HIROSHI SAITO

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HIROSHI SAITO

Department of Mathematics, College of General Education, Kyoto University, Kyoto, Japan

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1. Introduction

For a local field $F$ of characteristic 0, let $\pi$ an irreducible admissible representation of $GL_2(F)$ of infinite dimension, and $\omega_\pi$ its central character, $\chi_\pi$ its character respectively. For each quadratic extension $L$ of $F$, we fix an embedding of $L^\times$ into $GL_2(F)$, and consider $L^\times$ as the set of $F$-rational points of a Cartan subgroup of $GL_2(F)$. In [T], Tunnell gave an expression for the restriction of $\chi_\pi$ to $L^\times$ as a sum of quasicharacters of $L^\times$, in which the coefficients are written in terms of $\varepsilon$-factors of the base change lifting of $\pi$ to $L$. His proof is based on the case by case computation of $\varepsilon$-factors and characters and is not transparent. Furthermore the case of residual characteristic 2 was only partially treated. The purpose of this paper is to give a natural proof of the formula of Tunnell including the case of residual characteristic 2.

We state the theorem more exactly. Let $\psi$ be a nontrivial additive character of $F$, and set $\psi_L = \psi \circ tr_{L/F}$, where $tr_{L/F}$ is the trace from $L$ to $F$. Let $\Pi$ be the base change lifting of $\pi$ to $L$ (cf. [L]). Then the central character $\omega_\Pi$ of $\Pi$ is given by $\omega_\Pi = n_{L/F}$, where $n_{L/F}$ is the norm from $L$ to $F$. For each quasicharacter $\lambda$ of $L^\times$, let $\varepsilon(\Pi \otimes \lambda, \psi_L)$ be the $\varepsilon$-factor of the representation $\Pi \otimes \lambda$ of $GL_2(L)$ with respect to $\psi_L$. For $\lambda$ whose restriction to $F^\times$ is $\omega_\pi$, it is shown in [T] that $\varepsilon(\Pi \otimes \lambda, \psi_L)$ is independent of $\psi$ and is equal to 1 or $-1$. In this notation, we can state the result as follows.

THEOREM. Let $\pi$ be an infinite dimensional irreducible admissible representation of $GL_2(F)$, and $\chi_\pi$ its character. For a quadratic extension $L$ of $F$, let $\Pi$ be the base change lifting of $\pi$ to $L$. Then one has

$$
\chi_\pi|_{L^\times} = \sum_{\lambda} \frac{1 + \varepsilon(\Pi \otimes \lambda, \psi_L)\omega_\pi(-1)}{2} \lambda,
$$

where $\lambda$ runs through all quasicharacters of $L^\times$ whose restriction to $F^\times$ coincide with $\omega_\pi$.

Here the functions on both sides are considered as continuous functions on...
The idea of the proof is simple. Let $\sigma$ be the generator of the Galois group $\text{Gal}(L/F)$. For $g \in \text{GL}_2(L)$, we set

$$N(g) = g^\sigma g.$$ 

Here $^\sigma g$ denotes the componentwise action of $\sigma$ on $g$. For $a \in L^\times$, to distinguish components of $\text{GL}_2(L)$ and elements of $\text{GL}_2(F)$ under the embedding fixed above, we write $\tilde{a}$ when considered as an element of $\text{GL}_2(F)$. Then we have

$$N \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \right)^w = \begin{pmatrix} a & 0 \\ 0 & ^\sigma a \end{pmatrix} \sim \tilde{a} \in \text{GL}_2(F), \quad (1.1)$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\sim$ denotes the conjugacy in $\text{GL}_2(L)$. The idea is to reduce the calculation of $L(x)$ to that in $\text{GL}_2(L)$ by means of (1.1) and the theory of base change lifting.

### 2. Proof of the theorem

Besides the notations introduced above, we use the following ones. For a local field $L$, let $\mathfrak{o}_L$ be the maximal compact ring of $L$, and $p_L$ the maximal ideal of $\mathfrak{o}_L$. Let $v_L$ be the additive valuation of $L$ such that $v_L(\mathfrak{p}_L) = 1$ for a prime element $\mathfrak{p}_L$ of $L$. Let $|\mathfrak{o}_L/p_L| = q_L$, and let $||$ be the absolute value of $L$ such that $|\mathfrak{p}_L| = q_L^{-1}$.

