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Foliation of phase space for the cubic non-linear Schrödinger equation

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1. Introduction and theorems

Consider the defocussing cubic non-linear Schrödinger equation (NLS)

\[ i \frac{\partial \psi}{\partial t}(x, t) = -\frac{\partial^2 \psi}{\partial x^2}(x, t) + 2|\psi(x, t)|^2\psi(x, t) \]

for complex valued function \( \psi \) with periodic boundary conditions \( \psi(x + 1, t) = \psi(x, t) \). It is well known that (NLS) is a completely integrable infinite dimensional Hamiltonian system. The periodic eigenvalues of the corresponding self-adjoint AKNS-system are invariant under the flow of (NLS), where the AKNS-system is given by

\[ (H(p, q)F)(x) = \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -q(x, t) & p(x, t) \\ p(x, t) & q(x, t) \end{pmatrix} \right] F(x) \]

with \( \psi(x, t) = p(x, t) - iq(x, t) \). Define for \( N \in \mathbb{N} \)

\[ \mathcal{H}^N = \{(p, q) \in H^N_p([0, 1])^2/p^{(j)}(0) = p^{(j)}(1), \quad q^{(j)}(0) = q^{(j)}(1) \quad \text{for} \quad j = 0, \ldots, N - 1 \}. \]

For \( N \geq 1 \) the Liouville tori of (NLS) in the phase space \( \mathcal{H}^N \) are the isospectral sets

\[ \text{Iso}_N(p, q) = \{ (\tilde{p}, \tilde{q}) \in \mathcal{H}^N / H(\tilde{p}, \tilde{q}) \text{ has the same periodic spectrum as } H(p, q) \}. \]

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For every $N$, $\text{Iso}_N(p, q)$ is compact, connected and generically an infinite product of circles.

For $(p, q) \in \mathcal{H}^N$ ($N = 0, 1$) let $\{\lambda_k(p, q)\}_{k \in \mathbb{Z}}$ be the periodic and antiperiodic spectrum of $H(p, q)$. One knows that the gap length map $\gamma$ from $\mathcal{H}^1$ into $l_N^2$ defined as

$$(p, q) \xrightarrow{\gamma} (\gamma_k(p, q) = \lambda_{2k}(p, q) - \lambda_{2k-1}(p, q))_{k \in \mathbb{Z}}$$

is continuous (but not analytic), onto and $\gamma^{-1}(\gamma(p, q)) = \text{Iso}_1(p, q)$, where $l_N^2 = \{(a_k)_{k \in \mathbb{Z}}/\sum_{k \in \mathbb{Z}} k^{2N}|a_k|^2 < \infty\}$ ($N \geq 0$). (see [Gre-Gui]).

In Appendix A we prove

**Theorem 1.1.** (1) The gap-length map $\gamma: \mathcal{H}^0 \to l^2$ is continuous and

$\gamma^{-1}(\gamma(p, q)) = \text{Iso}_0(p, q)$

(2) $\|(p, q)\|_{\mathcal{H}^0}$ is a spectral invariant, i.e. constant on $\text{Iso}_0(p, q)$.

Knowing the Dirichlet-spectrum $\{\mu_k(t)\}_{k \in \mathbb{Z}}$ of the operator $H(T\,p, T\,q)$, where $(T\,f)(x) = f(x + t)$ one can reconstruct $p$ and $q$ by the trace formulas

$$p(t) = -\sum_{k \in \mathbb{Z}} \frac{1}{2}(\lambda_{2k} + \lambda_{2k-1}) - \tilde{\mu}_k(t)$$

$$q(t) = \sum_{k \in \mathbb{Z}} \frac{1}{2}(\lambda_{2k} + \lambda_{2k-1}) - \mu_k(t).$$

Here $\{\tilde{\mu}_k(t)\}_{k \in \mathbb{Z}}$ is the Dirichlet-spectrum of $H(T\,q, -T\,p)$. The dependence of $t$ of $\{\mu_k(t)\}_{k \in \mathbb{Z}}$ is given (see [Gre-Gui]) by a system of singular differential equations. For finite gap potentials $\mu_k(t)$ can be explicitly calculated by geometric methods (see [Pre]). In this article we compute the image of $\mu_k(\cdot)$, or equivalently the image of the flow by translation $T_t$ on $\text{Iso}(p, q)$, for non-finite gap potentials. To do this we introduce the space

$$\mathcal{M}^N = \{(R_k)_{k \in \mathbb{Z}}/R_k \text{ is a } 2 \times 2 \text{ symmetric, real, trace-free matrix with } \sum_{k \in \mathbb{Z}} k^{2N}\|R_k\|^2 < \infty\}.$$ 

and a map $\text{det}_N$ from $\mathcal{M}^N$ into $l_N^2$ defined as

$$(R_k)_{k \in \mathbb{Z}} \xrightarrow{\text{det}_N} \{2(-\text{det} R_k)^{1/2}\}_{k \in \mathbb{Z}}.$$

We will prove

**Theorem 1.2.** For $N = 0, 1$ there exists a real analytic one-to-one map $\Phi$ from
\( \mathcal{H}^N \) into \( \mathcal{M}^N \) with \( \Phi(\text{Iso}_N(p, q)) = \det_N^{-1}(\det_N(\Phi(p, q))) \). For \( N = 1 \), \( \Phi \) is onto and bianalytic.

This theorem gives a geometrical description of the “foliation” \( \text{Iso}_N(p, q) \) in \( \mathcal{H}^N \). A similar theorem for the KdV equation has been proved by T. Kappeler in [Kp]. In section 2 we construct the map \( \Phi \) using results from [Gre-Gui] and [Kp]. Theorem 1.2 follows immediately as in [Kp] using arguments from [Gar-Tru, 1, 2] and

**THEOREM 1.3.** The derivative of \( \Phi \) at \( (p, q) \) is an isomorphism from \( \mathcal{H}^N \) to \( \mathcal{M}^N \) \((N = 0, 1)\).

Theorem 1.3 is proven in section 3.

Let \( \Phi = (\Phi_k)_{k \in \mathbb{Z}} \). The above mentioned result concerning the flow by translation is now a consequence of Theorem 1.2 and proved at the end of Section 2:

**THEOREM 1.4.** Suppose \((p, q) \in \mathcal{H}^0\) (resp. \( \mathcal{H}^1 \)). Then for every \( k \) with \( \lambda_{2k-1}(p, q) < \lambda_{2k}(p, q) \) there exists a continuous (resp. cont. differentiable) function \( \varphi_k(\cdot) : \mathbb{R} \to \mathbb{R} \) such that

\[
\Phi_k(T_p, T_q) = \frac{\gamma_k(p, q)}{2} \begin{pmatrix} \cos 2\varphi_k(t) & \sin 2\varphi_k(t) \\ \sin 2\varphi_k(t) & -\cos 2\varphi_k(t) \end{pmatrix}
\]

with \( \varphi_k(t + 1) = \varphi_k(t) + k\pi \) for every \( t \in \mathbb{R} \).

This shows that the image of \( \mu_k(\cdot) \) by the flow of translation consists, for all \( k \neq 0 \), of the whole gap \([\lambda_{2k-1}(p, q), \lambda_{2k}(p, q)]\).

Similarly as in [Kp] for KdV Theorem 1.2 can be applied to the so called finite gap potentials. Define, for a finite subset \( J \subseteq \mathbb{Z} \),

\[
\text{Gap}_J := \{(p, q) \in \mathcal{H}^0 : \lambda_{2n-1}(p, q) = \lambda_{2n}(p, q), n \notin J \}
\]

and

\[
\text{Gap}_{J,r} := \{(p, q) \in \text{Gap}_J : \lambda_{2n-1}(p, q) < \lambda_{2n}(p, q), n \in J \}.
\]

Elements in \( \text{Gap}_{J,r} \) are called regular \( J \)-gap potentials. It is well known that the potentials in \( \text{Gap}_J \) are, in fact, real analytic. Further, observe that \( \text{Gap}_J = \Phi^{-1}\{R \in (R_k)_{k \in \mathbb{Z}} \in \mathcal{M}^0 : R_k = 0 \forall k \notin J\} \) and thus \( \text{Gap}_J \) is a \( 2N \)-dimensional manifold where \( N = \# J \). Clearly \( \text{Gap}_{J,r} \) is open in \( \text{Gap}_J \) and \( \Phi(\text{Gap}_{J,r}) = (\mathbb{R}^+)^N \times T^N \) (diffeomorphically) where \( \mathbb{R}^+ := \{x : x > 0\} \) and \( T^N \) denotes the \( N \)-torus \((S^1)^N\). Obviously \( \text{Gap}_{J,r} \) is invariant by \( NLS \). Therefore, with the symplectic structure coming from \( NLS \), it follows from Theorem 1.2 that \((\mathbb{R}^+)^N \times T^N \) is a symplectic manifold of dimension \( 2N \) with a trivial fibration by Lagrangian tori \( T^N \). We thus obtain (cf. [Dui])

**COROLLARY 1.5.** When restricted to \( \text{Gap}_{J,r} \), NLS admits global action-angle variables.
2. Global coordinates on $\mathcal{H}^N$

We first define the map $\Phi$ mentioned in the introduction.

If $\lambda_{2k-1}(p, q) \neq \lambda_{2k}(p, q) (k \in \mathbb{Z})$ one denotes by $F_{2k-1}(\cdot; p, q)$ and $F_{2k}(\cdot; p, q)$ the two corresponding eigenfunctions of $H(p, q)$ such that, for $j = 2k-1, 2k$

(i) $\|F_j(\cdot; p, q)\|_{L^2(\mathbb{R}, \mathcal{H}^2)} = 1$

(ii) If $F_j^{(1)}(0; p, q) \neq 0$ then $F_j^{(1)}(0; p, q) > 0$

If $F_j^{(1)}(0; p, q) = 0$ then $F_j^{(2)}(0; p, q) > 0$

If $\lambda_{2k-1}(p, q) = \lambda_{2k}(p, q)$ then $F_{2k-1}(\cdot; p, q)$ and $F_{2k}(\cdot; p, q)$ are two orthonormal eigenfunctions such that

(i) $F_{2k-1}^{(1)}(0; p, q) = 0$ and $F_{2k-1}^{(2)}(0; p, q) > 0$

(ii) If $F_{2k}^{(1)}(0; p, q) \neq 0$ then $F_{2k}^{(1)}(0; p, q) > 0$

If $F_{2k}^{(1)}(0; p, q) = 0$ then $F_{2k}^{(2)}(0; p, q) > 0$

As the eigenvalues $\lambda_j$ are periodic or antiperiodic one has

$$F_j(x + 1; p, q) = (-1)^k F_j(x; p, q).$$

Let $E_k(p, q)$ be the two-dimensional subspace of $L^2$ generated by $F_{2k-1}$ and $F_{2k}$.

As in [Kp], in order to introduce an orthonormal basis $(G_{2k-1}(\cdot; p, q), G_{2k}(\cdot; p, q))$ of $E_k(p, q)$ depending analytically on $(p, q) \in \mathcal{H}^0$ one needs the following lemma.

**Lemma 2.1.** For $(p, q) \in \mathcal{H}^0$ and for every $k \in \mathbb{Z}$ the map

$$F \mapsto (F^{(1)}(0), F^{(2)}(0))$$

from $E_k(p, q)$ into $\mathbb{R}^2$ is a linear isomorphism.

Before proving Lemma 2.1, let us introduce some more notations and a few elementary results from [Gre-Gui] which will be used later.

Denote by

$$F_j(x, \lambda; p, q) = \begin{pmatrix} Y_j(x, \lambda; p, q) \\ Z_j(x, \lambda; p, q) \end{pmatrix} \quad j = 1, 2$$

the fundamental solutions to $H(p, q)F_j = \lambda F_j$ such that

$$F_1(0, \lambda; p, q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad F_2(0, \lambda; p, q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The $\mu_k(p, q)$'s (resp. $v_k(p, q)$'s) are the simple zeroes of $Z_1(1, \cdot; p, q)$ (resp. $Y_2(1, \cdot; p, q)$).
\[ \lambda_{2k-1}(p, q) \leq \mu_k(p, q), \nu_k(p, q) \leq \lambda_{2k}(p, q), \quad k \in \mathbb{Z}. \]

Denote by \( \Delta(\lambda) \) the discriminant
\[ \Delta(\lambda) = \Delta(\lambda; p, q) = Y_1(1, \lambda; p, q) + Z_2(1, \lambda; p, q). \]

The collection of periodic and antiperiodic eigenvalues \( (\lambda_k(p, q))_{k \in \mathbb{Z}} \) written in increasing order and with multiplicities have the following asymptotics
\[ \lambda_{2k}(p, q) = k\pi + l^2(k) \]
and
\[ \lambda_{2k-1}(p, q) = k\pi + l^2(k) \]
where the error terms are uniform on bounded sets of potentials \( (p, q) \in L^2([0, 1])^2 \).

