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Some remarks on the moduli space of principally polarized abelian varieties with level (2, 4)-structure

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Introduction

We shall first explain the moduli space of principally polarized abelian varieties with level $(n, 2n)$-structure. For a positive integer n, we define subgroups of the modular group $\Gamma_g(1) = \mathrm{Sp}_{2g}(\mathbb{Z})$:

$$\Gamma_g(n) = \left\{ \sigma \in \Gamma_g(1) \,\middle|\, \sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\},$$

$$\Gamma_g(n, 2n) = \left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g(n) \,\middle|\, \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \bmod 2n \right\}.$$

Let $\mathfrak{S}_g$ denote the Siegel upper half-space of degree g on which $\Gamma_g(1)$ acts by the map: $\tau \to \sigma \circ \tau = (a\tau + b)(c\tau + d)^{-1}$. We denote by $A_g(n, 2n)$ the quotient space of $\mathfrak{S}_g$ by $\Gamma_g(n, 2n)$, which we call the moduli space of principally polarized abelian varieties with level-$(n, 2n)$ structure. For the moduli theoretic meaning of this space, we refer to [13].

If $n \geqslant 2$, then we have the holomorphic map of $\mathfrak{S}_g$ to the projective space $\mathbb{P}^N$, $N = n^g - 1$, defined by $\tau \to (\cdots, \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(n\tau \mid 0), \cdots)$, where $\theta\begin{bmatrix} a \\ 0 \end{bmatrix}(n\tau \mid 0)$ are theta constants and a runs over a complete set of representatives of $n^{-1}\mathbb{Z}^g$ modulo $\mathbb{Z}^g$. It induces

$$\Phi_n : A_g(n, 2n) \to \mathbb{P}^N.$$

Igusa ([7], [8]) proved that Φ_n is an immersion for $n \geqslant 4$ and $4 \mid n$. Moreover Mumford proved ([11], [13]) in a purely algebraic situation that Φ_n is an immersion for all $n \geqslant 4$. In this paper we treat the map Φ_2. Very few facts about the injectivity of Φ_2 is known. The main result of this paper is:

THEOREM. *If* $x \in A_g(2, 4)$ *corresponds to the period matrix of a hyperelliptic* *curve of genus g, then* $\Phi_2^{-1}(\Phi_2(x)) = \{x\}$; *hence* $\Phi_2(x)$ *is a non-singular point of the Zariski closure of* $\Phi_2(A_g(2,4))$.

For a good application of the above result, we refer to B. van Geemen's works [3] and [4]. As another application, we have the following:

THEOREM. *If* $g \leqslant 3$, *then* Φ_2 *is injective.*

The contents of this paper are as follows. In Section 1 we discuss the local injectivity of the map Φ_2, and in Section 2 we prove our main result. In the last section 3 we prove the injectivity of Φ_2 for $g \leqslant 3$.

1. Local injectivity of Φ_2 and irreducibility of a point of $\mathfrak{S}_g$

Let $m = \begin{pmatrix} m' \\ m'' \end{pmatrix}$ denote an element in $1/2 \cdot \mathbb{Z}^{2g}$ (m' and $m'' \in 1/2 \cdot \mathbb{Z}^g$). Then we define the *theta function* $\theta[m](\tau \mid z)$ of characteristic m and of modulus $\tau \in \mathfrak{S}_g$ by

$$\theta[m](\tau \mid z) = \sum_{p \in \mathbb{Z}^g} \mathbf{e}(1/2 \cdot {}^t(m'+p)\tau(m'+p) + {}^t(m'+p)(z+m''))$$

where z is a variable in $\mathbb{C}^g$ and $\mathbf{e}(*) = \exp(2\pi\sqrt{-1}\,*)$. $\theta[m](\tau) = \theta[m](\tau \mid 0)$ is called a *theta constant* of characteristic m. We call an element $[m]$ in $1/2 \cdot \mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ a theta characteristic. We say that a theta characteristic $[m]$ is even or odd according as $\mathbf{e}(2{}^tm'm'') = e(m) = \pm 1$. The number of even theta characteristics is $M = 2^{g-1}(2^g+1)$. Since $\theta[m](\tau \mid -z) = e(m)\theta[m](\tau \mid z)$, it follows that $[m]$ is odd if and only if $\theta[m](\tau \mid z)$ is an odd function. Moreover $[m]$ is odd if and only if $\theta[m](\tau) \equiv 0$; cf. [8], Th. 6.

