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On the cohomology of a general fiber of a polynomial map

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Introduction

Let $X = \mathbf{C}^n$, $S = \mathbf{C}$, and $f: X \to S$ a map defined by a polynomial which is also denoted by f. Then f induces a topological fibration over a Zariski-open subset of S. It is interesting whether we can compute algebraically the cohomology of a generic fiber $F = f^{-1}(t)$ using the polynomial ring $\mathcal{O} := \mathbf{C}[x_1, \dots, x_n]$ and the polynomial f. Of course, we can calculate the cohomology using the de Rham cohomology of the (scheme theoretic) generic fiber of f, but it is not quite computable.

In the weighted homogeneous case, the answer was given by [3]. Let $\Omega^{\cdot}$ denote the complex of global algebraic differential forms on X (i.e., Ω^p is a free $\mathcal{O}$-module with a basis $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$ $(i_1 < \cdots < i_p)$). Define a differential D_f on $\Omega^{\cdot}$ by

$$D_f(\omega) = \mathrm{d}\omega - \mathrm{d}f \wedge \omega \quad \text{for } \omega \in \Omega^p. \tag{0.1}$$

Then we have an isomorphism

$$H^{k+1}(\Omega^{\cdot}, D_f) \cong \tilde{H}^k(F, \mathbf{C}) \quad \text{for any } k, \tag{0.2}$$

if f is weighted homogeneous, where $\tilde{H}$ denotes reduced cohomology. (In [loc. cit.], D_f was denoted by $\bar{D}_f$. See also (2.10) below.) In this paper, we prove

(0.3) THEOREM. *The isomorphism* (0.2) *holds for any polynomial* f.

The proof uses the theory of *algebraic* Gauss–Manin system which is a generalization of [4] (see, for example, [1]), and also the theory of monodromical algebraic $\mathcal{D}$-modules. Let $\int_f \mathcal{O}_X$ denote the algebraic Gauss–Manin system, which is defined by the direct image of $\mathcal{O}_X$ by f as algebraic $\mathcal{D}$-module. Let t be the coordinate of S, and $\partial_t = \partial/\partial t$. Then we have a natural quasi-isomorphism (see (2.7)):

$$R\Gamma\left(S, \text{Cone}\left(\partial_t - \text{id}\colon \int_f \mathcal{O}_X \to \int_f \mathcal{O}_X\right)\right) \xrightarrow{\sim} (\Omega^{\cdot}, D_f)[n]. \tag{0.4}$$

Here $\partial_t - \text{id}$ is *analytically* equivalent to ∂_t (because $\partial_t - 1 = e^t \partial_t e^{-t}$ in $\mathcal{D}_{S^{an}}$). Let $\int_f^p \mathcal{O}_X$ denote the pth cohomology of $\int_f \mathcal{O}_X$. Its restriction to a Zariski open subset of S is a vector bundle (i.e., a locally free sheaf) whose fiber is isomorphic to the cohomology of the fiber of f (see (2.3)). We take the direct image of $\int_f^p \mathcal{O}_X$ by the compactification $S \to \mathbf{P}^1$, and compute its *analytic* local cohomology at infinity (see (2.8)). Then we get the assertion using the theory of monodromical $\mathcal{D}$-modules (see (2.9)).

It should be noted that Theorem (0.3) is essentially *of algebraic nature*, and the local analytic version of (0.3) does not hold. For example, $(\Omega^{\cdot}_{X^{an}}, D_f)$ [1] is not quasi-isomorphic to Deligne's vanishing cycle sheaf, because D_f is *analytically* equivalent to the natural differential d using e^f.

1. Monodromical *D*-modules of one variable

In this section, we gather some elementary facts from the theory of monodromical algebraic $\mathcal{D}$-modules of one variable, which should be well known to specialists.

(1.1) Let S denote the affine line $\mathbf{C}$ with coordinate t (i.e., $S = \operatorname{Spec} \mathbf{C}[t]$). Let $S^* = S \setminus \{0\}$ with a natural inclusion $j\colon S^* \to S$. Let $\mathcal{D}_S$ be the sheaf of algebraic differential operators on S [1], [5]. We denote by R the global sections of $\mathcal{D}_S$, which is the Weyl algebra $\mathbf{C}[t, \partial_t]$. Let $M_{\text{coh}}(\mathcal{D}_S)$ be the category of coherent $\mathcal{D}_S$-modules, and $M_{\text{fin}}(R)$ the category of finite R-modules. We have an equivalence of categories

$$M_{\text{coh}}(\mathcal{D}_S) = M_{\text{fin}}(R) \tag{1.1.1}$$

by the global section functor $\Gamma(S, *)$.