For a quasicharacter $\lambda$ of $L^\times$, we denote by $f(\lambda)$ the exponent of the conductor of $\lambda$. For quasicharacters $\lambda_1, \lambda_2$ of $L^\times$, we define $\lambda_1 \sim \lambda_2$ if $\lambda_1 \lambda_2^{-1}$ is unramified.

Let $L$ and $F$ be as in Section 1, and let $n(\psi_L)$ the largest integer which satisfies $\psi_L(p_L^{-n(\psi_L)}) = \{1\}$. For a positive integer $n$, set

$$\Gamma_n = \left( \begin{pmatrix} 1 + p_L^n & p_L^n \\ p_L^n & 1 + p_L^n \end{pmatrix} \right) \cap \text{GL}_2(\mathfrak{o}_L).$$

For an irreducible admissible representation $\Pi$ of $\text{GL}_2(L)$, let $p_L^{(\Pi)}$ the conductor of $\Pi$. For $\Pi$, let $L(s, \Pi)$ and $\varepsilon(s, \Pi, \psi_L)$ be as in [J-L], and set $\varepsilon(\Pi, \psi_L) = \varepsilon(1/2, \Pi, \psi_L)$.

The proof of the theorem in the cases of principal series and special representations is easy and treated completely in [T], and in the following we assume $\pi$ is supercuspidal. Let $\mathcal{W}(\Pi)$ be the Whittaker model of $\Pi$ with respect to the additive character $\psi_L$. Then $\mathcal{W}(\Pi)$ consists of functions $W$ on $\text{GL}_2(L)$
which satisfy
\[ W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi_L(x)W(g), \quad x \in L, \]
and the right translation \( \rho \) gives \( \Pi \). Let \( ^*\Pi \) be the representation defined by \( ^*\Pi(g) = \Pi(^*g) \) for \( g \in GL_2(L) \). We set
\[ ^*\mathcal{W}(\Pi) = \{ W(\sigma g) | W \in \mathcal{W}(\Pi) \}, \]
and \( (I_\sigma W)(g) = W(\sigma^{-1} g) = W(\sigma g) \). Then \( I_\sigma \) gives an isomorphism from \( ^*\mathcal{W}(\Pi) \) to \( \mathcal{W}(\Pi) \), and for \( W' \in ^*\mathcal{W}(\Pi) \), we see
\[ W'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi_L(x)W'(g) \]
and
\[ \rho(g)W' = I_\sigma^{-1} \Pi(\sigma g) I_\sigma W'. \]
By the uniqueness of Whittaker models, we have \( \mathcal{W}(^*\Pi) = ^*\mathcal{W}(\Pi) \). Since \( \Pi \) is the base change lifting of \( \pi \) to \( L \), \( \Pi \) is equivalent to \( ^*\Pi \) and \( \mathcal{W}(\Pi) = \mathcal{W}(^*\Pi) = ^*\mathcal{W}(\Pi) \). We see also that \( I_\sigma \) gives an intertwining operator of \( \Pi \) to \( ^*\Pi \) which satisfies \( I_\sigma^2 = 1 \), and fixes the linear form \( L(W) = W(1) \) on \( \mathcal{W}(\Pi) \). Setting \( \Pi((g, \sigma)) = \Pi(g)I_\sigma \), we can extend \( \Pi \) to the semidirect product of \( GL_2(L) \) and \( \text{Gal}(L/F) \) by the action of \( \text{Gal}(L/F) \) on \( GL_2(L) \). Let \( \chi_{\Pi,\sigma} \) be the twisted character of this representation, namely, the distribution on \( C^\infty_c(GL_2(L)) \) defined by
\[ \chi_{\Pi,\sigma}(\varphi) = \text{trace}(\Pi(\varphi)I_\sigma), \]
for \( \varphi \in C^\infty_c(GL_2(L)) \). Then the twisted character is given by a function which is locally integrable and locally constant on the set of \( \sigma \)-regular elements of \( GL_2(L) \) and satisfies
\[ \chi_{\Pi,\sigma}(\sigma g) = \chi_{\Pi}(Ng), \]
for \( \sigma \)-regular elements (cf. [L], [A-C]). By (1.1), to obtain \( \chi_\pi|_{L^*} \), it is enough to compute \( \chi_{\Pi,\sigma}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) \). We carry out this calculation in the Kirillov model \( \mathcal{K}(\Pi) \) of \( \Pi \), that is, on \( \mathcal{K}(\Pi) = \left\{ W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) | W \in \mathcal{W}(\Pi) \right\} \). An element \( f \) of \( \mathcal{K}(\Pi) \) is a locally constant function on \( L^* \) which satisfies
\[ f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} x\right) = \psi_L(bx)f(ax), \]
and the action of $I_\sigma$ can be written as $(I_\sigma f)(x) = f(\sigma^{-1} x) = f(\sigma x)$.