We shall recall the *transformation formula* of theta functions: if $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma_g(1)$, then we have

$$\begin{aligned} &\theta[\sigma \circ m](\sigma \circ \tau \mid {}^t(c\tau+d)^{-1}z) \\ &\quad = \kappa(\sigma)\det(c\tau+d)^{1/2}\mathbf{e}(\phi_m(\sigma))\mathbf{e}(1/2 \cdot {}^tz(c\tau+d)^{-1}cz)\theta[m](\tau \mid z) \end{aligned}$$

where

$$\sigma \circ m = {}^t\sigma^{-1}m + \frac{1}{2}\begin{pmatrix} \operatorname{diag}(c\,{}^td) \\ \operatorname{diag}(a\,{}^tb) \end{pmatrix}$$

$$\begin{aligned} \phi_m(\sigma) = &-1/2({}^tm'\,{}^tbdm' + {}^tm''\,{}^tacm'' - 2\,{}^tm'\,{}^tbcm'' \\ &- {}^t\operatorname{diag}(a\,{}^tb)(dm' - cm'')) \end{aligned}$$

and $\kappa(\sigma)$ is an eighth root of unity depending only on σ and on the choice of the square root sign in $\det(c\tau + d)^{1/2}$; the correspondence $m \to \sigma \circ m$ gives rise to an action of $\Gamma_g(1)$ on $1/2 \cdot \mathbb{Z}^{2g}/\mathbb{Z}^{2g}$. For details we refer to [8].

A point $\tau \in \mathfrak{S}_g$ is said to be *reducible* if there exists $\sigma \in \Gamma_g(1)$ such that

$$\sigma \circ \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \tau_1 \in \mathfrak{S}_{g_1} \quad \text{and} \quad \tau_2 \in \mathfrak{S}_{g_2}.$$

Otherwise it is said to be *irreducible*. Let (A_τ, Θ_τ) denote the principally polarized abelian variety associated with $\tau \in \mathfrak{S}_g$, i.e., $A_\tau = \mathbb{C}/(\tau, 1_g)\mathbb{Z}^{2g}$ and Θ_τ is the zero divisor of the theta function $\theta[0](\tau \mid z)$. Then τ is reducible if and only if (A_τ, Θ_τ) is a product of principally polarized abelian varieties of smaller dimension. For $\tau \in \mathfrak{S}_g$, we denote by $\mathscr{L}(\tau)$ the $2^g \times (\frac{1}{2}g(g+1)+1)$ matrix:

$$\mathscr{L}(\tau) = \left(\theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau \mid 0) \ \cdots \ \frac{\partial^2}{\partial z_i \partial z_j} \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau \mid 0) \ \cdots \right)$$

where a runs over $1/2 \cdot \mathbb{Z}^g/\mathbb{Z}^g$ and $1 \leqslant i \leqslant j \leqslant g$. Since the theta series satisfies the heat equation:

$$\frac{\partial^2}{\partial z_i \partial z_j} \theta[m](\tau \mid z) = 2\pi\sqrt{-1}(1 + \delta_{ij}) \frac{\partial}{\partial \tau_{ij}} \theta[m](\tau \mid z),$$

$\mathscr{L}(\tau)$ is a non-zero constant multiple of

$$\left(\theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau \mid 0) \ \cdots \ \frac{\partial^2}{\partial \tau_{ij}} \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau \mid 0) \ \cdots \right)$$

As a criterion for irreducibility, we have the following, which is a combination of [2] Cor. 3.23 or [18] Lem. 1.6 and [16] Th. 1.

PROPOSITION 1.1. *Let $\tau \in \mathfrak{S}_g$; then the following are equivalent:*

(1) *τ is irreducible.*
(2) *The theta divisor Θ_τ on A_τ is irreducible.*
(3) rank $\mathscr{L}(\tau) = \frac{1}{2}g(g + 1) + 1$.
(4) rank $\mathscr{L}(\sigma \circ \tau) = \frac{1}{2}g(g+1)+1$ *for all $\sigma \in \Gamma_g(1)$.*

The following two propositions are proved by A. Seyama [19]. Let (A, Θ) be a principally polarized abelian variety with an irreducible theta divisor Θ. Then the restriction homomorphism:

$\{\sigma \in \mathrm{Aut}(A) \mid \sigma^{-1}\Theta$ is algebraically equivalent to $\Theta\} \to \mathrm{Aut}(A_2)$

is injective, where A_2 is the kernel of $2 \cdot 1_A$. This fact yields

PROPOSITION 1.2. *Let $\tau \in \mathfrak{S}_g$ be irreducible; then the point $\tau \bmod \Gamma_g(2,4)$ in $A_g(2,4)$ is non-singular.*