It follows that for \( j = 2k - 1, 2k \)
\[ F_1(x, \lambda_j; p, q) = \begin{pmatrix} \cos \lambda_j x \\ -\sin \lambda_j x \end{pmatrix} + l^2(k) \]
and
\[ F_2(x, \lambda_j; p, q) = \begin{pmatrix} \sin \lambda_j x \\ \cos \lambda_j x \end{pmatrix} + l^2(k). \]

Finally, for \( \lambda_{2k-1}(p, q) < \lambda_{2k}(p, q) \) one has \( j = 2k - 1, 2k \)
\[ F_j(x; p, q) = \left( \frac{-Y_2(1, \lambda_j(p, q))}{\Delta(\lambda_j(p, q))} \right)^{1/2} F_1(x, \lambda_j(p, q); p, q) \]
\[ + \varepsilon_j(p, q) \left( \frac{Z_1(1, \lambda_j(p, q))}{\Delta(\lambda_j(p, q))} \right)^{1/2} F_2(x, \lambda_j(p, q); p, q) \]
where \( \varepsilon_j(p, q) = \pm 1. \)

Proof of Lemma 2.1.
Fix $k$ and $(p, q)$. It suffices to show that

$$W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(0) \neq 0.$$  

where

$$W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(x) = F_{2k}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q) - F_{2k}^{(2)}(x; p, q)F_{2k-1}^{(1)}(x; p, q)$$

is the Wronskian of $F_{2k}$ and $F_{2k-1}$. Using the equation $H(p, q)F_j = \lambda_j F_j$ one derives

$$\frac{d}{dx} W(F_{2k}, F_{2k-1})(x)$$

$$= (\lambda_{2k} - \lambda_{2k-1})F_{2k}^{(1)}(x)F_{2k-1}^{(1)}(x) + F_{2k}^{(2)}(x)F_{2k-1}^{(2)}(x)$$

(cf. [Gre-Gui]).

Thus, if $\lambda_{2k} = \lambda_{2k-1}$, we conclude that $W(F_{2k}, F_{2k-1})$ is constant. As $F_{2k}$ and $F_{2k-1}$ are linearly independent, this constant is different from zero. In the case where $\lambda_{2k-1} < \lambda_{2k}$ we first show that $W(F_{2k}, F_{2k-1})(x)$ has at most simple zeroes. Assume that this is not the case. Then there exists $0 \leq x_0 \leq 1$ and $0 \leq \varphi(x_0) \leq 2\pi$ such that

$$F_{2k}^{(1)}(x_0)F_{2k-1}^{(2)}(x_0) - F_{2k}^{(2)}(x_0)F_{2k-1}^{(1)}(x_0)$$

$$= |F_{2k}(x_0)||F_{2k-1}(x_0)|\sin \varphi(x_0) = 0$$

and

$$F_{2k}^{(1)}(x_0)F_{2k-1}^{(1)}(x_0) + F_{2k}^{(2)}(x_0)F_{2k-1}^{(2)}(x_0)$$

$$= |F_{2k}(x_0)||F_{2k-1}(x_0)|\cos \varphi(x_0) = 0$$

where here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$.

But both $|F_{2k}(x_0)| \neq 0$ and $|F_{2k-1}(x_0)| \neq 0$ which leads to a contradiction.

Let us consider the smooth path $(tp, tq)$ in $\mathcal{X}$. Denote by $t_0 = \max\{0 \leq t \leq 1; \lambda_{2k}(tp, tq) = \lambda_{2k-1}(tp, tq)\}$. Then $0 \leq t_0 < 1$. Choose $L^2$-normalized eigenfunctions $\tilde{F}_{2k}(\cdot, tp, tq)$ and $\tilde{F}_{2k-1}(\cdot, tp, tq)$ such that for $t = 1$, $\tilde{F}_{2k}(\cdot, p, q) = F_{2k}(\cdot, p, q)$ and $\tilde{F}_{2k-1}(\cdot, p, q) = F_{2k-1}(\cdot, p, q)$ and $\tilde{F}_{2k}$ and $\tilde{F}_{2k-1}$ are continuous in $t$, i.e. $\tilde{F}_{2k}$ and $\tilde{F}_{2k-1} \in C([t_0, 1], (H^1[0, 1])^2)$. In particular we conclude that $\tilde{F}_{2k}(\cdot; t_0p, t_0q)$ and $\tilde{F}_{2k-1}(\cdot; t_0p, t_0q)$ are $L^2$-normalized orthogonal eigenfunctions for $\lambda_{2k}(t_0p, t_0q)$. We conclude that for $t = t_0$
$W(\tilde{F}_{2k}, \tilde{F}_{2k-1})$ is constant and different from zero. Clearly $W(t, x) := W(\tilde{F}_{2k}(\cdot, tp, tq), \tilde{F}_{2k-1}(\cdot, tp, tq))(x)$ is continuous in $0 \leq x \leq 1$ and $t_0 \leq t \leq 1$. To simplify notation assume that $W(t_0, x) > 0$ ($0 \leq x \leq 1$). For fixed $t_0 \leq t \leq 1$, $W(t, x)$ can have at most simple zeroes and thus by a classical argument from homotopy theory we conclude that $W(t, x)$ can never vanish for $0 \leq x \leq 1$ and $t_0 \leq t \leq 1$ and Lemma 2.1 is proved.

We use Lemma 2.1 to define $G_{2k-1}(\cdot; p, q)$ as the unique function in $E_k(p, q)$ satisfying

(i) $\|G_{2k-1}(\cdot; p, q)\|_{L^2([0, 1])^2} = 1$

(ii) $G_{2k-1}^{(1)}(0; p, q) = 0$ and $G_{2k-1}^{(2)}(0; p, q) > 0$.

$G_{2k}(\cdot; p, q)$ is then defined to be the unique function in $E_k(p, q)$ such that

(i) $\|G_{2k}(\cdot; p, q)\|_{L^2([0, 1])^2} = 1$; $G_{2k}^{(1)}(0; p, q) > 0$

(ii) $(G_{2k}(\cdot; p, q), G_{2k-1}(\cdot; p, q))_{L^2([0, 1])^2} = 0$

Clearly, $G_{2k}$ and $G_{2k-1}$ can be expressed in terms of $F_{2k}$ and $F_{2k-1}$. There exist a unique $\theta_k(p, q) \in [0, 2\pi)$ such that

$$
\begin{pmatrix}
G_{2k}(\cdot; p, q) \\
G_{2k-1}(\cdot; p, q)
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_k(p, q) & -\sin \theta_k(p, q) \\
\sin \theta_k(p, q) & \cos \theta_k(p, q)
\end{pmatrix}
\begin{pmatrix}
F_{2k}(\cdot; p, q) \\
F_{2k-1}(\cdot; p, q)
\end{pmatrix},
$$

where $\epsilon_k = \text{sign} W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(0)$.

Using a perturbation argument (cf. [Ka]) one proves as in [Kp] that $G_{2k}(\cdot; p, q)$ and $G_{2k-1}(\cdot; p, q)$ are both analytic functions of $(p, q)$ as maps from $(L^2([0, 1]))^2$ into $(H^1_0([0, 1]))^2$.

$F_{2k}$ and $F_{2k-1}$ are eigenfunctions of $H(p, q)$ but they cannot depend analytically on $(p, q)$ due to possible multiplicity of the eigenvalue $\lambda_{2k}$. $G_{2k}$ and $G_{2k-1}$ are not necessarily eigenfunctions but they depend analytically on $(p, q)$.

For $(p, q) \in \mathcal{X}^N (N = 0, 1)$ and for $k \in \mathbb{Z}$ define

$$
\Phi_k(p, q) = \begin{pmatrix}
(G_{2k}(\cdot; p, q), (H - \tau_k)G_{2k}(\cdot))_{L^2([0, 1])^2} \\
(G_{2k-1}(\cdot; p, q), (H - \tau_k)G_{2k-1}(\cdot))_{L^2([0, 1])^2}
\end{pmatrix},
\begin{pmatrix}
(G_{2k}(\cdot; p, q), (H - \tau_k)G_{2k-1}(\cdot))_{L^2([0, 1])^2} \\
(G_{2k-1}(\cdot; p, q), (H - \tau_k)G_{2k-1}(\cdot))_{L^2([0, 1])^2}
\end{pmatrix}
$$

where $\tau_k = (\lambda_{2k} + \lambda_{2k-1})/2$. One easily shows that

$$
\Phi_k(p, q) = \frac{\gamma_k(p, q)}{2} \begin{pmatrix}
\cos 2\theta_k(p, q) & \sin 2\theta_k(p, q) \\
\sin 2\theta_k(p, q) & -\cos 2\theta_k(p, q)
\end{pmatrix},
$$

where $\gamma_k(p, q) = \lambda_{2k}(p, q) - \lambda_{2k-1}(p, q)$.

The matrix $\Phi_k(p, q)$ is symmetric and its trace is zero. Its eigenvalues are
For every \( k \in \mathbb{Z} \), \( \Phi_k(\cdot, \cdot) \) is a compact map from \( \mathcal{H}^0 \) into the space of real symmetric trace free matrices. (See [Kp] for a proof.)

Furthermore it is proved in [Gre-Gui] that \( (\gamma_k(p, q))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) (resp. \( l_1^2(\mathbb{Z}) \)) for \((p, q) \in \mathcal{H}^0 \) (resp. \( \mathcal{H}^1 \)) and, for \( N = 0, 1 \), \( \sum_k \gamma_k(p, q)^2 k^{2N} < \infty \) uniformly on bounded sets of potentials in \( \mathcal{H}^N \).

**DEFINITION 2.2.** For \((p, q) \in \mathcal{H}^N \) set

\[
\Phi(p, q) = (\Phi_k(p, q))_{k \in \mathbb{Z}}.
\]

It follows that \( \Phi(\cdot, \cdot) \) is a bounded map from \( \mathcal{H}^N \) (\( N = 0, 1 \)) into \( \mathcal{M}^N \).

As in [Kp] one shows that \( \Phi(\cdot, \cdot) \) is real analytic. Furthermore \( \Phi(\cdot, \cdot) \) preserves isospectrality in the following sense: \( \Phi(p, q) \) and \( \Phi(p', q') \) are isospectral, i.e., \( \text{spec} \Phi_k(p, q) = \text{spec} \Phi_k(p', q') \) for every \( k \), if and only if \( \gamma_k(p, q) = \gamma_k(p', q') \) for every \( k \). It is shown in [Gre-Gui] that, for \((p, q) \) and \((p', q') \) in \( \mathcal{H}^1 \), \( \gamma_k(p, q) = \gamma_k(p', q') \) for every \( k \) implies \( \lambda_k(p, q) = \lambda_k(p', q') \) for every \( k \). For \((p, q) \) and \((p', q') \) in \( \mathcal{H}^0 \) the same conclusion follows from Appendix A (see Corollary A.4) by the same argument given for the case \( N = 1 \) in [Gre-Gui].

**REMARK 2.3.** \( \mathcal{M}^0 \) (resp. \( \mathcal{M}^1 \)) can be identified with \( l^2(\mathbb{Z}) \) (resp. \( l_1^2(\mathbb{Z}) \)) by the map

\[
\begin{pmatrix}
\frac{\gamma_k(p, q)}{2} \\
\cos 2\theta_k(p, q) \frac{\gamma_k(p, q)}{2} \\
\sin 2\theta_k(p, q)
\end{pmatrix}
\rightarrow c_k(p, q) = \frac{\gamma_k(p, q)}{2} e^{2i\theta_k(p, q)} \quad k \in \mathbb{Z}.
\]

It then follows that for \((p, q) \in \mathcal{H}^N \) with \( N = 0, 1 \)

\[
\sum_{k \in \mathbb{Z}} k^{2N} \|\Phi_k(p, q)\|^2 = \sum_{k \in \mathbb{Z}} k^{2N} |c_k|^2 < \infty.
\]

In particular \( \Phi(\cdot, \cdot) \) coordinatizes \( \mathcal{H}^N \) globally.

It follows that for \((p_0, q_0) \in \mathcal{H}^N \)

\[
\Phi(\text{Iso}_N(p_0, q_0)) = \{(c_k)_{k \in \mathbb{Z}} \in l_1^2(\mathbb{Z}) | |c_k| = |c_k(p_0, q_0)|, k \in \mathbb{Z}\}.
\]

One recovers the well-known result that \( \text{Iso}_N(p_0, q_0) \) is a compact set, generically an infinite product of circles, the radii of which are in \( l_1^2(\mathbb{Z}) \).