First we treat the case where $\Pi$ is supercuspidal. Then $\mathcal{H}(\Pi)$ coincides with the space $\mathcal{H}(L^\ast)$ of Schwartz-Bruhat functions on $L^\ast$ and a basis of this space is given by the set of the following functions:

$$\zeta(x) = \begin{cases} \lambda(x) & \text{if } \nu_L(x) = -n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $n$ is extended over all integers and $\lambda$ is extended over a complete system of representatives of all quasicharacters of $L^\ast$ modulo $\sim$ defined above. Later in the proof we choose representatives suitably. On this basis, the action of $w$ is described by means of $\epsilon$-factor as follows.

**Lemma 2.1.** Let $\Pi$ be a supercuspidal representation of $GL_2(L)$ and $\zeta(\lambda)$ as above. Then one has

$$\Pi(w)\zeta(\lambda) = \epsilon(\Pi \otimes \lambda^{-1}, \psi_L)\zeta(\mu)\omega_\Pi\lambda^{-1},$$

where $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$.

This is the formula (9) of [Y] and can be deduced from local functional equations of $GL_2(L)$.

We determine the subspace $\mathcal{H}(\Pi)^n$ of $\mathcal{H}(\Pi)$ consisting of elements invariant under $\Gamma_n$. Since an element of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in o_L^+$, normalizes $\Gamma_n$, $\mathcal{H}(\Pi)^n$ has a basis consisting of elements of the form

$$v = \sum_m a_m \zeta(\mu),$$

with $\lambda$ such that $f(\lambda) \leq n$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L)$ with $d \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (ad - bc)/d & b \\ c & d \end{pmatrix} w^{-1} \begin{pmatrix} 1 & -c/d \\ 0 & 1 \end{pmatrix} w.$$

Hence for $n$ such that $f(\omega_\Pi) \leq n$, $v$ is invariant under $\Gamma_n$ if and only if $v$ and $\Pi(w)v$ are invariant under $\begin{pmatrix} 1 & 0 \\ 0 & p_L^n \end{pmatrix}$. This condition is equivalent to that the supports of $v$ and $\Pi(w)v$ are contained in $p_L^{-n(\psi_L)-n}$, and that $a_m = 0$ unless $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leq m \leq n(\psi_L) + n$ by the above lemma. Let $B_n$ be the set of $\zeta(\mu)$ such that $f(\lambda) \leq n$ and $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leq m \leq n(\psi_L) + n$. Then $B_n$
gives a basis of $\mathcal{H}(\Pi)^n$ for $n$ sufficiently large and the union $\bigcup_n B_n$ gives a basis of $\mathcal{H}(\Pi)$. Let $P_n$ be the projection of $\mathcal{H}(\Pi)$ onto $\mathcal{H}(\Pi)^n$ defined by

$$\int_{\Gamma_n} \Pi(g)dg/\int_{\Gamma_n} dg,$$

where $dg$ is a Haar measure on $GL_2(L)$. Then we can calculate the value of $\chi_{\Pi,\sigma}(\tilde{a})$ as trace $\left( \Pi \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w \right) I_\sigma P_n \right)$ with respect to this basis for a sufficiently large $n$ (cf. [T] Lemma 2.2).

By the above lemma and the relation $e(\Pi \otimes \sigma \lambda^{-1}, \psi_L) = e(\sigma \Pi \otimes \lambda^{-1}, \psi_L) = e(\Pi \otimes \lambda^{-1}, \psi_L)$, which follows from $\sigma \Pi \sim \Pi$, we have

**LEMMA 2.2.** The notation being as above, for $a \in L_{\text{reg}}^\times$ one has

$$\Pi \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w \right) I_\sigma \xi_{\lambda}(a) = e(\Pi \otimes \sigma \lambda^{-1}, \psi_L) e(\sigma \Pi \otimes \lambda^{-1}, \psi_L) (-a)^{m + v_L(a)}.$$

Here $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$.