If $\tau \in \mathfrak{S}_g$ is of the form:

$$\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \tau_i \in \mathfrak{S}_{g_i}$$

then theta constants enjoy the following vanishing property:

$$(P) \quad \theta[m](\tau) = 0 \text{ for all } m = \begin{bmatrix} m'_1 \\ m'_2 \\ m''_1 \\ m''_2 \end{bmatrix} \in 1/2 \cdot \mathbb{Z}^{2g}$$

with $e(m_1) = e(m_2) = -1$

where m'_1 and m''_1 (resp. m'_2 and m''_2) are the first g_1 (resp. the last g_2) coefficients of m' and m''. Conversely we have the following:

PROPOSITION 1.3. *Let $\tau \in \mathfrak{S}_g$. Assume τ satisfy the property* (P). *Then there exists $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g(1)$ such that*

$$\sigma \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \tau_i \in \mathfrak{S}_{g_i}$$

and $a_{ij} \equiv b_{ij} \equiv c_{ij} \equiv d_{ij} \equiv 0 \bmod 2$ where

$$a = \begin{pmatrix} a_1 & a_{12} \\ a_{21} & a_2 \end{pmatrix} \qquad a_i \in M_{g_i}(\mathbb{Z}), \text{ etc.}$$

2. Main results

We define two holomorphic maps:

$$\tilde{\Phi}_2 : \mathfrak{S}_g \to \mathbb{P}^N, \ N = 2^g - 1$$

and

$$\tilde{\Psi}_2 : \mathfrak{S}_g \to \mathbb{P}^M, \ M = 2^{g-1}(2^g + 1) - 1$$

by

$$\tilde{\Phi}_2(\tau) = \left(\cdots, \theta\begin{bmatrix} a \\ 0 \end{bmatrix}(\tau), \cdots\right), \qquad a \in 1/2 \cdot \mathbb{Z}^g/\mathbb{Z}^g$$

and

$$\tilde{\Psi}_2(\tau) = (\cdots, \theta^2[m](\tau), \cdots),$$

where $[m]$ runs over the set of even theta characteristics. They induce the maps:

$$\Phi_2 : A_g(2,4) \to \mathbb{P}^N \quad \text{and} \quad \Psi_2 : A_g(2,4) \to \mathbb{P}^M.$$

By the addition formula of theta functions; cf. [8] IV Th. 2, we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{S}_g/\Gamma_g(2,4) = A_g(2,4) & \xrightarrow{\Phi_2} & \mathbb{P}^N \\ {\scriptstyle \Psi_2}\downarrow & & \downarrow{\scriptstyle v} \\ \mathbb{P}^M & \xrightarrow[L]{} & \mathbb{P}^M \end{array}$$

where v is the Veronese map and L is an appropriate linear transformation. Since the map $\Psi_4 : A_g(4,8) = S_g/\Gamma_g(4,8) \to \mathbb{P}^M$ induced by the map $\tau \to (\cdots, \theta[m](\tau \mid 0), \cdots)$ is an immersion; cf. [8] V Cor. of Th. 4, it follows that any fiber of Ψ_2 is a finite set.

Now we shall utilize the Satake compactification $\bar{A}_g(2,4)$ of $A_g(2,4)$; cf. [1]. It is known that $\bar{A}_g(2,4)$ is a complete, normal algebraic variety and contains $A_g(2,4)$ as an open algebraic subvariety and that the boundary $\bar{A}_g(2,4) - A_g(2,4)$ is a finite disjoint union of $A_k(2,4)$'s with $0 \leqslant k \leqslant g-1$. The action of $\Gamma_g(1)/\Gamma_g(2,4)$ on $A_g(2,4)$ can be extended on $\bar{A}_g(2,4)$ naturally. Moreover the maps Φ_2 and Ψ_2 can be extended to $\bar{A}_g(2,4)$ naturally. Let $\bar{B}_g(2,4)$ denote the Zariski closure of $B_g(2,4) = \Phi_2(A_g(2,4))$; then Φ_2 induces the map $\bar{A}_g(2,4) \to \bar{B}_g(2,4)$, which we denote the same letter.

PROPOSITION 2.1. *The map* $\Phi_2 : \bar{A}_g(2,4) \to \bar{B}_g(2,4)$ *is a finite surjective morphism,* $B_g(2,4)$ *is a Zariski open subset of* $\bar{B}_g(2,4)$ *and* $\Phi_2^{-1}(B_g(2,4)) = A_g(2,4)$.