We now prove Theorem 1.4. Following [Kp, Thm. 4] one easily shows that there exists a continuous (resp. continuously differentiable in the case
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$(p,q) \in \mathcal{H}^1$ function $\psi_k(t,s)$ such that

$$G_{2k-1}(x; sT_t p, sT tq) = \cos \psi_k(t,s)\tilde{F}_{2k-1}(x + t; sp, sq)$$

$$+ \sin \psi_k(t,s)\tilde{F}_{2k}(x + t; sp, sq)$$

$$G_{2k}(x; sT_t p, sT tq) = -\sin \psi_k(t,s)\tilde{F}_{2k-1}(x + t; sp, sq)$$

$$+ \cos \psi_k(t,s)\tilde{F}_{2k}(x + t; sp, sq)$$

for $(t, s) \in [0, 1]^2$ where, for $s_0 \leq s \leq 1$, $\tilde{F}_{2k}(\cdot; sp, sq)$ and $\tilde{F}_{2k-1}(\cdot; sp, sq)$ are chosen as in the proof of Lemma 2.1 with $s_0 = \max\{0 \leq s < 1; \lambda_{2k}(sp, sq) = \lambda_{2k-1}(sp, sq)\}$. Taking possible crossings of the eigenvalues $\lambda_{2k}(sp, sq)$ and $\lambda_{2k-1}(sp, sq)$ into account (cf. [Ka]), $\tilde{F}_{2k}(\cdot; sp, sq)$ and $\tilde{F}_{2k-1}(\cdot; sp, sq)$ can be chosen to depend smoothly on $s$, $0 \leq s \leq s_0$, if one allows $\tilde{F}_{2k}(\cdot; sp, sq)$ to be either a (normalized) eigenfunction for $\lambda_{2k}(sp, sq)$ or $\lambda_{2k-1}(sp, sq)$ and similarly for $\tilde{F}_{2k-1}(\cdot; sp, sq)$.

Define $\varphi_k(t) := \psi_k(t, 1)$ and the winding numbers $h_k(s) := (\psi_k(1 + t,s) - \psi_k(t,s))/\pi, h_k(\cdot)$ being a continuous function of $s$ with values in $\mathbb{Z}$. Therefore $h_k(s) = h_k(0) = k$ for every $s \in [0, 1]$ and thus $\varphi_k(1 + t) - \varphi_k(t) = k\pi$.

REMARK 2.4. For $(p,q) \in \mathcal{H}^1$ one shows that

$$\text{sign} \frac{d\varphi_k}{dt} (t) = \text{sign}(\lambda_{2k-1} + q(t)).$$

Then, for $|k|$ sufficiently large, one has

$$\frac{d\varphi_k}{dt} (t) > 0 \text{ if } k > 0 \text{ and } \frac{d\varphi_k}{dt} (t) < 0 \text{ if } k < 0$$

i.e. $\Phi_k(T_t p, T_t q)$ winds $|k|$ times around the origin without stopping, clockwise if $k < 0$ and counterclockwise if $k > 0$.

3. The derivative of $\Phi$

In this section we compute the derivative of $\Phi$ and show that it is a linear isomorphism from $\mathcal{H}^N$ onto $\mathcal{M}^N$ for $N = 0, 1$.

As in [Kp] it is convenient to write $\Phi$ in a slightly different form. One writes $\Phi$ as a map $\Psi$ from $\mathcal{H}^N$ into $l_N^2(\mathbb{Z})$ (see Remark 2.3) with $\Psi(p,q) = (\Psi_k(p,q))_{k\in\mathbb{Z}}$

$$k\in\mathbb{Z}$$
where

\[ \Psi_{2k-1}(p, q) = (G_{2k-1}(\cdot; p, q), (H - \tau_k(p, q))G_{2k-1}(\cdot; p, q))_{L^2([0,1])^2} \]
\[ \Psi_{2k}(p, q) = (G_{2k}(\cdot; p, q), (H - \tau_k(p, q))G_{2k-1}(\cdot; p, q))_{L^2([0,1])^2}. \]

Let \( d_{(p,q)}\Psi_{2k} \) (resp. \( d_{(p,q)}\Psi_{2k-1} \)) denote the derivative of \( \Psi_{2k}(\cdot, \cdot) \) (resp. \( \Psi_{2k-1}(\cdot, \cdot) \)).

**THEOREM 3.1.** Suppose \((u, v) \in H^0.\) Then

\[
d_{(p,q)}\Psi_{2k-1}[(u, v)] = 2\Psi_{2k}(p, q) \int_0^1 d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](x) \cdot G_{2k}(x; p, q) \, dx
\]
\[
+ \frac{1}{2} \int_0^1 (G_{2k-1}^{(2)}(x; p, q))^2 - G_{2k-1}^{(1)}(x; p, q)^2 + G_{2k}^{(1)}(x; p, q)^2
\]
\[
- G_{2k}^{(1)}(x; p, q)G_{2k}^{(2)}(x; p, q)u(x) \, dx
\]
\[
d_{(p,q)}\Psi_{2k}[(u, v)] = -2\Psi_{2k-1}(p, q) \int_0^1 d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](x)
\]
\[
\cdot G_{2k}(x; p, q) \, dx + \int_0^1 (-G_{2k}^{(1)}(x; p, q)G_{2k-1}^{(1)}(x; p, q)
\]
\[
+ G_{2k}^{(2)}(x; p, q)G_{2k-1}^{(2)}(x; p, q))v(x) \, dx
\]
\[
+ \int_0^1 (G_{2k}^{(1)}(x; p, q)G_{2k-1}^{(1)}(x; p, q)
\]
\[
+ G_{2k}^{(2)}(x; p, q)G_{2k-1}^{(2)}(x; p, q))u(x) \, dx
\]

where \( \cdot, \cdot \) denotes the scalar product in \( \mathbb{R}^2. \)

**Proof of Theorem 3.1.** The derivative \( d_{(p,q)}\Psi_{2k-1}[(u, v)] \) is given by

\[
d_{(p,q)}\Psi_{2k-1}[(u, v)] = (d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)], (H - \tau_k)G_{2k-1}(\cdot; p, q))
\]
\[
+ (G_{2k-1}(\cdot; p, q), (H - \tau_k)d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](\cdot))
\]
\[
+ (G_{2k-1}(\cdot; p, q), d_{(p,q)}(H - \tau_k)[(u, v)](\cdot) \cdot G_{2k-1}(\cdot; p, q)).
\]
The chosen normalization of $G_k$ imply that

$$(d_{(p,q)}G_k(\cdot; p, q), G_k(\cdot; p, q)) = 0, \quad k \in \mathbb{Z}.$$ 

Further

$$(H - \tau_k(p, q))G_{2k-1}(x; p, q) = -\frac{\gamma_k(p, q)}{2} \cos 2\theta_k(p, q)G_{2k-1}(x; p, q)$$

$$+\frac{\gamma_k(p, q)}{2} \sin 2\theta_k(p, q)G_{2k}(x; p, q).$$

One then gets

\begin{align*}
d_{(p,q)}\Psi_{2k-1}[(u, v)] &= \Psi_{2k}(p, q)(G_{2k}(\cdot; p, q), d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)])(\cdot) \\
&\quad+ \Psi_{2k}(p, q)(d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)])(\cdot), G_{2k}(\cdot; p, q) \\
&\quad+(G_{2k-1}(\cdot; p, q), \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix}) G_{2k-1}(\cdot; p, q) \\
&\quad-d_{(p,q)}\tau_k[(u, v)].
\end{align*}

Hence one finally obtains

\begin{align*}
d_{(p,q)}\Psi_{2k-1}[(u, v)] &= 2\Psi_{2k-1}(p, q)(G_{2k}(\cdot; p, q), d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)])(\cdot) \\
&\quad+(G_{2k-1}(\cdot; p, q), \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix}) G_{2k-1}(\cdot; p, q) \\
&\quad-d_{(p,q)}\tau_k[(u, v)].
\end{align*}

Let us now compute $d_{(p,q)}\tau_k[(u, v)]$.

Define, for fixed $k \in \mathbb{Z}$, the open set $U_k \subseteq \mathcal{H}^0$

$$U_k = \{(p, q) \in \mathcal{H}^0; \lambda_{2k}(p, q) \text{ simple}\}.$$ 

$\lambda_{2k}(\cdot, \cdot)$ and $\lambda_{2k-1}(\cdot, \cdot)$ are continuously differentiable on $U_k$.

Using $H(p, q)F_j = \lambda_j(p, q)F_j$ ($j = 2k - 1, 2k$) one obtains for $(p, q) \in U_k$

$$d_{(p,q)}\lambda_j[(u, v)] = (F_j(\cdot; p, q), \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix}) F_j(\cdot; p, q).$$
Thus

\[
d_{(p,q)} \tau_k [(u, v)] = \frac{1}{2} \int_0^1 (F^{(2)}_{2k}(x; p, q)^2 + F^{(2)}_{2k-1}(x; p, q)^2 - F^{(1)}_{2k}(x; p, q)^2
\]

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where $E_k = \text{sign } W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))$. We now study the asymptotics of $d(p, q)_{2k}$ and $d(p, q)_{2k-1}$. First of all it will be useful to bring into another form.

**Lemma 3.3.**

The proof of Lemma 3.3 follows as in [Kp; Lemma 5.3].

In order to bound $F_{2k-1}(\cdot)$ and $F_{2k}(\cdot)$ uniformly with respect to $k$ we use the following lemma.

**Lemma 3.4.** For $(p, q) \in \mathcal{H}$ and $k \in \mathbb{Z}$ denote $I_k(\cdot)$ the unique function in $E_k(p, q)$ such that $\|I_k(\cdot)\|_{L^2([0, 1]^2)} = 1$ with $I_k^{(1)}(0) > 0$ and $I_k^{(2)}(0) = 0$. Then for

\[
+ \left( \int_0^1 (F_{2k-1}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q)) u(x) dx \right) \begin{pmatrix} -\sin \theta_k(p, q) \\ \cos \theta_k(p, q) \end{pmatrix} \\
+ \varepsilon_k \left( \int_0^1 (F_{2k}^{(1)}(x; p, q)F_{2k}^{(2)}(x; p, q)) u(x) dx \right) \begin{pmatrix} \cos \theta_k(p, q) \\ \sin \theta_k(p, q) \end{pmatrix} \\
+ \gamma_k(p, q) \left( \int_0^1 d(p, q) G_{2k-1}(\cdot; p, q)(u, v) \right) (x) \\
\cdot G_{2k}(x; p, q) dx \begin{pmatrix} \cos \theta_k(p, q) \\ \sin \theta_k(p, q) \end{pmatrix}
\]

where $\varepsilon_k = \text{sign } W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(0)$.

We now study the asymptotics of $d_{(p, q)} \Psi_{2k}$ and $d_{(p, q)} \Psi_{2k-1}$. First of all it will be useful to bring

\[
\int_0^1 d(p, q) G_{2k-1}(\cdot; p, q)(u, v) \cdot G_{2k}(x; p, q) dx
\]

into another form.

**Lemma 3.3.**

\[
\int_0^1 d(p, q) G_{2k-1}(\cdot; p, q)(u, v) \cdot G_{2k}(x; p, q) dx
\]

\[
= \sum_{j \neq 2k, 2k-1} F_j^{(1)}(0) \begin{pmatrix} -v & u \\ u & v \end{pmatrix} F_{2k} \sin \theta_k \frac{1}{\lambda_{2k} - \lambda_j} \\
+ \sum_{j \neq 2k, 2k-1} F_j^{(1)}(0) \begin{pmatrix} -v & u \\ u & v \end{pmatrix} F_{2k-1} \varepsilon_k \cos \theta_k \frac{1}{\lambda_{2k-1} - \lambda_j}.
\]

The proof of Lemma 3.3 follows as in [Kp; Lemma 5.3].

In order to bound $F_{2k-1}(\cdot)$ and $F_{2k}(\cdot)$ uniformly with respect to $k$ we use the following lemma.

**Lemma 3.4.** For $(p, q) \in \mathcal{H}$ and $k \in \mathbb{Z}$ denote $I_k(\cdot)$ the unique function in $E_k(p, q)$ such that $\|I_k(\cdot)\|_{L^2([0, 1]^2)} = 1$ with $I_k^{(1)}(0) > 0$ and $I_k^{(2)}(0) = 0$. Then for
The error terms are uniform with respect to $0 \leq x \leq 1$ and $(p, q)$ in any bounded set of $\mathcal{H}^0$.

REMARK. We present a proof of Lemma 3.4 which generalizes easily to a situation encountered in Lemma 3.14 below.

Proof of Lemma 3.4. (1) Assume that $j = 2k$. Observe that (see [Gre-Gui])

$$F_1(0, \lambda_{2k}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad F_1(1, \lambda_{2k}) = \begin{pmatrix} (-1)^k \\ 0 \end{pmatrix} + I_2(k).$$

Existence and uniqueness of $I_k(\cdot)$ follow from Lemma 2.1. Then there exist $\alpha_k$ and $\beta_k$ satisfying

$$I_k(\cdot) = \alpha_k F_{2k-1}(\cdot) + \beta_k F_{2k}(\cdot)$$

with $\alpha_k^2 + \beta_k^2 = 1$.