Hence if $\xi_{\lambda}(a)$ contributes to the $\chi_{\Pi,\sigma}(\tilde{a})$, then it holds

1. $n = n(\psi_L) + \frac{1}{2} f(\Pi \otimes \lambda^{-1}) + v_L(a)$,
2. $\lambda^{|L_{\sigma}|} = \omega_{\Pi} |L_{\sigma}|^\sigma \lambda^{-1}|L_{\sigma}|$.

First we assume $L/F$ is unramified. Then (b) implies that $\lambda^{|L_{\sigma}|} = \omega_{\Pi} |L_{\sigma}|^\sigma \lambda^{-1}|L_{\sigma}|$. As a representative of the class of $\lambda$, we take $\lambda$ such that $\lambda|_{\mathfrak{o}_F} = \omega_{\sigma} |\mathfrak{o}_F|$. Then we have $\omega_{\Pi} |L_{\sigma}|^\sigma \lambda^{-1} = \lambda$. Let $\lambda'$ be the quasicharacter of $L^\times$ defined by the condition $\lambda'|_{L_{\sigma}} = \lambda^{|L_{\sigma}|}$ and $\lambda|_{\mathfrak{o}_F} = -\lambda'(\mathfrak{o}_F)$. Then we see the contribution to $\chi_{\Pi,\sigma}$ of $\xi_{\lambda}(a)$ for the above $\lambda$ is equal to

$$\begin{cases} e(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-a) & \text{if } v_L(a) \equiv f(\Pi \otimes \lambda^{-1}) \pmod{2}, \\ 0 & \text{otherwise}, \end{cases}$$

and noticing $e(\Pi \otimes \lambda'^{-1}, \psi_L) = e(\Pi \otimes \lambda^{-1}, \psi_L)(-1)^{f(\Pi \otimes \lambda^{-1})}$, we see this is equal to

$$\frac{1}{2} (e(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-a) + e(\Pi \otimes \lambda'^{-1}, \psi_L)\lambda'(a)).$$

Let $\chi_{L/F}$ be the unramified character of $F^\times$ corresponding to the quadratic
extension $L/F$. Then we have
\[
\chi_\pi(\alpha) = \chi_{\Pi, \sigma}(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}) = \sum_{\lambda_{L/F} = \omega_x} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a) \\
+ \sum_{\lambda_{L/F} = \omega_x\chi_{L/F}} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a) \\
= \sum_{\lambda_{L/F} = \omega_x} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\omega_x(-1)}{2} \lambda(a) \\
+ \sum_{\lambda_{L/F} = \omega_x\chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\omega_x(-1)}{2} \lambda(a),
\]

since $\lambda(-1) = \omega_x(-1)$ and $\sum_{\lambda_{L/F} = \omega_x} \lambda = 0$, $\sum_{\lambda_{L/F} = \omega_x\chi_{L/F}} \lambda = 0$. We note
\[
\chi_\pi(\alpha) = \chi_\pi(\alpha) \chi_\pi(\beta)
\]
for $r \in F^\times$. Therefore the second factor of the above sum vanishes and our assertion has been proved in this case.

Now assume $L/F$ is ramified. Let $\sigma_F$ be a prime element of $F$ which is contained in the norm of $L$. Let $\chi_{L/F}$ be the quadratic character of $F^\times$ corresponding to the extension $L/F$ as above. In this case the condition (b) implies only that $\lambda_{L/F} = \omega_x\chi_{L/F}$, hence that $\lambda_{L/F} = \omega_x\chi_{L/F}$ or $\omega_x\chi_{L/F}$. In the class of $\lambda$ satisfying (b), there exists exactly two characters satisfying $\lambda_i(\sigma_F) = \omega_x(\sigma_F)$, $i = 1, 2$, and they satisfy $\omega_x\chi_{L/F} = \lambda_i$ and $\lambda_1(\sigma_F) = -\lambda_2(\sigma_F)$. In the same way as in the unramified case, we obtain
\[
\chi_\pi(\alpha) = \sum_{\lambda_{L/F} = \omega_x} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)}{2} \lambda(a) \\
+ \sum_{\lambda_{L/F} = \omega_x\chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)}{2} \lambda(a).
\]

Also as in the unramified case, the second sum vanishes and for $\lambda$ in the first sum, one has $\lambda(-1) = \omega_x(-1)$. This completes the proof of the theorem in the case where $\Pi$ is supercuspidal.