Proof. It is well known that Φ_2 is a proper *algebraic* morphism. By Prop. 1.1 and 1.2, we see that Φ_2 is a locally immersion at every irreducible point of $A_g(2,4)$; hence we have $\dim \bar{B}_g(2,4) = \dim B_g(2,4) = \frac{1}{2}g(g+1)$. It follows that Φ_2 is surjective. Since $\Phi_2(\bar{A}_g(2,4) - A_g(2,4))$ is closed in $\bar{B}_g(2,4)$, it suffices to show

$\Phi_2^{-1}(B_g(2,4)) = A_g(2,4)$. Let P denote any point of $\bar{A}_g(2,4)$. Then there exists $\sigma \in \Gamma_g(1)$ such that $\sigma^{-1} \cdot P$ is a *special point* defined by a sequence in $\mathfrak{S}_g$:

$$\tau_n = \begin{pmatrix} \tau_{1n} & * \\ * & \tau_{2n} \end{pmatrix}, \quad \tau_{in} \in \mathfrak{S}_{g_i}; n = 1, 2, 3, \ldots,$$

where $\{\tau_{1n}\}_n$ converges to $\tau_{10} \in \mathfrak{S}_{g_1}$, $\mathscr{I}m\,\tau_{2n} \to \infty$ and the other entries remain bounded. Let $\tau \in \mathfrak{S}_g$ such that $\Phi_2(\bar{\tau}) = \Phi_2(P)$, where $\bar{\tau}$ is the point in $A_g(2,4)$ induced by τ; hence we have $\Psi_2(\bar{\tau}) = \Psi_2(P)$. The point $\Psi_2(P) \in \mathbb{P}^M$ is given by

$$(\cdots, \mathbf{e}(2\phi_{\sigma^{-1}\circ m}(\sigma)) \times \lim_{n\to\infty} \theta^2[\sigma^{-1} \circ m](\tau_n \mid 0), \cdots).$$

Suppose $g_2 = g - g_1 \geqslant 1$. Then we have

$$\lim \theta^2[\sigma^{-1} \circ m](\tau_n \mid 0) = \begin{cases} \theta^2[(\sigma^{-1} \circ m)_1](\tau_{10} \mid 0) & \text{if } (\sigma^{-1} \circ m)'_2 \equiv 0 \bmod 1 \\ 0 & \text{otherwise} \end{cases}$$

where $(\sigma^{-1} \circ m)'_1$ (resp. $(\sigma^{-1} \circ m)'_2$) is the first g_1 (resp. the last g_2) coefficients of $(\sigma^{-1} \circ m)'$ and $(\sigma^{-1} \circ m)''_1$ is the first g_1 coefficients of $(\sigma^{-1} \circ m)''$; cf. [8] V Lem. 28. We put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let a', b', c', d' and a'', b'', c'', d'' denote matrices of size $g \times g_1$ and $g \times g_2$ such that $a = (a', a'')$, etc. Then we have

$$(\sigma^{-1} \cdot m)'_2 = {}^t a'' m' + {}^t c'' m'' - \tfrac{1}{2} \cdot \operatorname{diag}({}^t a'' c'').$$

If $\theta[m](\tau \mid 0) \neq 0$ then the corresponding coordinate of $\Phi(P)$ is different from 0. In particular we must have $(\sigma^{-1} \cdot m)'_2 \equiv 0 \bmod 1$. Thus we see that ${}^t a'' m' + {}^t c'' m'' \equiv \frac{1}{2} \cdot \operatorname{diag}({}^t a'' c'') \bmod 1$ is independent of m for which $\theta[m](\tau \mid 0) \neq 0$. By the Lemma below, we have $a'' \equiv c'' \equiv 0 \bmod 2$. This contradicts that σ is contained in $\Gamma_g(1)$; hence $g_2 = 0$. Thus we see that P is contained in $A_g(2,4)$. Since $\Phi_2\colon A_k(2,4) \to B_k(2,4)$, $0 \leqslant k \leqslant g$, has finite fibers, we see that $\Phi_2\colon \bar{A}_g(2,4) \to \bar{B}_g(2,4)$ is a finite morphism. □

REMARK. The essential part of the above proof is given by Geemen [3].

REMARK. Combining with Th. 2.4 below, we see that the degree of Φ_2 is in fact one.

The following lemma is proved by Igusa; [9] Lem. 7.