Further

$$H(p, q)I_k(\cdot) = \lambda_{2k} I_k(\cdot) - \alpha_k \gamma_k F_{2k-1}(\cdot)$$

with $(\alpha_k \gamma_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$.

Define

$$f_k(\cdot) = I_k(\cdot) - I_k^{1}(0)F_1(\cdot, \lambda_{2k}).$$

Then $f_k(\cdot)$ satisfies

$$H(p, q)f_k(\cdot) = \lambda_{2k} f_k(\cdot) - \alpha_k \gamma_k F_{2k-1}(\cdot)$$

with

$$f_k(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set

$$K(x) = \begin{pmatrix} F_1^{(1)}(x, \lambda_{2k}) & F_1^{(1)}(x, \lambda_{2k}) \\ F_1^{(2)}(x, \lambda_{2k}) & F_1^{(2)}(x, \lambda_{2k}) \end{pmatrix}.$$
We then obtain
\[ f_k(x) = -\int_0^x K(x)^{-1} K(x')(\alpha_k \gamma_k F_{2k-1}(x')) \, dx'. \]

It follows from the estimates of \( F_1(\cdot, \lambda) \) and \( F_2(\cdot, \lambda) \) in [Gre-Gui; Section 1] that there is a constant \( C > 0 \) independent of \( k \) such that
\[ \| f_k \|_\infty \leq C|\alpha_k|\gamma_k \leq C\gamma_k. \]

Therefore we get
\[ \| F_1(\cdot, \lambda_{2k}) \|_{L^2([0,1])^2} I_{k}^{(1)}(0) = 1 + l^2(k). \]

Further we get from [Gre-Gui; Section 1]
\[ \| F_1(\cdot, \lambda_{2k}) \|_{L^2([0,1])^2} = 1 + l^2(k). \]

Thus
\[ I_{k}^{(1)}(0) = 1 + l^2(k) \]

and (i) is proved with \( j = 2k \). The case \( j = 2k - 1 \) follows exactly in the same way.

To prove (ii) remark that
\[ F_2(0, \lambda_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad F_2(1, \lambda_j) = \begin{pmatrix} 0 \\ (-1)^k \end{pmatrix} + l^2(k). \]

Further
\[ \| G_{2k-1}(\cdot) \|_{L^2([0,1])^2} = 1 \quad \text{and} \quad G_{2k-1}^{(2)}(0) > 0. \]

Thus (ii) follows in the same way as (i) and Lemma 3.4 is proved.

Let us deduce from Lemma 3.4 that
\[ \| F_k(\cdot) \|_{L^\infty([0,1])^2} \leq C \]
(highlighted) uniformly with respect to \( k \).

Consider \( F_{2k} \). For \( |k| \) sufficiently large it follows from Lemma 3.4 that \( W(I_k, G_{2k-1})(\cdot) \neq 0 \) because \( W(F_1(\cdot, \lambda_{2k}), F_2(\cdot, \lambda_{2k})) = 1. \)

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Therefore

\[ F_{2k}(\cdot) = \alpha_k I_k(\cdot) + \beta_k G_{2k-1}(\cdot), \quad \alpha_k, \beta_k \in \mathbb{R} \]

for \(|k| \) sufficiently large.

From \( \|F_{2k}(\cdot)\|_{L^2([0,1])^2} = 1 \) we deduce that

\[ 1 = \alpha_k^2 + \beta_k^2 + 2\alpha_k\beta_k(I_k(\cdot), G_{2k-1}(\cdot))_{L^2([0,1])^2} \]

with \( |(I_k, G_{2k-1})| \leq 1 \) and \( (I_k(\cdot), G_{2k-1}) \in l^2(k) \) because \( (F_1(\cdot, \lambda_{2k}), F_2(\cdot, \lambda_{2k})) \in l^2(k) \).

We then get

\[ |\alpha_k| \leq C \quad \text{and} \quad |\beta_k| \leq C \]

uniformly with respect to \( k \). (3.1) then follows from Lemma 3.4.

We now study the asymptotics of \( d_{(p,q)} \Psi_{2k} \) and \( d_{(p,q)} \Psi_{2k-1} \). One easily shows that

\[ G_{2k}(x; p, q) = \begin{pmatrix} \cos k\pi x \\ -\sin k\pi x \end{pmatrix} + l^2(k) \]
\[ G_{2k-1}(x; p, q) = \begin{pmatrix} \sin k\pi x \\ \cos k\pi x \end{pmatrix} + l^2(k) \]

where the error terms are uniform with respect to \( 0 \leq x \leq 1 \). Furthermore since \( G_{2k}(\cdot; p, q) \) and \( G_{2k-1}(\cdot; p, q) \) are real analytic functions of \( (p, q) \) as maps from \( \mathcal{H}^0 \) into \( H_k^0([0,1])^2 \) it follows that \( d_{(p,q)} G_{2k}(\cdot; p, q) \) and \( d_{(p,q)} G_{2k-1}(\cdot; p, q) \) are bounded linear maps from \( \mathcal{H}^0 \) into \( H_k^0([0,1])^2 \) which are still real analytic functions of \( (p, q) \).

It follows from Lemma 3.3 and (3.1) that the norm of the linear map

\[ (u, v) \mapsto \int_0^1 d_{(p,q)} G_{2k-1}(\cdot; p, q)[(u, v)](x) \cdot G_{2k}(x; p, q) \, dx \]

is uniformly bounded with respect to \( (p, q) \) on bounded sets of \( \mathcal{H}^0 \) and to \( k \in \mathbb{Z} \) (See [Kp; Prop. 5.4]).

It then follows from Theorem 3.1 and from the fact that \( (\Psi_k(p, q))_{k \in \mathbb{Z}} \) is in \( l^2(\mathbb{Z}) \) that we obtain

**THEOREM 3.5.**

\[ \begin{pmatrix} d_{(p,q)} \Psi_{2k}[u, v] \\ d_{(p,q)} \Psi_{2k-1}[u, v] \end{pmatrix} = \int_0^1 \begin{pmatrix} \cos 2k\pi x & -\sin 2k\pi x \\ \sin 2k\pi x & \cos 2k\pi x \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \, dx + l^2(k) \]
where the error term is bounded uniformly with respect to \((u, v)\) and \((p, q)\) in any bounded subset of \(\mathcal{H}^0\).

We need to introduce some more notation. For \((p, q) \in \mathcal{H}^0\) set

\[
J = \{k \in \mathbb{Z}; \lambda_{2k-1}(p, q) < \lambda_{2k}(p, q)\}.
\]

Then, for \(k \in \mathbb{Z}\), define

\[
H_{2k}(x; p, q) = \left( \frac{F_{2k-1}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q)}{(F_{2k}^{(1)}(x; p, q))^2 - F_{2k}^{(2)}(x; p, q)^2 + F_{2k-1}^{(1)}(x; p, q)^2 - F_{2k-1}^{(2)}(x; p, q)^2} \right)
\]

For \(k \notin J\) set

\[
H_{2k-1}(x; p, q) = \varepsilon_k \left( F_{2k-1}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q) + F_{2k}^{(1)}(x; p, q)F_{2k}^{(2)}(x; p, q) \right)
\]

and for \(k \in J\) define

\[
H_{2k-1}(x; p, q) = \varepsilon_k \left( F_{2k}^{(1)}(x; p, q)F_{2k}^{(2)}(x; p, q) + F_{2k-1}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q) \right)
\]

\[
+ \gamma_k(p, q) \left( \int_0^1 \left\{ G_{2k}^{(1)}(y; p, q) \frac{\partial G_{2k}^{(1)}}{\partial p}(x) + G_{2k}^{(2)}(y; p, q) \frac{\partial G_{2k}^{(2)}}{\partial q}(x) \right\} \, dy \right)
\]

Then, from Corollary 3.2, it follows that

\[
\begin{pmatrix}
  d_{(p, q)} \Psi_{2k}[u, v] \\
  d_{(p, q)} \Psi_{2k-1}[u, v]
\end{pmatrix}
\]

\[
= (H_{2k}(\cdot; p, q), (u(\cdot), v(\cdot)) \begin{pmatrix}
  -\sin 2\theta_k(p, q) \\
  \cos 2\theta_k(p, q)
\end{pmatrix}
\]

\[
+ (H_{2k-1}(\cdot; p, q), (u(\cdot), v(\cdot)) \begin{pmatrix}
  \cos 2\theta_k(p, q) \\
  \sin 2\theta_k(p, q)
\end{pmatrix}
\]

**THEOREM 3.6.** Suppose \((p, q) \in \mathcal{H}^0\). Then \(d_{(p, q)} \Phi\) is a linear isomorphism form \(\mathcal{H}^0\) onto \(\mathcal{H}^0\).
The proof of Theorem 3.6 is rather long and several steps are needed.

Theorem 3.5 shows that $d_{(p,q)}\Psi$ is a Fredholm operator of index zero. Therefore it suffices to show that $d_{(p,q)}\Psi$ is one to one in order to prove Theorem 3.6.

Assume that $d_{(p,q)}\Psi[(u, v)] = 0$ where $(u, v) \in \mathcal{K}^0$. From the above formula we conclude that $(H_k(\cdot; p, q), (u(\cdot), v(\cdot))) = 0$ for every $k \in \mathbb{Z}$. Therefore, in order to prove that $d_{(p,q)}\Psi$ is one to one, one must prove that $\{H_k(\cdot; p, q)\}_{k \in \mathbb{Z}}$ is a Riesz basis of $\mathcal{K}^0$. Using the definition of the $H_k$’s and the asymptotic behavior of the $G_k$’s one shows that $\{H_k(\cdot; p, q)\}_{k \in \mathbb{Z}}$ is quadratically close to the orthonormal basis $(T_k(\cdot; p, q))$ of $\mathcal{K}^0$ where

$$T_{2k}(x; p, q) = -\sin 2\theta_k(p, q) \begin{pmatrix} \cos 2k\pi x \\ -\sin 2k\pi x \end{pmatrix} + \cos 2\theta_k(p, q) \begin{pmatrix} \sin 2k\pi x \\ \cos 2k\pi x \end{pmatrix}$$

$$T_{2k-1}(x; p, q) = \cos 2\theta_k(p, q) \begin{pmatrix} \cos 2k\pi x \\ -\sin 2k\pi x \end{pmatrix} + \sin 2\theta_k(p, q) \begin{pmatrix} \sin 2k\pi x \\ \cos 2k\pi x \end{pmatrix}$$

Thus to prove that $(H_k(\cdot; p, q))_{k \in \mathbb{Z}}$ is a basis of $\mathcal{K}^0$ it remains to prove that the $H_k$’s are linearly independent, i.e., if $(\alpha_k)_{k \in \mathbb{Z}}$ is a sequence of real numbers such that

(i) $\sum_{k \in \mathbb{Z}} \alpha_k^2 \|H_k(\cdot; p, q)\|_{L^2([0, 1])}^2 < \infty$ and

(ii) $\sum_{k \in \mathbb{Z}} \alpha_k H_k = 0$,

then $\alpha_k = 0$ for all $k$.

First, let us recall that the set $\text{Iso}_o(p, q)$ of isospectral potentials is a countable intersection of manifolds and that one can define the normal space $N(p, q)$ and the tangent space $T(p, q)$ of $\text{Iso}_o(p, q)$ at $(p, q)$. Using results of [Gre-Gui], an easy computation shows that $\{H_{2k}(\cdot; p, q)\}_{k \in \mathbb{Z}}$ and $\{H_{2k-1}(\cdot; p, q)\}_{k \in \mathbb{Z}}$ belong to the normal space $N(p, q)$ of the isospectral set $\text{Iso}_o(p, q)$ at $(p, q)$.

Set for $k' \in \mathbb{Z}$

$$(p_{k'}, q_{k'}) = (\nabla_{(p,q)}\Delta(\cdot; p, q)|_{\lambda = \nu_k(p, q)})^\perp$$

where $(a, b)^\perp = (-b, a)$, $(\nu_k(p, q))_{k \in \mathbb{Z}}$ is one of the two Dirichlet auxiliary spectra defined in section 2.

Clearly $(p_{k'}, q_{k'})$ is in the tangent space $T(p, q)$ of $\text{Iso}_o(p, q)$ at $(p, q)$. Hence it follows that for every $k'$

$$0 = \sum_{k \in \mathbb{Z}} \alpha_k(H_k(\cdot; p, q), (p_{k'}(\cdot), q_{k'}(\cdot))),$$

$$= \sum_{k \in J} \alpha_{2k}(H_{2k-1}(\cdot; p, q), (p_{k}(\cdot), q_{k}(\cdot))).$$

The proof of Theorem 3.6 consists of three steps. In the first one we show that
\( \alpha_{2k-1} = 0 \) for \( k \in J \). In the second one we prove that \( \alpha_{2k} = \alpha_{2k-1} = 0 \) for \( k \notin J \) and in the third one we finally show that \( \alpha_{2k} = 0 \) for every \( k \) in \( J \).

