \otimes_{\mathbf{Z}} k, (\Lambda^g \pi_*(\Omega^1_{\mathscr{A}/\mathscr{X}_g}))^{\otimes h}).$$

Here $\mathscr{X}_g$ denotes the moduli stack of principally polarized abelian varieties of dimension g, and $\pi\colon \mathscr{A} \to \mathscr{X}_g$ denotes the universal abelian scheme. By a result of Faltings ([2], §6), f has the Fourier expansion rational over k:

$$F(f) = \sum_{T=(t_{ij})} a(T) \prod_{i,j=1}^{g} q_{ij}^{t_{ij}} \quad (a(T) \in k),$$

where $T = (t_{ij})$ runs through all semi-integral (i.e., $2t_{ij} \in \mathbf{Z}$ and $t_{ii} \in \mathbf{Z}$) and positive semi-definite symmetric matrices of degree g (if $k = \mathbf{C}$, then $q_{ij} = \exp(2\pi\sqrt{-1}\cdot z_{ij})$, where $Z = (z_{ij})$ is in the Siegel upper half space of degree g). Then by the symmetry $q_{ij} = q_{ji}$ and the positive semi-definiteness of T, $F(f)$ belongs to

$$k\left[q_{ij}, \prod_{i<j} 1/q_{ij}\right][[q_{11}, \ldots, q_{gg}]] \quad (1 \leqslant i, j \leqslant g).$$

Let $\mathscr{M}_g$ denote the moduli stack of proper and smooth curves of genus g, and let $\tau\colon \mathscr{M}_g \to \mathscr{X}_g$ denote the Torelli map.

3.2. THEOREM. *Let A be as in* 2.1, *and let*

$$\rho\colon k\left[q_{ij}, \prod_{i<j} 1/q_{ij}\right][[q_{11}, \ldots, q_{gg}]] \to A \mathbin{\widehat{\otimes}}_{\mathbf{Z}} k$$

be the ring homomorphism over k defined by $\rho(q_{ij}) = p_{ij}$ $(1 \leqslant i, j \leqslant g)$. Then for any Siegel modular form f of degree g defined over k, the pull back $\tau^*(f)$ of f to $\mathscr{M}_g \otimes_{\mathbf{Z}} k$ is equal to 0 *if and only if $\rho(F(f)) = 0$ in $A \mathbin{\widehat{\otimes}}_{\mathbf{Z}} k$.*

Proof. We may assume that k is algebraically closed. First we assume that $\tau^*(f) = 0$. We regard $k((z))$ as the complete discrete valuation field over k with prime element z. Then by the result of Gerritzen quoted in 1.1, $\rho(F(f))$ vanishes for any $(x_i, x_{-i}, y_i)_{1 \leqslant i \leqslant g} \in (k \times k \times k[[z]])^g$ with $x_j \neq x_k$ $(j \neq k)$ and $y_i \in z \cdot k[[z]]$. Since k is an infinite field, $\rho(F(f)) = 0$. Next we assume that $\rho(F(f)) = 0$. Let Z be the closed subset of $\mathscr{M}_g \otimes_{\mathbf{Z}} k$ defined by $\tau^*(f) = 0$. Then Z can be identified with a closed subset of $M_g \otimes_{\mathbf{Z}} k$, where M_g denotes the coarse moduli scheme of proper and smooth curves of genus g. Since $M_g \otimes_{\mathbf{Z}} k$ is irreducible ([1], 5.15), there exist a complete discrete valuation field K containing k, and a proper and smooth curve over K with multiplicative reduction which corresponds to the generic point η of $M_g \otimes_{\mathbf{Z}} k$. Hence by the assumption, Z contains η. Therefore, by the irreducibility of $M_g \otimes_{\mathbf{Z}} k$, we have $Z = M_g \otimes_{\mathbf{Z}} k$. This completes the proof.

3.3. COROLLARY. *Let f be a Siegel modular form of degree g defined over k whose Fourier expansion is $\Sigma_{T=(t_{ij})} a(T) \Pi_{i,j} q_{ij}^{t_{ij}}$, and put*

$$S = \min\{Tr(T) \mid a(T) \neq 0\}.$$

If $\tau^*(f) = 0$, then for any set $\{s_1, \ldots, s_g\}$ of non-negative g-integers satisfying $\Sigma_{i=1}^g s_i = S$,

$$\sum_{t_{ii} = s_i} a(T) \prod_{i<j} [x_i, x_{-i}; x_j, x_{-j}]^{2t_{ij}} = 0.$$

Proof. This follows from 2.7 and 3.2.

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