LEMMA 2.2. *Let r be an even positive integer. Let τ denote any point of $\mathfrak{S}_g$ and ξ an element of $\mathbb{Z}^{2g}$; suppose that ${}^t\xi m \bmod 1$ is independent of m in $r^{-1}\mathbb{Z}^{2g}$ for which $\theta[m](\tau \mid 0) \neq 0$; then $\xi \equiv 0 \bmod r$.*

LEMMA 2.3. *Let τ denote any point of $\mathfrak{S}_g$ and σ an element of $\Gamma_g(2)$. If there exists a non-zero constant c satisfying*

$$\theta^2[m](\tau \mid 0) = c\theta^2[m](\sigma \circ \tau \mid 0)$$

for all $m \in 1/2 \cdot \mathbb{Z}^{2g}$, then σ is contained in $\Gamma_g(2, 4)$.

Proof. We put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\sigma \in \Gamma_g(2)$, we have

$$\theta^2[\sigma \circ m](\sigma \circ \tau \mid 0) = \theta^2[m](\sigma \circ \tau \mid 0)$$

and

$$\mathbf{e}(2 \cdot \phi_m(\sigma)) = \mathbf{e}(-{}^t m' {}^t bdm' + {}^t m'' {}^t acm'').$$

Hence, by the transformation formula, we get

$$\theta^2[m](\sigma \cdot \tau \mid 0) = \kappa(\sigma)^2 \det(c\tau + d)\mathbf{e}(-{}^t m' {}^t bdm' + {}^t m'' {}^t acm'')\theta^2[m](\tau \mid 0).$$

By the assumption, we see that $-{}^t m' {}^t bdm' + {}^t m'' {}^t acm'' \bmod 1$ is independent of m for which $\theta[m](\tau \mid 0) \neq 0$. Since ${}^t bd$ and ${}^t ac$ are symmetric, it follows that $-{}^t m' {}^t bdm' + {}^t m'' {}^t acm'' \equiv ({}^t\mathrm{diag}({}^t b'd), \ {}^t\mathrm{diag}({}^t ac'))m \bmod 1$, where $b = 2b'$ and $c = 2c'$. By Lem. 2.2, we have $\mathrm{diag}({}^t b'd) \equiv \mathrm{diag}({}^t ac') \equiv 0 \bmod 2$; hence $\mathrm{diag}({}^t bd) \equiv \mathrm{diag}({}^t ac) \equiv 0 \bmod 4$. Thus we have shown that

$$\sigma^{-1} = \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix} \in \Gamma_g(2, 4).$$

Since $\Gamma_g(2, 4)$ is a group, σ is contained in $\Gamma_g(2, 4)$. □

Following [14], we shall recall the definition of the period matrix of a hyperelliptic curve. Let C denote a hyperelliptic curve of genus g defined by an equation:

$$y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2g+1}), \quad a_i \neq a_j \in \mathbb{C}.$$

We denote by $\{A_i, B_i\}$ the *standard* homology basis on C; cf. [14] III §5 and $\{\omega_i\}$ the normalized basis of the space of the holomorphic 1 forms on C; hence we have

$$\left(\int_{A_i} \omega_j\right) = 1_g \quad \text{and} \quad \left(\int_{B_i} \omega_j\right) = \tau \in \mathfrak{S}_g.$$

We call τ a *standard* period matrix of C associated with the branch points $B = \{a_1, \ldots, a_{2g+1}, \infty\}$. Let $\mathfrak{H}_g$ denote the subset of $\mathfrak{S}_g$ consisting of points which are $\Gamma_g(1)$-equivalent to period matrices of hyperelliptic curves and let $\mathfrak{H}_g(2,4) = \mathfrak{H}_g/\Gamma_g(2,4)$.

THEOREM 2.4. *If x is a point of $\mathfrak{H}_g(2,4)$, then $\Phi_2^{-1}(\Phi_2(x)) = \{x\}$.*

Proof. Let τ denote a point of $\mathfrak{S}_g$ such that τ induces x. By definition, there exists a hyperelliptic curve C defined by an equation: $y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2g+1})$ such that the standard matrix τ_0 associated with $\{a_1, \ldots, a_{2g+1}, \infty\}$ is $\Gamma_g(1)$-equivalent to τ, i.e., $\sigma \circ \tau = \tau_0$ for some $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g(1)$. Let τ' be another point of $\mathfrak{S}_g$ such that $\tilde{\Phi}_2(\tau) = \tilde{\Phi}_2(\tau')$; hence $\tilde{\Psi}_2(\tau) = \tilde{\Psi}_2(\tau')$. By the transformation formula, we have