3.1. The first step

Let us begin with a computational lemma.

**Lemma 3.7.** If \((u, v) \in T(p, q)\) and \( k \) in \( J \) such that \( \lambda_{2k-1}(p, q) < \nu_k(p, q) \leq \lambda_{2k}(p, q) \), then

\[
(H_{2k-1}(\cdot; p, q), (u(\cdot), v(\cdot)))
\]

\[
= -\frac{\gamma_k(p, q)}{2} (G_{2k}^{(1)}(p, q) - \nu_k(p, q)) F_{2k-1}^{(1)}(p, q) - \frac{1}{\nu_k(p, q) - \lambda_{2k}(p, q)}
\]

\[
\cdot \left( \sum_{j \in J} \left( \frac{1}{v_j(p, q) - \lambda_{2k-1}(p, q)} - \frac{1}{\nu_j(p, q) - \lambda_{2k}(p, q)} \right) \right)
\]

\[
\cdot (\nabla_{(p,q)} v_j(p, q), (u, v)) .
\]

**Proof of Lemma 3.7.** We first prove that for \((u, v) \in T(p, q)\)

\[
\gamma_k(p, q)d_{(p,q)} \theta_k[(u, v)] = (H_{2k-1}(\cdot; p, q), (u(\cdot), v(\cdot)))
\]

as follows:

\[
\int_0^1 d_{(p,q)} G_{2k-1}(\cdot; p, q)[(u, v)](x) \cdot G_{2k}(x; p, q) \, dx
\]

\[
= d_{(p,q)} \theta_k[(u, v)] + \epsilon_k \cos \theta_k(p, q) \int_0^1 d_{(p,q)} F_{2k-1}(\cdot; p, q)[(u, v)](x)
\]

\[
\cdot G_{2k}(x; p, q) \, dx + \sin \theta_k(p, q) \int_0^1 d_{(p,q)} F_{2k}(\cdot; p, q)[(u, v)](x)
\]

\[
\cdot G_{2k}(x; p, q) \, dx
\]

\[
= d_{(p,q)} \theta_k[(u, v)] + \epsilon_k \int_0^1 d_{(p,q)} F_{2k-1}(\cdot; p, q)[(u, v)](x)
\]

\[
\cdot F_{2k}(x; p, q) \, dx.
\]

Using \( H(p, q)F_j = \lambda_j F_j \) one gets

\[
(d_{(p,q)} F_{2k-1}(\cdot; p, q)[(u, v)](\cdot), F_{2k}(\cdot; p, q))
\]

\[
= -\frac{1}{\gamma_k(p, q)} \left( F_{2k-1}(\cdot; p, q), \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix} \right) F_{2k}(\cdot; p, q).
\]
Thus (3.4) follows from the definition of $H_{2k-1}$. To compute $d(p,q)\theta_k[(u,v)]$ take the derivative of $0 = G_{2k-1}^{(1)}(0) = \sin \theta_k F_{2k}^{(1)}(0) + \epsilon_k \cos \theta_k F_{2k-1}^{(1)}(0)$ and use a similar argument as in [Kp, Lemma 6.8] to obtain

$$-G_{2k}^{(1)}(0; p, q)d_{(p,q)}\theta_k[(u,v)]$$

$$= \frac{1}{2} \epsilon_k \cos \theta_k (p, q) F_{2k}^{(1)}(0; p, q)$$

$$\times \sum_{j \in \mathbb{Z}} \left( \frac{1}{v_j(p, q) - \lambda_{2k-1}(p, q)} - \frac{1}{v_j(p, q) - \lambda_{2k}(p, q)} \right)$$

$$\cdot (\nabla_{(p,q)} v_j, (u, v)).$$

In the case where $v_k(p, q) \in \{\lambda_{2k}(p, q), \lambda_{2k-1}(p, q)\}$ the following result holds.

**LEMMA 3.8.** If $k \in J$ with $v_k(p, q) \in \{\lambda_{2k}(p, q), \lambda_{2k-1}(p, q)\}$, then, for $k' \in \mathbb{Z},$

$$(H_{2k-1}^{(1)}(\cdot; p, q), (p_k(\cdot), q_k(\cdot))) = \delta_{k'k} c_k \quad \text{with } c_k \neq 0.$$

The proof of Lemma 3.8 follows as in [Kp, Lemma 6.10], once the following result is proved:

"Every $(p, q) \in X_0$ with $v_k(p, q) \in \{\lambda_{2k}(p, q), \lambda_{2k-1}(p, q)\}$, for some $k \in J$, is the limit of a sequence $(p_j, q_j)_{j \in \mathbb{N}}$ in $\text{Iso}_0(p, q)$ with $\lambda_{2k-1}(p, q) < v_k(p_j, q_j) < \lambda_{2k}(p, q)."$

This result easily follows from Appendix A.

Thus using (3.3) and Lemma 3.8 one gets $\alpha_{2k-1} = 0$ for every $k \in J - J_1$ where $J_1 = \{k \in \mathbb{Z}; \lambda_{2k-1}(p, q) < v_k(p, q) < \lambda_{2k}(p, q)\}$. We now prove that $\alpha_{2k-1} = 0$ for $k \in J_1$. For that purpose define

$$A_{k', k} = (H_{2k-1}^{(1)}(\cdot; p, q), (p_k(\cdot), q_k(\cdot))), \quad k, k' \in J_1$$

where $(p_k, q_k)$ is given by (3.2). Define

$$B_{k', k} = A_{k', k} - A_{k', k} \delta_{k'k}$$

$$C_{k', k} = A_{k', k} \delta_{k'k}$$

where $\delta_{k'k}$ denotes the Kronecker delta function.

Let $A$ (resp. $B$, $C$) be the linear operator associated with the matrix $(A_{k', k})_{k', k \in J_1 \times J_1}$ (resp. $(B_{k', k})$, $(C_{k', k})$). Then $A$ (resp. $B$, $C$)$\in \mathcal{B}(l^2(J_1))$ has the following properties.
LEMMA 3.9.
(i) $B$ is of trace class.
(ii) $C$ is invertible with a bounded inverse.
(iii) $A$ is one-to-one.

It then follows that $\alpha_{2k-1} = 0$ for $k \in J_1$ since

$$\sum_{k \in J_1} \alpha_{2k-1}(H_{2k-1}(\cdot; p, q), (p_k', q_k')) = \sum_{k \in J_1} \alpha_{2k-1} A_{kk'}, \quad k' \in J_1.$$ 

Proof of Lemma 3.9. Use [Gre, part II Chap 3 Th. 5] to conclude that

$$(\nabla_{p,q} v_k, (p_k', q_k')) = \delta_{kk'}(Z_2(1, v_k') - Y_1(1, v_k')).$$

From Lemma 3.7, it follows that

$$A_{k', k} = \frac{1}{2} (G_{2k}^{(1)}(0))^{-1} e_k \cos \theta_k(p, q) F_{2k-1}^{(1)}(0; p, q)(Z_2(1, v_k') - Y_1(1, v_k'))$$

$$= \frac{\lambda_{2k}(p, q) - \lambda_{2k-1}(p, q)}{(v_k(p, q) - \lambda_{2k-1}(p, q))(\lambda_{2k}(p, q) - v_k(p, q))}.$$  (3.5)

Moreover as we have already observed

$$(G_{2k}^{(1)}(0; p, q))^{-1} = 1 + i^2(k), \quad G_{2k-1}^{(1)}(0; p, q) = i^2(k)$$

as well as $\cos^2 \theta_k = F_{2k}^{(1)}(0)^2/(F_{2k}^{(1)}(0)^2 + F_{2k-1}^{(1)}(0)^2)$, we conclude that

$$|\cos \theta_k(p, q) F_{2k-1}^{(1)}(0; p, q)|$$

$$= \frac{|F_{2k}^{(1)}(0; p, q) F_{2k-1}^{(1)}(0; p, q)|}{(F_{2k}^{(1)}(0; p, q)^2 + F_{2k-1}^{(1)}(0; p, q)^2)^{1/2}}$$

$$= \frac{|F_{2k}^{(1)}(0; p, q) F_{2k-1}^{(1)}(0; p, q)|}{(G_{2k}^{(1)}(0; p, q)^2 + G_{2k-1}^{(1)}(0; p, q)^2)^{1/2}}$$

$$= |F_{2k}^{(1)}(0; p, q) F_{2k-1}^{(1)}(0; p, q)|(1 + i^2(k))$$

$$= \left( - \frac{Y_2(1, \lambda_{2k}(p, q))}{\Delta(\lambda_{2k}(p, q))} \right)^{1/2} \left( - \frac{Y_2(1, \lambda_{2k-1}(p, q))}{\Delta(\lambda_{2k-1}(p, q))} \right)^{1/2} (1 + i^2(k))$$

(see the beginning of section 2). Using Lemma B.3 (Appendix B) we then obtain the estimate

$$|\cos \theta_k(p, q) F_{2k-1}^{(1)}(0; p, q)|$$

$$= \frac{((\lambda_{2k}(p, q) - v_k(p, q))^{1/2}(v_k(p, q) - \lambda_{2k-1}(p, q))^{1/2}}{\lambda_{2k}(p, q) - \lambda_{2k-1}(p, q)} (1 + i^2(k)).$$
Further (cf. [Gre, Part II, Ch. 3, Th. 5])

\[ |Z_2(1, v_k'(p, q)) - Y_1(1, v_k(p, q))| \]
\[ = (\Delta^2(v_k'(p, q)) - 4)^{1/2} \]
\[ = 2(\lambda_{2k}(p, q) - v_k'(p, q))^{1/2}(v_k(p, q) - \lambda_{2k-1}(p, q))^{1/2}(1 + l^2(k')) \]

where we used for the last equality the representation of \( \Delta^2 - 4 \) by an infinite product (cf. Appendix B). Thus, from (3.5), one obtains that \( |A_{k'k}| \) is given by

\[
\frac{(\lambda_{2k'} - v_k')^{1/2}(v_k' - \lambda_{2k-1})^{1/2}(\lambda_{2k} - v_k)\lambda_{2k-1})^{1/2}}{(v_k' - \lambda_{2k-1})(\lambda_{2k} - v_k)} (1 + l^2(k))(1 + l^2(k')).
\]

(3.6)

From the asymptotic behavior of the \( \lambda_k \)'s and \( v_k \)'s it follows that

\[
B_{k'k} = \frac{a_k b_k}{(k - k')^2}
\]

where \((a_k)_{k \in J_1}\) and \((b_k)_{k \in J_1}\) are in \( l^2(J_1) \). To prove (i) one must show that

\[
\sum_{k,k' \in J_1, k \neq k'} |B_{k'k}| < +\infty.
\]

By well known properties of the convolution this follows from the estimate

\[
\sum_{k,k' \in J_1, k \neq k'} |B_{k'k}| \leq \sum_{k \in J_1} |a_k| \sum_{k \in J_1, k \neq k'} \frac{|b_k|}{(k - k')^2}.
\]

From (3.6) we learn that

\[
|A_{kk}| = 1 + l^2(k).
\]

Furthermore \( A_{kk} \) is different from zero for any \( k \in J_1 \). Thus (ii) follows.

Towards (iii) we first observe that \( C^{-1}A = \text{Id} + C^{-1}B \) is a Fredholm operator of index zero. Thus in order to prove the first step we must show that \( C^{-1}A \) is one to one, or equivalently, that the Fredholm determinant of \( C^{-1}A \) is different from zero. Let \( \det C^{-1}A \) be this Fredholm determinant which is a limit of determinants of finite matrices, i.e., \( \det C^{-1}A = \lim_{J_2 \to J_1} \det(C^{-1}A)_{J_2} \) where \( (C^{-1}A)_{J_2} \) denotes the \( J_2 \times J_2 \) matrix \( (C^{-1}A)_{k,k' \in J_2} \) with \( J_2 \) a finite subset of \( J_1 \). As
$C^{-1}$ is diagonal, one has

$$\det(C^{-1}A)_{J_2} = \frac{\det A_{J_2}}{\det C_{J_2}} = \det \left( \frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}} \right)_{k', k \in J_2} \cdot \left[ \prod_{k \in J_2} \left( \frac{1}{v_k - \lambda_{2k-1}} - \frac{1}{v_k - \lambda_{2k}} \right) \right]^{-1}.$$ 

As in [Kp] one considers the sequence $x = (x_k)_{k \in J_2}$ with $x_k \in \{-\lambda_{2k-1}, -\lambda_{2k}\}$ and $\varepsilon = (\varepsilon_k)_{k \in J_2}$ with $\varepsilon_k = 0$ if $x_k = -\lambda_{2k-1}$ and $\varepsilon_k = 1$ if $x_k = -\lambda_{2k}$. From [P-S, p. 98] (cf. also [Mck-Tru, p. 207]) it follows that

$$\det \left( \frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}} \right)_{k', k \in J_2} = \sum_x (-1)^{|\varepsilon|} \det \left( \frac{1}{v_{k'} + x_k} \right)_{k', k \in J_2}$$

$$= \sum_x (-1)^{|\varepsilon|} \prod_{k' > k} (v_{k'} - v_k) \prod_{k' > k} (x_{k'} - x_k) \prod_{k, k'} (x_k + v_{k'}).$$

where $|\varepsilon| = \sum_{k \in J_2} \varepsilon_k$.