Next assume $\Pi$ is a principal series representation $\pi(\mu_1, \mu_2)$. Since $\Pi$ is a base change lifting and $\pi$ is neither principal series nor special, we have $\mu_2 = \sigma\mu_1$. In this case, we have to take care of the difference between $\mathcal{H}(\Pi)$ and $\mathcal{S}(L^\times)$. As a basis of $\mathcal{H}(\Pi)$, we employ the following: $\{\eta_1(n) \cup \eta_2(n)\}$. Here $n \in \mathbb{Z}$ and $\lambda$ is extended over all classes of quasicharacters with respect to $\sim$ which do not contain $\mu_1$ nor $\mu_2$. The element $\eta_i(n)$, for $i = 1, 2$, is defined by
\[
\eta_i(n)(x) = \begin{cases} 
\mu_i(x)|x|^{1/2} & \text{if } v_L(x) \geq -n \\
0 & \text{otherwise}
\end{cases}
\]
We note $\omega_\Pi = \omega_\pi \circ n_{L/F} = \mu_1 \mu_2 = \mu_1 \circ n_{L/F} = \mu_2 \circ n_{L/F}$. Hence we have $\mu_i = \omega_\pi$ or $\omega_\pi L/F$ on $F^\times$. The action of $w$ on this basis can be described by

**LEMMA 2.3.** The notation being as above, let $\mu = \mu_1 \mu_2^{-1}$. For $\lambda \neq \mu_1, \mu_2$, one has

$$\Pi(w)\zeta^{(n)}_\lambda = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)^{(f(\Pi \otimes \lambda^{-1}) + 2n(\phi_L) - n)}.$$ 

If $\mu$ is ramified, one has

$$\Pi(w)\eta_1^{(n)} = \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)|\sigma_L|^{f(\Pi \otimes \mu_1^{-1})/2 + n(\phi_L) - n}\eta_2^{(f(\Pi \otimes \mu_1^{-1}) + 2n(\phi_L) - n)}_2,$$

$$\Pi(w)\eta_2^{(n)} = \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)|\sigma_L|^{f(\Pi \otimes \mu_2^{-1})/2 + n(\phi_L) - n}\eta_1^{(f(\Pi \otimes \mu_2^{-1}) + 2n(\phi_L) - n)}_1,$$

and if $\mu$ is unramified, one has

$$\Pi(w)\eta_1^{(n)} = \frac{1}{2}|\sigma_L|^{n(\psi_L) - n}\{\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)(\eta_2^{(2n(\phi_L) - n)} + |\sigma_L|\eta_1^{(2n(\phi_L) - n + 1)})$$

$$+ (1 - 1)^n(\eta_2^{(2n(\phi_L) - n)} - |\sigma_L|\eta_1^{(2n(\phi_L) - n + 1)})\},$$

$$\Pi(w)\eta_2^{(n)} = \frac{1}{2}|\sigma_L|^{n(\psi_L) - n}\{\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)(\eta_1^{(2n(\phi_L) - n)} + |\sigma_L|\eta_1^{(2n(\phi_L) - n + 1)})$$

$$+ (1 - 1)^n(\eta_2^{(2n(\phi_L) - n)} - |\sigma_L|\eta_2^{(2n(\phi_L) - n + 1)})\}.$$ 

**Proof.** The case of $\zeta^{(n)}_\lambda$ can be proved in the same way as in the case of cuspidal representations, since the support of $\Pi(w)\zeta^{(n)}_\lambda$ is also compact. To treat the case of $\eta_1^{(n)}$, we recall the construction of Kirillov models in the case of principal series representations ($[G]$). Let $F$ be the space of locally constant functions $\phi$ on $L$ such that $\phi(x)\mu(x)|x|$ is constant for large $|x|$. For $\phi \in F$, we set

$$\hat{\phi}(x) = \sum_{n \in \mathbb{Z}} \int_{\nu_L(y) = m} \phi(y)\overline{\psi}_L(xy) dy,$$

where $dy$ is the Haar measure of $L$ such that $\int_{\mathbb{A}} dy = |\sigma_L|^{n(\psi_L)/2}$. Then the map $\phi \mapsto \mu_2(x)|x|^{1/2}\hat{\phi}$ gives an isomorphism from $F$ to $\mathcal{K}(\Pi)$. We denote this isomorphism by $\hat{\cdot}$. The action of $w$ in $F$ is given by $\Pi(w)\phi(x) = \mu(-x^{-1})|x|^{-1}\phi(-x^{-1})$. Let

$$\phi_1^{(n)}(x) = \begin{cases} 1 & v_L(x) \geq n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_2^{(n)}(x) = \begin{cases} \mu^{-1}(x)|x|^{-1} & v_L(x) \leq -n, \\ 0 & \text{otherwise}. \end{cases}$$