$$\frac{\theta^2[m](\sigma \circ \tau)}{\theta^2[m](\sigma \circ \tau')} = \frac{\det(c\tau + d)}{\det(c\tau' + d)} \cdot \frac{\theta^2[\sigma^{-1} \circ m](\tau)}{\theta^2[\sigma^{-1} \circ m](\tau')},$$

which does not depend on m. Therefore we have $\tilde{\Psi}_2(\tau_0) = \tilde{\Psi}_2(\sigma \cdot \tau) = \tilde{\Psi}_2(\sigma \cdot \tau')$. By Th. 1 in [17], we see that $\sigma \cdot \tau'$ is also the standard period matrix of a hyperelliptic curve defined by an equation: $y^2 = (x - a'_1)(x - a'_2) \cdots (x - a'_{2g+1})$. Since

$$\frac{\theta^4[m](\tau_0)}{\theta^4[n](\tau_0)} = \frac{\theta^4[m](\sigma \circ \tau')}{\theta^4[n](\sigma \circ \tau')},$$

we get, by III Cor. 8.13 in [14],

$$(a_k - a_l)/(a_k - a_m) = (a'_k - a'_l)/(a'_k - a'_m)$$

for all k, l and m; hence $\tau_0 = \sigma \circ \tau = (\sigma' \circ \tau')$ for some $\sigma' \in \Gamma_g(2)$ by III Lem. 8.12 in [14]. Since $\Gamma_g(2)$ is a normal subgroup, we have $\sigma_0 = \sigma^{-1} \cdot \sigma' \cdot \sigma \in \Gamma_g(2)$. Moreover we have $\tilde{\Phi}_2(\tau) = \tilde{\Phi}_2(\tau') = \tilde{\Phi}_2(\sigma_0 \circ \tau')$. By Lem. 2.3, we see $\sigma_0 \in \Gamma_g(2,4)$. □

3. The injectivity of Φ_2 for $g \leqslant 3$

In this section we discuss the injectivity of the canonical map:

$$\Phi_2^{(g)} = \Phi_2 \colon A_g(2,4) = \mathfrak{S}_g/\Gamma_g(2,4) \to \mathbb{P}^N, \; N = 2^g - 1.$$

LEMMA 3.1. *Assume that $\Phi_2^{(k)}$ is injective for $1 \leqslant k \leqslant g - 1$ and that $\Phi_2^{(g)}$ is injective on the irreducible points. Then $\Phi_2^{(g)}$ is injective.*

Proof. We shall prove that $\Phi_2^{(g)}$ is injective on the reducible points. Let τ and τ' be two points in $\mathfrak{S}_g$ such that $\tilde{\Phi}_2^{(g)}(\tau) = \tilde{\Phi}_2^{(g)}(\tau')$. Suppose τ is reducible; hence there exists an element $\sigma \in \Gamma_g(1)$ such that

$$\sigma \circ \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \tau_i \in \mathfrak{S}_{g_i};\ g_i > 0.$$

Since $\tilde{\Psi}_2^{(g)}(\tau) = \tilde{\Psi}_2^{(g)}(\tau')$, by the transformation formula, we have a non-zero constant c such that

$$\theta^2[\sigma \circ m](\sigma \circ \tau) = c \cdot \theta^2[\sigma \circ m](\sigma \circ \tau')$$

for all $m \in 1/2 \cdot \mathbb{Z}^{2g}$. It follows that the $\theta[m](\sigma \circ \tau')$'s satisfy the vanishing property (P) in Section 1. By Prop. 1.3, we get an element $\sigma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in $\Gamma_g(1)$ such that

$$\sigma' \circ (\sigma \circ \tau') = \begin{pmatrix} \tau'_1 & 0 \\ 0 & \tau'_2 \end{pmatrix}, \quad \tau'_i \in \mathfrak{S}_{g_i}$$

and that $a'_{ij} \equiv b'_{ij} \equiv c'_{ij} \equiv d'_{ij} \equiv 0 \bmod 2$, where

$$a' = \begin{pmatrix} a'_1 & a'_{12} \\ a'_{21} & a'_2 \end{pmatrix}, \quad a'_i \in M_{g_i}(\mathbb{Z}),$$

etc. Then we have

$$\begin{pmatrix} a'_i & b'_i \\ c'_i & d'_i \end{pmatrix} \bmod 2 \in \mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z}).$$