Let S^{an} denote the underlying complex analytic space of S. We have a functor

$$\text{An}\colon M_{\text{coh}}(\mathcal{D}_S) \to M_{\text{coh}}(\mathcal{D}_{S^{an}}) \tag{1.1.2}$$

by $M \to M^{an} := \mathcal{O}_{S^{an}} \otimes_{\mathcal{O}_S} M$, where the pull-back by the natural morphism $S^{an} \to S$ is omitted. Then the de Rham functor DR_S is given by

$$\text{DR}_S(M) = \text{Cone}(\partial_t\colon M^{an} \to M^{an}) \tag{1.1.3}$$

using the coordinate t to trivialize Ω_S^1 (see (2.1.2) below).

(1.2) For $M \in M_{\mathrm{coh}}(\mathscr{D}_S)$, let $M(S) = \Gamma(S, M)$, and

$$M(S)^\alpha = \bigcup_{i \geq 0} \mathrm{Ker}((t\partial_t - \alpha)^i : M(S) \to M(S)) \quad \text{for } \alpha \in \mathbf{C}. \tag{1.2.1}$$

Then

$$tM(S)^\alpha \subset M(S)^{\alpha+1}, \qquad \partial_t M(S)^\alpha \subset M(S)^{\alpha-1}, \tag{1.2.2}$$

and we have isomorphisms

$$t: M(S)^{\alpha-1} \xrightarrow{\sim} M(S)^\alpha, \qquad \partial_t: M(S)^\alpha \xrightarrow{\sim} M(S)^{\alpha-1} \quad \text{for } \alpha \neq 0. \tag{1.2.3}$$

In fact, $t\partial_t$ is bijective on $M(S)^\alpha$ for $\alpha \neq 0$, because $t\partial_t = \alpha$ on $\mathrm{Gr}_i^K M(S)^\alpha$ with $K_i M(S)^\alpha = \mathrm{Ker}(t\partial_t - \alpha)^{i+1}$ (similarly for $\partial_t t$).

(1.3) DEFINITION. We say that $M \in M_{\mathrm{coh}}(\mathscr{D}_S)$ is *monodromical* if M is generated by $M(S)^\alpha$ ($\alpha \in \mathbf{C}$) over $\mathscr{D}_S$. Let $M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ denote the full subcategory of $M_{\mathrm{coh}}(\mathscr{D}_S)$ consisting of monodromical $\mathscr{D}_S$-modules. Then $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ is called *meromorphic* (resp. *microlocal*) *type* if the action of t (resp. ∂_t) on $M(S)$ is bijective.

REMARK. The condition of monodromical $\mathscr{D}_S$-module is equivalent to that any element of $M(S)$ is annihilated by a polynomial of $t\partial_t$. So it is stable by extensions in $M_{\mathrm{coh}}(\mathscr{D}_S)$.

(1.4) LEMMA. *For $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$, we have a natural isomorphism*

$$\bigoplus_{\alpha \in C} M(S)^\alpha \xrightarrow{\sim} M(S), \tag{1.4.1}$$

and $M(S)^\alpha$ is finite dimensional over $\mathbf{C}$. *In particular, the functor $M \to M(S)^\alpha$ is exact.*

Proof. The injectivity of (1.4.1) is clear using the action of $t\partial_t$ on $M(S)$. Since the condition of monodromical $\mathscr{D}_S$-module is equivalent to the surjectivity of

$$\bigoplus_{\alpha \in \mathbf{C}} \mathscr{D}_S \otimes_{\mathbf{C}} M(S)^\alpha \to M, \tag{1.4.2}$$

the surjectivity of (1.4.1) follows from (1.2.3), taking the global section of (1.4.2). We have $\dim_{\mathbf{C}} M(S)^\alpha < \infty$, because $M(S)^\alpha$ is finitely generated over $\mathbf{C}[N]$ with $N = -(t\partial_t - \alpha)$.

REMARK. By (1.2.3) and (1.4.1), $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ is meromorphic (resp.

microlocal) type if and only if

$$t: M(S)^{-1} \to M(S)^{0} \text{ (resp. } \partial_t: M(S)^{0} \to M(S)^{-1}) \tag{1.4.3}$$

is bijective.

(1.5) LEMMA. *Let $M \in M_{\mathrm{coh}}(\mathscr{D}_S)$ such that* supp $M \subset \{0\}$. *Then M is monodromical, and $M(S)^{\alpha} = 0$ except for negative integers α.*

Proof. The assumption is equivalent to that any element of $M(S)$ is annihilated by a sufficiently high power of t. Then we can check the assertion using $\partial_t^i t^i = \Pi_{0<j\leq i}(t\partial_t + j)$.

REMARK. For M as above, M is a finite direct sum of $\mathscr{B}$ in the proof of (1.8) by (1.2.3). This is a special case of Kashiwara's equivalence (see [1]).