Then

$$\det \left( \frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}} \right)_{k', k \in J_2}$$

$$= \sum_x \left( \prod_{k' \in J_2} \frac{1}{|v_{k'} + x_k|} \right) \prod_{k \in J_2} \left( 1 - \frac{x_k + v_k}{x_k + v_{k'}} \right) \left( 1 - \frac{x_k + v_k}{x_k + v_k'} \right)$$

$$= \sum_x \left( \prod_{k' \in J_2} \frac{1}{|v_{k'} + x_k|} \right) \prod_{k \in J_2} \left( 1 - \frac{(x_k + v_k)(x_{k'} + v_{k'})}{(v_{k'} + x_k)(x_{k'} + v_k)} \right).$$

(3.7)

Note that

$$1 - D_{k, k'} = 1 - \frac{(x_k + v_k)(x_{k'} + v_{k'})}{(x_k + v_{k'})(x_{k'} + v_k)} > 0 \quad \text{for } k \neq k'.$$

Furthermore $D_{kk'}$ is of the form

$$D_{k,k'} = \frac{a_k b_{k'}}{(k - k')^2}$$
with \((a_k)_{k \in \mathbb{Z}} \) and \((b_{k'})_{k' \in \mathbb{Z}} \) in \(l^2(\mathbb{Z})\). Thus

\[
\sum_{k, k' \in \mathbb{Z}, k \neq k'} D_{k, k'} < \infty
\]

and there exists an integer \(N > 0\) independent of \(J_2\) such that

\[
\sum_{|k|, |k'| \geq N, k \neq k'} D_{k, k'} < \frac{1}{2}.
\]

One deduces that

\[
\prod_{k, k' \in J_2, k \neq k', |k|, |k'| \geq N} (1 - D_{k, k'}) \geq 1 - \sum_{j \geq 1} (\Sigma_N)^j = K' > 0.
\]

On the other hand one has

\[
\prod_{k, k' \in J_2, k > k', |k|, |k'| < N} (1 - D_{k, k'}) \geq K'' > 0.
\]

These two estimates lead to

\[
\prod_{k, k' \in J_2, k > k'} (1 - D_{k, k'}) \geq K = K'K'' > 0
\]

where \(K\) does not depend on the finite subset \(J_2\) of \(J_1\). Moreover

\[
\det C_{J_2} = \sum_x \prod_{k \in J_2} \frac{1}{v_k + x_k}.
\]

This implies together with (3.7) and (3.8) that \(\det(C^{-1}A)_{J_2} \geq K\) uniformly with respect to \(J_2 \subset J_1\). Thus \(\det C^{-1}A \geq K > 0\) and \(A\) is one-to-one.

3.2. The second step

We must show that \(\alpha_{2k} = \alpha_{2k-1} = 0\) for every \(k \notin J\).

The main ingredient of the proof is the following

**LEMMA 3.10.** (i) \(\langle H_{2k}(\cdot; p, q), H_{2k'}(\cdot; p, q) \rangle = 0, k, k' \in \mathbb{Z}\).

(ii) For \(k \notin J\) and \(k' \in \mathbb{Z}\)

\[
\langle H_{2k-1}(\cdot; p, q), H_{2k}(\cdot; p, q) \rangle = -\frac{1}{2} \delta_{kk'} W(F_{2k}, F_{2k-1})(0).
\]
Proof of Lemma 3.10. The proof is the same as in [Gre-Gui, Th. 1.7, assertions (i) and (ii)].

To prove Step 2 we argue as follows. For \( k' \notin J \) one deduces from the first step and Lemma 3.10 that

\[
0 = \sum_{k \in \mathbb{Z}} \alpha_{2k}(H_{2k}(\cdot; p, q), H_{2k}(\cdot; p, q)^{\perp})
+ \sum_{k \in J} \alpha_{2k-1}(H_{2k-1}(\cdot; p, q), H_{2k}(\cdot; p, q)^{\perp})
= -\frac{1}{2} \alpha_{2k-1} W(F_{2k'}, F_{2k'-1})(0).
\]

As \( W(F_{2k'}, F_{2k'-1})(0) \neq 0 \) (Lemma 2.1) we conclude that \( \alpha_{2k-1} = 0 \) for every \( k' \in J \).

Next, again for \( k' \notin J \)

\[
0 = \sum_{k \in \mathbb{Z}} \alpha_{2k}(H_{2k}(\cdot; p, q), H_{2k-1}(\cdot; p, q)^{\perp})
= -\frac{1}{2} \alpha_{2k'} W(F_{2k'}, F_{2k'-1})(0)
\]

and therefore \( \alpha_{2k'} = 0 \) for \( k' \notin J \). Thus step 2 is proved.

3.3. The third step

Here we show that \( \alpha_{2k} = 0 \) for every \( k \in J \). One already knows that

\[
\sum_{k \in J} \alpha_{2k} H_{2k}(\cdot; p, q) = 0. \tag{3.9}
\]

Thus it suffices to show that \( \{H_{2k}(\cdot; p, q)\}_{k \in J} \) is linearly independent. Note that
\( H_{2k}(x; T_t p, T_t q) = H_{2k}(x + t; p, q) \). Therefore it suffices to prove that
\( (H_{2k}(\cdot; T_t p, T_t q))_{k \in J} \) is linearly independent for some \( t \). The following result is easy to prove.

**Lemma 3.11.** There exists \( t_0 \) such that for all \( k \in J \)

\[
\lambda_{2k-1}(p, q) < v_k(T_{t_0} p, T_{t_0} q) < \lambda_{2k}(p, q).
\]

To make notation easier, we assume that \( t_0 = 0 \).
It remains to prove that $\alpha_{2k} = 0$ for $k \in J_1 = \{k \in \mathbb{Z}; \lambda_{2k-1}(p, q) < v_k(p, q) < \lambda_{2k}(p, q)\}$. Define

$$A_{k',k} = \frac{1}{2} \frac{\partial^2 Y_k}{\partial \lambda} (1, v_k)(\lambda_{2k} - \lambda_{2k-1})$$

A straightforward computation using [Gre-Gui] and [Gre] leads to

$$A_{k',k} = \frac{(\Delta v_k)^2 - 4}{2(\lambda_{2k} - v_k)^2} \left( \frac{F^{(1)}_{2k'-1}(0)^2 F^{(2)}_{2k'-1}(0)^2 - F^{(1)}_{2k}(0)^2 F^{(2)}_{2k}(0)^2}{v_k - \lambda_{2k-1}} \right). \tag{3.10}$$

Define

$$B_{k',k} = A_{k',k} - A_{k',k} \delta_{k,k}$$

$$C_{k',k} = A_{k',k} \delta_{k,k}.$$

Let $A$ (resp. $B$, $C$) denote the linear operator associated with the matrix $(A_{k',k}(k',k)_{k',k} \in J_1 \times J_1)$ (resp. $(B_{k',k}), (C_{k',k})$). Then $A$ (resp. $B$, $C$) $\in \mathcal{B}(l^2(J_1))$. The proof of the third step follows from

**Lemma 3.12.**

(i) $B$ is a Hilbert-Schmidt operator.

(ii) $C$ is invertible with a bounded inverse.

(iii) $A$ is one-to-one.

**Proof of Lemma 3.12.** Clearly

$$F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0) + F^{(1)}_{2k}(0)F^{(2)}_{2k}(0) = G^{(1)}_{2k'-1}(0)G^{(2)}_{2k'-1}(0) + G^{(1)}_{2k}(0)G^{(2)}_{2k}(0) = l^2(k').$$

Thus

$$(F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0))^2 = (F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2 + l^2(k')$$
and $A_{k',k}$ is given by

$$\frac{1}{2} (\lambda_{2k} - \lambda_{2k-1}) (\Delta(v_k)^2 - 4)^{1/2} \left[ (F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2 \times \left( \frac{1 + l^2(k')}{v_k - \lambda_{2k-1}} - \frac{1}{v_k - \lambda_{2k'}} \right) + \frac{l^2(k')}{v_k - \lambda_{2k' - 1}} \right].$$

(3.11)

Using formulas expressing the $F_k$'s in terms of $F_1$ and $F_2$ (see the beginning of Section 2) and Appendix B one shows that

$$(F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2 = - \frac{Y_2(1, \lambda_{2k})Z_1(1, \lambda_{2k})}{(\Delta(\lambda_{2k}))^2} \frac{(\lambda_{2k'} - v_k)(\lambda_{2k'} - \mu_k)}{(\lambda_{2k'} - \lambda_{2k' - 1})^2} (1 + l^2(k')).$$

Further

$$(\Delta(v_k)^2 - 4)^{1/2} = 2(\lambda_{2k} - v_k)^{1/2}(v_k - \lambda_{2k-1})^{1/2}(1 + l^2(k))$$

and hence

$$A_{k',k} = \frac{\lambda_{2k} - \lambda_{2k-1}}{(\lambda_{2k'} - \lambda_{2k' - 1})^2} \frac{(\lambda_{2k'} - v_k)(\lambda_{2k'} - \mu_k)}{(\lambda_{2k} - v_k)(v_k - \lambda_{2k'} - \lambda_{2k' - 1}) + \frac{l^2(k')}{v_k - \lambda_{2k' - 1}} (1 + l^2(k))(1 + l^2(k')) + \frac{\lambda_{2k} - \lambda_{2k'-1}}{v_k - \lambda_{2k' - 1}} l^2(k').$$

It follows from the asymptotic behavior of $\lambda_k$, $\mu_k$ and $v_k$ for large $|k|$ that for $k' \neq k$

$$|A_{k',k}| \lesssim \left( \frac{(\lambda_{2k} - \lambda_{2k-1})(\lambda_{2k'} - \lambda_{2k' - 1})}{(k - k')^2 \pi^2} + \frac{(\lambda_{2k} - \lambda_{2k-1})}{|k' - k| \pi} l^2(k') \right) \times (1 + l^2(k))(1 + l^2(k')).$$

Thus, for $k' \neq k$, we obtain

$$|A_{k',k}| \lesssim \frac{l^2(k)l^2(k')}{(k - k')^2} + \frac{l^2(k)l^2(k')}{|k - k'|} (1 + l^2(k)).$$
and therefore
\[ \sum_{k', k \in J_1} |B_{k', k}|^2 = \sum_{k', k \in J_1, k' \neq k} |A_{k', k}|^2 < \infty. \]

Thus (i) is proved.

To show (ii) observe that
\[ \frac{(F_{2k-1}^{(1)}(0)F_{2k-1}^{(2)}(0))^2}{v_k - \hat{\lambda}_{2k-1}} - \frac{(F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2}{v_k - \hat{\lambda}_{2k}} = \frac{1}{\hat{\lambda}_{2k} - \hat{\lambda}_{2k-1}} (1 + l^2(k)). \]

Hence
\[ A_{k,k} = 1 + l^2(k). \]

As \( A_{kk} \) is different from zero for every \( k \in J_1 \), (ii) follows.

In order to prove (iii) we must show that \( C^{-1} A \) is one-to-one. Lemma 3.10 shows that \( C^{-1} A = \text{Id} + C^{-1} B \) where \( C^{-1} B \) is a Hilbert-Schmidt operator. In order to show that \( C^{-1} A \) is one-to-one it suffices to prove that the regularized determinant \( \det_2 C^{-1} A \) is different from zero (see [Sim] for the definition and properties of \( \det_2 \)). As in the first step one estimates \( \det_2 C^{-1} A \) by the regularized determinants of finite matrices \( (C^{-1} A)_{J'} \) associated with a finite subset \( J' \) of \( J_1 \).