Since the canonical homomorphism $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ is surjective, there exists $\sigma_i \in \Gamma_{g_i}(1)$, $i = 1, 2$, such that $\sigma' \equiv \sigma_1 \oplus \sigma_2 \bmod \Gamma_g(2)$, where

$$\sigma_1 \oplus \sigma_2 = \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}$$

if $\sigma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 1, 2.$

Since $\sigma' \circ m \equiv (\sigma_1 \oplus \sigma_2) \circ m \bmod 1$, we have

$$\begin{aligned}
&\theta^2[\sigma' \circ m](\sigma' \circ (\sigma \circ \tau')) \\
&\quad = \theta^2[(\sigma_1 \oplus \sigma_2) \circ m]\left(\begin{pmatrix} \tau_1' & 0 \\ 0 & \tau_2' \end{pmatrix}\right) \\
&= \theta^2[\sigma_1 \circ m_1](\tau_1') \cdot \theta^2[\sigma_2 \circ m_2](\tau_2') \\
&= \prod_{i=1}^{2} \kappa(\sigma_i)^2 \det(c_i(\sigma_i^{-1} \circ \tau_i') + d_i) \mathbf{e}(2 \cdot \phi_{m_i}(\sigma_i)) \theta^2[m_i](\sigma_i^{-1} \circ \tau_i').
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\theta^2[\sigma' \circ m](\sigma' \circ (\sigma \circ \tau')) \\
&\quad = \kappa(\sigma)^2 \det(c'(\sigma \circ \tau') + d') \mathbf{e}(2 \cdot \phi_m(\sigma')) \theta^2[m](\sigma \circ \tau') \\
&\quad = \kappa(\sigma)^2 \det(c'(\sigma \circ \tau') + d') \mathbf{e}(2 \cdot \phi_{m_1}(\sigma_1) + 2 \cdot \phi_{m_2}(\sigma_2)) \theta^2[m](\sigma \circ \tau').
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\prod_{i=1}^{2} \theta^2[m_i](\sigma_i^{-1} \circ \tau_i') = c_1 \cdot \theta^2[m](\sigma \circ \tau') \\
&\quad = c_2 \prod_{i=1}^{2} \theta^2[m_i](\tau_i)
\end{aligned}$$

where c_1 and c_2 are non-zero constants independent of m. By these equalities, we have $\tilde{\Psi}_2^{(g_i)}(\tau_i) = \tilde{\Psi}_2^{(g_i)}(\sigma_i^{-1} \circ \tau_i')$, $i = 1, 2$. By the assumption we have $\mu_i \in \Gamma_{g_i}(2, 4)$ such that $\sigma_i^{-1} \circ \tau_i' = \mu_i \circ \tau_i$, $i = 1, 2$. Then we have $(\mu_1 \oplus \mu_2) \circ (\sigma \circ \tau) = (\sigma_1 \oplus \sigma_2)^{-1} \circ (\sigma' \circ \sigma \circ \tau)$. Since both of $(\mu_1 \oplus \mu_2)$ and $(\sigma_1 \oplus \sigma_2)^{-1} \sigma'$ are elements of $\Gamma_g(2)$, $\sigma^{-1} \circ ((\mu_1 \oplus \mu_2)^{-1} \circ (\sigma_1 \oplus \sigma_2)^{-1} \circ \sigma') \circ \sigma$ is also contained in $\Gamma_g(2)$; hence it is contained in $\Gamma_g(2, 4)$ by Lemma 2.4. Thus we have shown the injectivity of $\Phi_2^{(g)}$. □

THEOREM 3.2. *$\Phi_2^{(g)}$ is injective for $g \leqslant 3$.*

Proof. $\Phi_2^{(1)}$ is injective by Th. 2.4. Hence by Lemma 3.1 and Th. 2.4, $\Phi_2^{(2)}$ is injective. By Lemma 3.1 and Th. 2.4, the injectivity of $\Phi_2^{(3)}$ comes from the following lemma, which is proved in [6].

LEMMA 3.3. *Let τ and τ' be two points in $\mathfrak{S}_3$ such that τ is the period matrix of a non-hyperelliptic curve. If $\tilde{\Phi}_2(\tau) = \tilde{\Phi}_2(\tau')$, then $\tau = \sigma \circ \tau'$ for some $\sigma \in \Gamma_3(2, 4)$.*