Then these functions belong to $F$ and satisfy $\Pi(w)\phi_1^{(n)} = \mu(-1)\phi_2^{(n)}$. 
First assume $\mu$ is ramified. Then by some calculations, we obtain
\[
\mathcal{F}(\phi_1^{(n)}) = |\sigma_L|^{n(\psi_1)/2} + n_{\eta_2}^{(n+n(\psi_1))},
\]
\[
\mathcal{F}(\phi_\mu^{(n)}) = a\eta_1^{f(\mu)+n(\psi_1)-n}, \quad a = \int_{\psi_L(y) = -n(\psi_1)-f(\mu)} \mu^{-1}(y)\psi_L(y) d^x y,
\]
where $d^x y = |y|^{-1} dy$. Hence we have
\[
\Pi(w)\eta_2^{(n)} = a\mu(-1)|\sigma_L|^{n(\psi_1)/2} - n_{\eta_1}^{(f(\mu)+2n(\psi_1)-n)}.
\]

To determine the relation between $a$ and $\varepsilon$-factors, we use local functional equations. By [J-L], we have
\[
\int_{L^\times} (\Pi(w)\xi(x)\omega_{\Pi}^{-1}(x)\chi^{-1}(x)|x|^{1/2-s}d^x x
\]
\[
= \varepsilon(s, \Pi \otimes \chi, \psi_L) \frac{L(1-s, \Pi \otimes \chi)}{L(s, \Pi \otimes \chi)} \int_{L^\times} \xi(x)\chi(x)|x|^{s-1/2}d^x x.
\]
for $\xi \in \mathcal{H}(\Pi)$ and a quasicharacter $\chi$ of $L$. Here $d^x$ is the Haar measure of $L^\times$ such that $\int_{L^\times} d^x x = 1$. We take $\xi = \xi_2(0), \chi = \mu_2^{-1}$. Then we see the second integral is equal to $(1 - |\sigma_L|^{-1}) = L(s, \Pi \otimes \mu_2^{-1})$ and the first one is equal to
\[
a\mu(-1)|\sigma_L|^{n(\psi_1)/2 - (f(\mu)+2n(\psi_1))(1-s)(1-|\sigma_L|^{-s})^{-1}
\]
\[
= a\mu(-1)|\sigma_L|^{n(\psi_1)/2 - (f(\mu)+2n(\psi_1))(1-s)} L(1-s, \Pi \otimes \mu_2^{-1}).
\]
Since $f(\mu) = f(\Pi \otimes \mu_2^{-1})$ for $i = 1, 2$, we obtain
\[
\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) = a\mu(-1)|\sigma_L|^{-(f(\Pi \otimes \mu_2^{-1})+n(\psi_1))/2}.
\]
This shows our result on $\eta_2^{(n)}$. Interchanging $\mu_1$ and $\mu_2$, we obtain the equation for $\eta_1^{(n)}$.

Next assume $\mu$ is unramified and $\mu = |\sigma_F|$. If $L/F$ is unramified, then $\mu(\sigma_F) = 1$, and $\mu_1 = \mu_2 = \sigma \mu_1$. Hence $\Pi$ is a lifting of a principal series representation of $GL_2(F)$, which is contrary to our assumption. Therefore $L/F$ is ramified, and by the relation $\mu(\sigma_L) = |\sigma_L|^{\sigma_0}$, we see $|\sigma_L|^{2\sigma_0} = 1$. If $|\sigma_L|^{\sigma_0} = 1$, then again $\Pi$ is a lifting of a principal series representation. Hence we may assume $\mu(\sigma_L) = -1$. For $\phi_1^{(n)}$, in the same way as above, we have
\[
\mathcal{F}(\phi_1^{(n)}) = |\sigma_L|^{n(\psi_1)/2 + n_{\eta_2}^{(n+n(\psi_1))}}.
\]
By (131) of [G], we have

\[ \mathcal{F}(\phi_{\mu}^{(n)}) = |\sigma_L|^{n(\psi_L)/2} |x|^{1/2} \mu_2(x) (F_{x_0}^{(n)}(x) - |\sigma_L| F_{x_0}^{(n)}(x)), \]

where

\[ F_{x_0}^{(n)}(x) = \sum_{n(\psi_L) - v_L(x) \leq m \leq n} |\sigma_L|^{-ms_0} = \frac{1}{2}(1)^{n(\psi_L)} \mu(x) + (1)^n \phi_1^{-n(\psi_L)}(x). \]