First, recall that
\[ \det_2(C^{-1} A)_{J'} = \det(C^{-1} A)_{J'} e^{-\text{Tr}(C^{-1} B)_{J'}} = \det(C^{-1} A)_{J'}. \]

because \( \text{Tr}(C^{-1} B)_{J'} = 0 \) by the definition of \( B \). Further
\[
\det(C^{-1} A)_{J'} = \det\left( \frac{(F_{2k-1}^{(1)}(0)F_{2k-1}^{(2)}(0))^2}{v_k - \hat{\lambda}_{2k-1}} + \frac{(F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2}{\hat{\lambda}_{2k} - v_k} \right)_{(k', k) \in J' \times J'} \times \prod_{k \in J'} \left( \frac{(F_{2k-1}^{(1)}(0)F_{2k-1}^{(2)}(0))^2}{v_k - \hat{\lambda}_{2k-1}} + \frac{(F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2}{\hat{\lambda}_{2k} - v_k} \right)^{-1}.
\]

and, similar as above,
\[
\det\left( \frac{(F_{2k-1}^{(1)}(0)F_{2k-1}^{(2)}(0))^2}{v_k - \hat{\lambda}_{2k-1}} + \frac{(F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2}{\hat{\lambda}_{2k} - v_k} \right)_{k', k \in J' \times J'} = \sum_x (-1)^{|x|} \prod_{x_k = -\hat{\lambda}_{2k}} (F_{2k}^{(1)}(0)F_{2k}^{(2)}(0))^2 \prod_{x_k = -\hat{\lambda}_{2k-1}} (F_{2k-1}^{(1)}(0)F_{2k-1}^{(2)}(0))^2.
\]

\[
\cdot \det\left( \frac{1}{v_k + x_k} \right)_{(k', k) \in J' \times J'}.
\]
where $x = (x_k)_{k \in J'}$, $e = (e_k)_{k \in J'}$ and $|e|$ are defined as in the first step.

For $\det C_{J'}$, we obtain the following expression

$$
\prod_{k \in J'} \left( \frac{(F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2}{v_k - \bar{\lambda}_{2k-1}} + \frac{(F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2}{\bar{\lambda}_{2k} - v_k} \right)
$$

$$
= \sum_x (-1)^{|x|} \prod_{x_k = -\bar{\lambda}_{2k}} (F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2 \prod_{x_k = -\bar{\lambda}_{2k-1}} (F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2 \prod_{k \in J'} \frac{1}{v_k + x_k}.
$$

(3.14)

As in the first step using (3.12)–(3.14) we conclude

$$
\det(C^{-1}A)_{J'} = \det_{2}(C^{-1}A)_{J'} \geq K > 0
$$

for every finite subset $J' \subset J_1$, where $K$ is independent of $J'$. Therefore

$$
\det_{2}C^{-1}A \geq K > 0.
$$

Theorem 3.6 can be improved in the case where $(p, q) \in \mathcal{H}^1$.

**THEOREM 3.13.** For $(p, q) \in \mathcal{H}^1$, $d_{(p, q)}\Phi$ is a linear isomorphism from $\mathcal{H}^1$ onto $\mathcal{M}^1$.

For this purpose we need the following

**LEMMA 3.14.** If $(p, q) \in \mathcal{H}^1$ then

$$
G_{2k-1}(x) = \begin{bmatrix} \sin k\pi x \\
\cos k\pi x \end{bmatrix} + \frac{1}{2\pi k} \left( -q(x) \sin k\pi x + \cos k\pi x(p(x) - p(0)) \right)
$$

$$
\left. \sin k\pi x(p(0) + p(x)) + q(x) \cos k\pi x \right) + \frac{1}{2k\pi} \left( \int_0^x (p(t)^2 + q(t)^2) \, dt - x \int_0^1 (p(t)^2 + q(t)^2) \, dt \right)
$$

$$
\times \begin{bmatrix} -\cos k\pi x \\
\sin k\pi x \end{bmatrix} + l_1^2(k)
$$

(3.15)

and

$$
G_{2k}(x) = \begin{bmatrix} \cos k\pi x \\
-\sin k\pi x \end{bmatrix} + \frac{1}{2\pi k} \left( (p(0) - p(x)) \sin k\pi x - q(x) \cos k\pi x \right)
$$

$$
\left. (p(0) - p(x)) \sin k\pi x + (p(x) + p(0)) \cos k\pi x \right) + \frac{1}{2k\pi} \left( \int_0^x (p(t)^2 + q(t)^2) \, dt - x \int_0^1 (p(t)^2 + q(t)^2) \, dt \right)
$$

$$
\times \begin{bmatrix} \sin k\pi x \\
\cos k\pi x \end{bmatrix} + l_1^2(k)
$$

(3.16)
where the error terms are uniformly bounded in \(0 \leq x \leq 1\) and with respect to \((p, q)\) in any bounded set of \(\mathcal{H}_1\).

**Proof of Lemma 3.14.** From [Gre-Gui; Section 1] we get for \(j \in \{2k - 1, 2k\}
\)

\[
F_1(x, \lambda_j) = \left( \cos k\pi x, -\sin k\pi x \right) + \frac{1}{2k\pi} \left( -\left(p(x) + p(0)\right) \sin k\pi x + \left(q(0) - q(x)\right) \cos k\pi x \right)
- \frac{1}{2k\pi} \left( \int_0^x (p(t)^2 + q(t)^2) \, dt - x\left(\|p\|^2 + \|q\|^2\right) \right) \left( \sin k\pi x, \cos k\pi x \right) + l_1^2(k)
\]

(3.17)

and

\[
F_2(x, \lambda_j) = \left( \sin k\pi x, \cos k\pi x \right) + \frac{1}{2k\pi} \left( \left(p(x) - p(0)\right) \cos k\pi x - \left(q(x) + q(0)\right) \sin k\pi x \right)
- \frac{1}{2k\pi} \left( \int_0^x (p(t)^2 + q(t)^2) \, dt - x\left(\|p\|^2 + \|q\|^2\right) \right) \left( -\cos 2k\pi x, \sin 2k\pi x \right) + l_1^2(k)
\]

(3.18)

Then for \(j \in \{2k - 1, 2k\} \) and for \(k \neq 0\)

\[
F_1(0, \lambda_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad F_1(1, \lambda_j) = \begin{pmatrix} (-1)^k \\ 0 \end{pmatrix} + l_1^2(k),
\]

\[
\|F_1(\cdot, \lambda_j)\|_{L^2([0,1])^2} = 1 + \frac{q(0)}{k\pi} + l_1^2(k)
\]

(3.19)

and

\[
F_2(0, \lambda_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F_2(1, \lambda_j) = \begin{pmatrix} 0 \\ (-1)^k \end{pmatrix} + l_1^2(k),
\]

\[
\|F_2(\cdot, \lambda_j)\|_{L^2([0,1])^2} = 1 - \frac{q(0)}{k\pi} + l_1^2(k).
\]

(3.20)

Further

\[
(F_1(\cdot, \lambda_j), F_2(\cdot, \lambda_j))_{L^2([0,1])^2} = -\frac{p(0)}{k\pi} + l_1^2(k).
\]

(3.21)
Following the proof of Lemma 3.4 we now obtain for \( j \in \{2k - 1, 2k\} \)

\[
I_k(\cdot) = \frac{F_1(\cdot, \lambda_j)}{\|F_1(\cdot, \lambda_j)\|_{L^2([0,1])^2}} + l_1^2(k) \tag{3.22}
\]

\[
G_{2k-1}(\cdot) = \frac{F_2(\cdot, \lambda_j)}{\|F_2(\cdot, \lambda_j)\|_{L^2([0,1])^2}} + l_1^2(k). \tag{3.23}
\]

The error terms are in \( l_1^2(\mathbb{Z}) \) because, for \( (p, q) \in \mathcal{H}^1 \), \( (\gamma_k(p, q))_{k \in \mathbb{Z}} \in l_1^2(\mathbb{Z}) \).

Define for \(|k|\) sufficiently large

\[
L_k(\cdot) = \frac{\|F_1(\cdot, \lambda_{2k-1})\|_{L^2([0,1])^2} + (p(0)/k\pi)G_{2k-1}(\cdot)}{\|F_1(\cdot, \lambda_{2k-1})\|_{L^2([0,1])^2}} \tag{3.24}
\]

Thus \( L_k(\cdot) \in E_k(p, q) \) and \( \|L_k(\cdot)\|_{L^2([0,1])^2} = 1 \). It follows from (3.19), (3.21), (3.22) and (3.24) that

\[
(G_{2k-1}(\cdot), L_k(\cdot))_{L^2([0,1])^2} = l_1^2(k) \tag{3.25}
\]

for \(|k|\) sufficiently large.

Thus for \(|k|\) sufficiently large, there exist \( \alpha_k \) and \( \beta_k \) such that

\[
G_{2k}(\cdot) = \alpha_k L_k(\cdot) + \beta_k G_{2k-1}(\cdot).
\]

From \( \|G_{2k}(\cdot)\| = 1 \) and \( (G_{2k}(\cdot), G_{2k-1}(\cdot)) = 0 \) we deduce that

\[
1 = \alpha_k^2 + \beta_k^2 + 2\alpha_k \beta_k (L_k(\cdot), G_{2k-1}(\cdot))
\]

and

\[
0 = \alpha_k (L_k(\cdot), G_{2k}(\cdot)) + \beta_k.
\]

It then follows from (3.25) that

\[
\beta_k = l_1^2(k) \quad \text{and} \quad \alpha_k = 1 + l_1^2(k).
\]

We then obtain

\[
G_{2k}(\cdot) = L_k(\cdot) + l_1^2(k). \tag{3.26}
\]

Finally (3.15) and (3.16) are deduced from (3.17)–(3.23) and (3.26) and Lemma 3.14 is proved.
We then obtain

**Lemma 3.15.** If \((p, q) \in \mathcal{H}^{1}\) and \((u, v) \in \mathcal{H}^{0}\) then

\[
d_{(p,q)}\Psi_{2k}[(u, v)] = - \int_0^1 \sin 2k\pi x \ v(x) \, dx + \int_0^1 \cos 2k\pi x \ u(x) \, dx + l_1^2(k)
\]

\[
d_{(p,q)}\Psi_{2k-1}[(u, v)] = \int_0^1 \cos 2k\pi x \ v(x) \, dx + \int_0^1 \sin 2k\pi x \ u(x) \, dx + l_1^2(k)
\]

where the error terms are uniform with respect to \((u, v)\) on any bounded set of \(\mathcal{H}^{0}\).

**Proof of Lemma 3.15.** As \((p, q) \in \mathcal{H}^{1}\), the gap sequence \((\gamma_k)_{k \in \mathbb{Z}}\) is in \(l_1^2(\mathbb{Z})\). Lemma 3.15 then follows from Theorem 3.1 and the asymptotic estimates (3.15) and (3.16).

**Proof of Theorem 3.13.** It follows from Theorem 3.6 that \(d_{(p,q)}\Phi\) is one-to-one. To prove that \(d_{(p,q)}\Phi\) is onto it is equivalent to show that the linear map \(d_{(p,q)}\Psi\) from \(\mathcal{H}^{1}\) into \(l_1^2(\mathbb{Z}) \times l_1^2(\mathbb{Z})\) given by

\[
d_{(p,q)}\Psi[(u, v)] = (d_{(p,q)}\Psi_{2k}[(u, v)], d_{(p,q)}\Psi_{2k-1}[(u, v)])_{k \in \mathbb{Z}}.
\]

is onto.

Let \((a_k)_{k \in \mathbb{Z}}\) and \((b_k)_{k \in \mathbb{Z}}\) be in \(l_1^2(\mathbb{Z})\). From Theorem 3.6 it follows that there exist \(u(\cdot)\) and \(v(\cdot)\) in \(L^2([0, 1])\) such that

\[
d_{(p,q)}\Psi[(u, v)] = (a_k, b_k)_{k \in \mathbb{Z}}.
\]

It is to prove that \((u, v)\) is in \(\mathcal{H}^{1}\). Lemma 3.15 shows that each of the sequences

\[
\left( \int_0^1 \cos 2n\pi x \ v(x) \, dx \right)_{n \in \mathbb{N}}, \quad \left( \int_0^1 \cos 2n\pi x \ u(x) \, dx \right)_{n \in \mathbb{N}}
\]

\[
\left( \int_0^1 \sin 2n\pi x \ v(x) \, dx \right)_{n \in \mathbb{N}}, \quad \left( \int_0^1 \sin 2n\pi x \ u(x) \, dx \right)_{n \in \mathbb{N}}
\]

are in \(l_1^2(\mathbb{N})\). Then, as in the proof of Theorem I.18 of [Gre-Gui], this implies that \(u(\cdot)\) and \(v(\cdot)\) are in \(H^1([0, 1])\) with \(u(1) - u(0) = v(1) - v(0) = 0\).

**Appendix A**

In this appendix we generalize Theorem 3.7 of [Gre-Gui].

Let \(\pi(\cdot, \cdot)\) be the map from \(\mathcal{H}^{0}\) into \(\mathbb{R}^Z \times \mathbb{R}^Z\) defined by

\[
\pi(p, q) = ((\mu_k(p, q))_{k \in \mathbb{Z}}, (\chi_k(p, q))_{k \in \mathbb{Z}})
\]
where the \( \mu_k(p, q) \)'s are the zeroes of the map \( \lambda \mapsto Z_1(1, \lambda; p, q) \) and \( \chi_k(p, q) = \log \{(−1)^k Y_1(1, \mu_k(p, q))\} \). Let for \( (p, q) \in \mathcal{H}^0 \)

\[
\mathcal{F}_{(p,q)} = \left\{ \left( (\xi_k)_{k \in \mathbb{Z}}, (\eta_k)_{k \in \mathbb{Z}} \right) \in \left( \prod_{k \in \mathbb{Z}} [\lambda_{2k-1}(p, q), \lambda_{2k}(p, q)] \right) \times \mathbb{R}^2 \right\}.
\]

\[
\Delta (\xi_k; p, q) = 2(−1)^k \cosh \eta_k, \ k \in \mathbb{Z}.
\]

**THEOREM A.1.** Suppose \( (p_0, q_0) \in \mathcal{H}^0 \). Then \( \pi(\cdot, \cdot) \) is a homeomorphism from \( \text{Iso}_0(p_0, q_0) \) onto \( \mathcal{F}_{(p_0, q_0)} \).