Proof. We shall give a sketch of the proof. Since τ is the period matrix of a non-hyperelliptic curve, no even theta constants $\theta[m](\tau)$ vanishes. The number of even theta characteristics is $M + 1 = 2^{g-1}(2^g + 1) = 36$. We recall that the map: $\Psi_4\colon \mathfrak{S}_3/\Gamma_g(4, 8) \to \mathbb{P}^{35}$ defined by $\tau \bmod \Gamma_g(4, 8) \to (\cdots, \theta[m](\tau), \cdots)$ is injective. Since $\tilde{\Psi}_2(\tau) = \tilde{\Psi}_2(\tau')$, we have a non-zero constant c such that $\theta^2[m](\tau) = c^2 \cdot \theta^2[m](\tau')$ for all $m \in 1/2 \cdot \mathbb{Z}^{2g}$; hence $\theta[m](\tau) = c\varepsilon(m)\theta[m](\tau')$ with $\varepsilon(m) = \pm 1$. Using a set of generators for the group $\Gamma_3(2,4)/\Gamma_3(4,8)$; cf. [7], we see that there exist a non-zero constant d independent of m, an element $\sigma \in \Gamma_3(2,4)$ and 29 even theta characteristics $[m_i]$, $1 \leqslant i \leqslant 29$, satisfying $\theta[m_i](\tau) = d\varepsilon(m_i)\theta[m_i](\sigma \circ \tau)$ for $1 \leqslant i \leqslant 29$. Therefore we have $\theta[m_i](\tau') = (d/c)\theta[m_i](\sigma \circ \tau)$ for $1 \leqslant i \leqslant 29$. By theta relations; cf. e.g., [15] II Th. 18, we have $\theta[m](\tau') = (d/c)\theta[m](\sigma \circ \tau)$ for all even theta characteristics $[m]$. Hence, by the injectivity of Ψ_4, there is an element $\mu \in \Gamma_3(4,8)$ such that $\tau' = \mu \circ (\sigma \circ \tau)$. Then $\mu \circ \sigma \in \Gamma_3(2,4)$. This completes the proof. □

REMARK. The extended morphism

$$\Phi_2^{(g)}: \bar{A}_g(2, 4) \to \mathbb{P}^N$$

to the Satake compactification $\bar{A}_g(2,4)$ of $A_g(2,4)$ is also injective for $g \leqslant 3$. This is pointed out by the referee. I appreciate here the unknown referee's kind advice.

References

1. H. Cartan, *Fonctions automorphes*, Seminaire E.N.S. 1957/1958.
2. C. H. Clemens and P. A. Griffith, The intermediate Jacobian of the cubic threefold, *Ann. of Math.* 95 (1972), 281–356.
3. B. van Geemen, Siegel modular forms vanishing on the moduli space of curves, *Inv. Math.* 78 (1984), 329–349.
4. B. van Geemen, *The Schottky problem and moduli spaces of Kummer varieties*, Thesis, Utrecht, Netherland, 1985.
5. B. van Geemen and G. van der Geer, Kummer varieties and the moduli spaces of abelian varieties, *Amer. J. Math.* 108 (1986), 615–642.
6. J. P. Glass, Theta constants of genus three. *Compositio Math.* 40 (1980), 123–137.
7. J. Igusa, On the graded ring of theta constants, *Amer. J. Math.* 86 (1964), 219–246.
8. J. Igusa, *Theta functions*, Grund, Math. Wiss. 194, Springer-Verlag, 1972.
9. J. Igusa, On the variety associated with the ring of Thetanullwerte, *Amer. J. Math.* 103 (1981).
10. D. Mumford, *Introduction to algebraic geometry*, Lecture Notes, Harvard Univ. 1967.
11. D. Mumford, Varieties defined by quadratic equations, Questioni sulle varieta algebraiche. Corsi dal C.I.M.E. Edizioni Cremonese Roma, 1969.
12. D. Mumford, *Abelian varieties*, Tata studies in Math. Oxford Univ. Press, 1969.
13. D. Mumford and J. Fogarty, *Geometric invariant theory*, Ergebnisse der Math. 34, Springer-Verlag, 1982.
14. D. Mumford, *Lectures on theta II*, Progr. in Math. 43 Birkhäuser (1984).
15. H. Rauch and H. Farkas, *Theta functions with applications to Riemann surfaces*, Williams and Wilkins, 1974.
16. R. Sasaki, Modular forms vanishing at the reducible points of the Siegel upper-half space, *J. für die reine angew. Math.* 345 (1983), 1'1–123.
17. R. Sasaki, Moduli space of hyperelliptic period matrices with level 2 structure, to appear.
18. T. Sekiguchi, On the cubics defining abelian varieties, *J. Math. Soc. Japan* 30 (1978), 703–721.
19. A. Seyama, A characterization of reducible abelian varieties, to appear.