Since \( \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) = \varepsilon(1, \psi_L) \varepsilon(\mu, \psi_L) = \mu(\sigma_L^{n(\psi_L)}) = (1)^{n(\psi_L)} \), we have

\[ |x|^{1/2} \mu_2(x) F_{x_0}^{(n)}(x) = \frac{1}{2} \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) \eta_1^{n(\psi_L) - n} + (1)^n \eta_2^{n(\psi_L) - n}. \]

From these formulas, we obtain our formula for \( \Pi(w) \eta_2^{(n)} \). The case of \( \eta_1^{(n)} \) can be proved in the same way. This completes the proof of Lemma 2.3.

By this lemma, we can proceed in the same way as in the case where \( \Pi \) is cuspidal. We give a proof for the case where \( \mu \) is unramified. The contribution from \( \xi_{\lambda x}^{(n)} \) for \( \lambda \neq \mu_1, \mu_2 \) is equal to

\[
\sum_{\lambda \neq \mu_1, \mu_2} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a). \tag{2.1}
\]

For \( \eta_1^{(n)} \), we have

\[
\Pi\left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \right) L_\lambda \eta_1^{(n)} = \frac{|\sigma_L|^{n(\psi_L) - n}}{2} \{ \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) \mu_1(-a)|a|^{1/2} \\
\times (\eta_1^{2n(\psi_L) - n + v_L(a)} + |\sigma_L|^{n(\psi_L) - n + 1 + v_L(a)}) \\
+ (1)^n \mu_2(-a)|a|^{1/2}(\eta_2^{2n(\psi_L) - n + 1 + v_L(a)} - |\sigma_L|^{n(\psi_L) - n + 1 + v_L(a)}) \}, \]

Hence if \( \eta_1^{(n)} \) contributes to the trace, \( n \) is equal to \( n(\psi_L) + v_L(a)/2 \) or \( n(\psi_L) + (v_L(a) + 1)/2 \), and the contribution is equal to

\[
\begin{cases} 
\frac{1}{2} \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) \mu_1(-1) \mu_1(a) & \text{if } v_L(a) \text{ is even}, \\
\frac{1}{2} \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) \mu_1(-1)|\sigma_L|^{1/2} \mu_1(a) & \text{if } v_L(a) \text{ is odd}. 
\end{cases}
\]
In the same way the contribution from the elements of the form $\eta^{(n)}_2$ is equal to
\[
\begin{cases}
\frac{1}{2}\sigma(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a) & \text{if } v_L(a) \text{ is even}, \\
\frac{1}{2}\sigma(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\sigma_L^{1/2}\mu_2(a) & \text{if } v_L(a) \text{ is odd}.
\end{cases}
\]
The sum of these contributions is equal to
\[
\frac{1}{2}\sigma(\Pi \otimes \mu_1^{-1}, \psi_L)\mu_1(-1)\mu_1(a) + \frac{1}{2}\sigma(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a)).
\]
(2.2)

Here we used the fact that when $v_L(a)$ is odd, $\mu_1(a)+\mu_2(a)=0$ and that $\sigma(\Pi \otimes \mu_1^{-1}, \psi_L)=\sigma(\Pi \otimes \mu_2^{-1}, \psi_L)$. The sum of (2.1) and (2.2) can be transformed into the form in the theorem as in the supercuspidal case. The case $\Pi$ special does not occur, since $\pi$ is supercuspidal. This completes the whole proof of the theorem.

As a corollary of the proof, we see

**COROLLARY 2.4.** Let $\pi$ be a supercuspidal representation of $GL_2(F)$ with the central character $\omega_\pi$, and let $\Pi$ be the base change lifting of $\pi$ to $GL_2(L)$. Then for quasicharacters $\lambda$ of $L^\times$ which satisfy $\lambda|_{F^\times} = \omega_\pi\chi_L/F^\times$, $\sigma(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)$ is independent of $\lambda$.

References