In [Gre-Gui] Theorem A.1 is proved for \( (p_0, q_0) \in \mathcal{H}^1 \) using the isospectral flows \((k \in \mathbb{Z})\)

\[
\frac{d}{dt} \begin{pmatrix} p(\cdot, t) \\ q(\cdot, t) \end{pmatrix} = V_0(p(\cdot, t), q(\cdot, t))
\]

\[
p(x, 0) = p_0(x) \quad \text{and} \quad q(x, 0) = q_0(x)
\]

where

\[
V_0(p(\cdot), q(\cdot)) = \begin{pmatrix}
\frac{\partial \Delta}{\partial q(\cdot)} (\lambda; p(\cdot), q(\cdot)) |_{\lambda = \mu_k(p(\cdot), q(\cdot))} \\
- \frac{\partial \Delta}{\partial p(\cdot)} (\lambda; p(\cdot), q(\cdot)) |_{\lambda = \mu_k(p(\cdot), q(\cdot))}
\end{pmatrix}.
\]

According to [Gre-Gui], the ordinary differential equation (A.1) has a unique solution in \( H^1([-t_0, t_0], \mathcal{H}^0) \) for initial values in \( \mathcal{H}^0 \) with \( t_0 > 0 \) chosen sufficiently small, and for this solution to exist globally in \( t \), it suffices to prove the following

**LEMMA A.2.** Let \( (p(\cdot, t), q(\cdot, t)) \) be a solution of (A.1) defined on a compact interval \( I \subseteq \mathbb{R}, 0 \in I \), in \( H^1(I; \mathcal{H}^0) \). Then

\[
\|p(\cdot, t), q(\cdot, t)\|_{\mathcal{H}^0} = \| p_0(\cdot), q_0(\cdot)\|_{\mathcal{H}^0}, \ t \in I.
\]

**REMARK A.3.** If the potentials \( (p_0(\cdot), q_0(\cdot)) \in \mathcal{H}^1 \), it is easy to show that \( \| (p(\cdot, t), q(\cdot, t)) \|_{\mathcal{H}^0} \) is independent of \( t \) as this quantity is a spectral invariant appearing in the asymptotic expansion of the \( \lambda_k \)'s (cf. [Gre-Gui]).

**Proof of Lemma A.2.** Define \( u(x, t) = (p(x, t), q(x, t)) \) and \( u_0(x) = (p_0(x), q_0(x)) \). Choose a sequence \( (u^{(n)}_0)_{n \geq 0} \) in \( \mathcal{H}^1 \) which converges to \( u_0 \) in \( \mathcal{H}^0 \). According to [Gre-Gui] there exists a unique solution \( u^{(n)}(x, t) \) of (A.1) in \( H^1(\mathbb{R}; \mathcal{H}^1) \). Moreover these solutions satisfy for a.e \( t \):

\[
\left\| \frac{d}{dt} u^{(n)}(\cdot, t) \right\|_{\mathcal{H}^0} \leq \beta(\| u^{(n)}(\cdot, 0)\|_{\mathcal{H}^0})
\]
where $\beta(\cdot)$ is a positive function on $\mathbb{R}$ which is independent of $n$ and $t$. (See [Gre; Thm. 2, p. 132]).

Thus $(u^{(n)})_{n \geq 0}$ is a bounded sequence in $H^1(I; \mathcal{H}^0)$. Hence there exists a subsequence, again denoted by $(u^{(n)})_{n \geq 0}$, which converges weakly in $H^1(I, \mathcal{H}^0)$ to a function $v \in H^1(I, \mathcal{H}^0)$, i.e.,

$$\lim_{n \to \infty} \frac{d^j}{dt^j} u^{(n)} = \frac{d^j}{dt^j} v$$
weakly in $L^2(I, \mathcal{H}^0)$ for $j = 0, 1$.

Furthermore it follows from [Gre, Part II, Chap. 3, Th. 2] and [Pô-Tru] that the vector fields $V_k$ are compact on $\mathcal{H}^0$. Thus $(V_k(u^{(n)}))_{n \geq 1}$ converges strongly to $V_k(v)$ in $L^2(I, \mathcal{H}^0)$. Hence

$$\frac{dv}{dt} = V_k(v) \text{ in } L^2(I, \mathcal{H}^0).$$

The trace theorem guarantees the weak-convergence of $(u^{(n)}(\cdot, 0))_{n \geq 0}$ weakly in $\mathcal{H}^0$ to $v(\cdot, 0)$ as $n$ tends to infinity and $(u^{(n)}(\cdot, 0))_{n \geq 0} = (u_0^{(n)}(\cdot))_{n \geq 0}$ converges to $u_0(\cdot)$ strongly in $\mathcal{H}^0$. Thus $v(x, 0) = u_0(x)$ for a.e. $x$ in $[0, 1]$.

By the uniqueness of the solution to (A.1) we get $u(x, t) = v(x, t)$ for a.e. $x \in [0, 1]$ and for every $t \in I$. Since $(u^{(n)}(\cdot, t))_{n \geq 0}$ converges to $u(\cdot, t)$ weakly in $\mathcal{H}^0$ and

$$\left(\frac{du^{(n)}}{dt}(\cdot, t)\right)_{n \geq 0} \text{ converges to } \frac{du}{dt}(\cdot, t)$$
strongly in $\mathcal{H}^0$ for every $t \in I$,

$$\left\{\left(u^{(n)}(\cdot, t), \frac{du^{(n)}}{dt}(\cdot, t)\right)\right\}_{n \geq 0} \text{ converges to } \left(u(\cdot, t), \frac{du}{dt}(\cdot, t)\right)$$
for a.e. $t$ in $I$.

Furthermore

$$\left(u^{(n)}(\cdot, t), \frac{d}{dt} u^{(n)}(\cdot, t)\right) = \frac{1}{2} \frac{d}{dt} \|u^{(n)}(\cdot, t)\|_\mathcal{H}^0$$
and it follows from Remark A.3 that

$$\frac{d}{dt} \|u^{(n)}(\cdot, t)\|_\mathcal{H}^0 = 0 \text{ for every } n \in \mathbb{N}.$$ 

Therefore

$$\frac{d}{dt} \|u(\cdot, t)\|_\mathcal{H}^0 = 0 \text{ for every } t \in I$$
and Lemma A.2 is proved.

As a corollary we obtain the following generalization of Theorem 3.7 in [Gre-Gui].

**COROLLARY A.4.** Suppose that \((p, q) \in \mathcal{H}^0\). Then

(i) \(\text{Iso}_0(p, q) = \{(p', q') \in \mathcal{H}^0; \gamma_k(p', q') = \gamma_k(p, q), k \in \mathbb{Z}\}\)

(ii) \(\|(p, q)\|_{\mathcal{H}^0}\) is a spectral invariant, i.e. is constant on \(\text{Iso}_0(p, q)\).

In particular, this proves Theorem 1.1 as stated in the introduction.

**Appendix B**

In this appendix we prove the asymptotic expansions used in the proof of Theorem 3.4. The first result concerns certain asymptotic properties of the discriminant \(\Delta(\lambda)\).

**LEMMA B.1.** Suppose \((p, q) \in \mathcal{H}^0\). Then, for every \(k \in \mathbb{Z}\),

(i) \(\hat{\Delta}(\lambda_{2k})(p, q) = (-1)^{k+1}\gamma_k(p, q)(1 + l^2(k))\)

(ii) \(\hat{\Delta}(\lambda_{2k-1})(p, q) = (-1)^{k}\gamma_k(p, q)(1 + l^2(k))\).

**Proof of Lemma B.1.** We only prove (i). Assertion (ii) follows by a similar argument. In [Gre-Gui] it is shown that

\[
\Delta(\lambda)^2 - 4 = -4(\lambda_0 - \lambda)(\lambda_{-1} - \lambda) \prod_{k \in \mathbb{Z}^*} \frac{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}{k^2 \pi^2}
\]

where \(\prod_{k \in \mathbb{Z}^*} a_k\) means \(\prod_{k \in \mathbb{N}^*} a_k \cdot a_{-k}\).

Thus, for \(k \in \mathbb{Z}^*\),

\[
2\Delta(\lambda_{2k})\hat{\Delta}(\lambda_{2k}) = -4(\lambda_0 - \lambda_{2k})(\lambda_{-1} - \lambda_{2k}) \frac{\gamma_k}{k^2 \pi^2} \prod_{\substack{l \in \mathbb{Z}^* \setminus \{k\}}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2 \pi^2}.
\]

Since \(\Delta(\lambda_{2k}) = 2(-1)^k\) this leads to

\[
\hat{\Delta}(\lambda_{2k}) = (-1)^{k+1}\gamma_k(1 + l^2(k)) \prod_{\substack{l \in \mathbb{Z}^* \setminus \{k\}}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2 \pi^2}.
\]

Further, using that the Hilbert transform is a bounded operator on \(l^2(\mathbb{Z})\),

\[
\prod_{\substack{l \in \mathbb{Z}^* \setminus \{k\}}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k-1})}{l^2 \pi^2} = \prod_{\substack{l \in \mathbb{Z}^* \setminus \{k\}}} \frac{(l\pi - \lambda_{2k})^2}{l^2 \pi^2} (1 + r(k, l))
\]
where the error term satisfies $|r(k, l)| \leq l^2(k)$ for every $l \in \mathbb{Z}^*, l \neq k$. Using the well-known product formula

$$\sin \frac{\lambda}{\lambda} = \prod_{l \geq 1} \frac{l^2 \pi^2 - \lambda}{l^2 \pi^2}$$

we finally obtain

$$\prod_{l \in \mathbb{Z}^*, l \neq k} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2 \pi^2} = \left(\frac{k \pi}{\lambda_{2k}} \left(\frac{\lambda_{2k}}{k \pi - \lambda_{2k}}\right)\right)^2 \left(1 + l^2(k)\right) = 1 + l^2(k).$$

**Lemma B.2.** Let $(p, q)$ be in $\mathcal{H}^0$. For every $k \in \mathbb{Z}$

(i) $Y_2(1, \lambda_{2k}(p, q)) = (-1)^k (\lambda_{2k}(p, q) \lambda_{2k}(p, q)) (1 + l^2(k))$

(ii) $Y_2(1, \lambda_{2k-1}(p, q)) = (-1)^k (\lambda_{2k-1}(p, q) \lambda_{2k-1}(p, q)) (1 + l^2(k))$.

**Proof of Lemma B.2.** In [Gre-Gui] it is proved that

$$Y_2(1, \lambda; p, q) = (\lambda - v_0(p, q)) \prod_{m \in \mathbb{Z}^*} \frac{v_m(p, q) - \lambda}{m \pi}.$$  

Thus for $k \in \mathbb{Z}^*$ and $j \in \{2k - 1, 2k\}$ we obtain

$$Y_2(1, \lambda_j(p, q); p, q)$$

$$= \frac{(\lambda_j(p, q) - v_0(p, q))}{2\pi} (\lambda_j(p, q) - v_k(p, q)) \prod_{m \in \mathbb{Z}^*, m \neq k} \frac{(v_m(p, q) - \lambda_j(p, q))}{m \pi}$$

$$= (-1)^k (\lambda_j(p, q) - v_k(p, q)) \left(\frac{\lambda_j(p, q) - v_0(p, q)}{k \pi} \prod_{m \in \mathbb{Z}^*, m \neq k} \frac{(v_m(p, q) - \lambda_j(p, q))}{m \pi}\right)$$

from which one deduces Lemma B.2, using similar arguments as in the proof of Lemma B.1.

Combining the two lemmas we obtain

**Lemma B.3.** Let $(p, q)$ be in $\mathcal{H}^0$. Then for every $k$ with $\lambda_{2k-1} < \lambda_{2k}$,

(i) $- \frac{Y_2(1, \lambda_{2k}(p, q))}{\Delta(\lambda_{2k}(p, q))} = \frac{\lambda_{2k}(p, q) - \gamma_k(p, q)}{\gamma_k(p, q)} (1 + l^2(k))$

(ii) $- \frac{Y_2(1, \lambda_{2k-1}(p, q))}{\Delta(\lambda_{2k-1}(p, q))} = \frac{\gamma_k(p, q) - \lambda_{2k-1}(p, q)}{\gamma_k(p, q)} (1 + l^2(k))$
References