(1.6) LEMMA. *Let Λ be a subset of* **C** *such that $0 \in \Lambda$ and the natural morphism $\Lambda \to \mathbf{C}/\mathbf{Z}$ is bijective. Let $\Lambda' = \Lambda \cup \{-1\}$. Let $\mathscr{C}$ be the category whose object is a family of* **C***-vector spaces $V^{\alpha}(\alpha \in \Lambda')$ with morphisms $u: V^0 \to V^{-1}$, $v: V^{-1} \to V^0$, and $N: V^{\alpha} \to V^{\alpha}$ $(\alpha \in \Lambda \backslash \{0\})$ such that $\bigoplus_{\alpha \in \Lambda'} V^{\alpha}$ is finite dimensional, and vu, uv and N are nilpotent. Then we have an equivalence of categories*

$$M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}} = \mathscr{C} \tag{1.6.1}$$

by associating $M(S)^{\alpha}$, ∂_t, t and $t\partial_t - \alpha$ to $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$.

Proof. This follows from (1.2.2–3) and (1.4.1).

(1.7) COROLLARY. *We have an equivalence of categories*

$$M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon},*} \text{ (resp. } M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon},!}) = V(\mathbf{C}, T), \tag{1.7.1}$$

where the left-hand side is the category of monodromical $\mathscr{D}_S$-modules of meromorphic (resp. microlocal) type, the right-hand side is the category of finite dimensional **C***-vector spaces with an automorphism T, and the functor is defined by $M \to \bigoplus_{\alpha \in \Lambda} M(S)^{\alpha}$ with $T = \exp(-2\pi i t\partial_t)$.*

REMARK. Using (1.6), we can show that the category $M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ is equivalent to the category of regular holonomic $\mathscr{D}_{S^{\mathrm{an}},0}$-modules (for which an equivalence of categories similar to (1.6.1) holds). The terms 'meromorphic' and 'microlocal' are originally used in this case (see [8]).

(1.8) PROPOSITION. *Let $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$. Then M is regular holonomic* [1].

Proof. Since the action of $t\partial_t - \alpha$ on $M(S)^{\alpha}$ is nilpotent, we may assume $\Sigma_{\alpha \in \Lambda'} \dim M(S)^{\alpha} = 1$ by (1.6), taking the graduation of a finite filtration on M (because regular holonomic $\mathscr{D}$-modules are stable by extensions [loc. cit.]). Then we can check that M is isomorphic to one of the following:

(i) $\mathcal{O}_S = \mathcal{D}_S/\mathcal{D}_S\partial_t$,
(ii) $\mathcal{B} := \mathcal{D}_S/\mathcal{D}_S t$,
(iii) $M(\alpha) := \mathcal{D}_S/\mathcal{D}_S(t\partial_t - \alpha)$ $(\alpha \in \Lambda\backslash\{0\})$,

depending on the α such that $M(S)^\alpha \neq 0$. So we get the assertion.

REMARK. We can show that a regular holonomic $\mathcal{D}_S$-module is monodromical, if and only if its restriction to S^* is finite over $\mathcal{O}_{S^*}$ (i.e., a vector bundle with connection [2]). In fact, we may assume that $M|_{S^*}$ is a vector bundle by (1.10) below. Then the assertion is reduced to case where the action of t on M is bijective using the localization of M by t (because $M_{\mathrm{coh}}(\mathcal{D}_S)_{\mathrm{mon}}$ is stable by extensions in $M_{\mathrm{coh}}(\mathcal{D}_S)$, see Remark after (1.3)). Then the assertion follows [2] (see also (1.11) below).

(1.9) PROPOSITION. *For $M \in M_{\mathrm{coh}}(\mathcal{D}_S)_{\mathrm{mon}}$, there exists uniquely $M' \in M_{\mathrm{coh}}(\mathcal{D}_S)_{\mathrm{mon}}$ of meromorphic (resp. microlocal) type with a morphism $M \to M'$ (resp. $M' \to M$) inducing an isomorphism on S^*.*

Proof. By (1.6) there exists uniquely $M' \in M_{\mathrm{coh}}(\mathcal{D}_S)_{\mathrm{mon}}$ with a morphism $M \to M'$ (resp. $M' \to M$), such that

$$M'(S)^\alpha = M(S)^\alpha \quad \text{for } -\alpha \notin \mathbf{N}\backslash\{0\},$$
$$\partial_t^i : M(S)^0 \xrightarrow{\sim} M(S)^{-i} \text{ (resp. } t^i : M(S)^{-i} \xrightarrow{\sim} M(S)^0) \tag{1.9.1}$$

for $i > 0$. Then the morphism induces an isomorphism on S^* by (1.5).

REMARK. In the standard notation (see [1]), M' is denoted by j_*j^*M (resp. $j_!j^*M$). Here j_* is really the direct image as Zariski sheaf (because M' is the localization of M by t), but $j_!$ is not. In fact, $j_!$ is defined by $\mathbf{D}j_*\mathbf{D}$ with $\mathbf{D}$ the dual functor (see [loc. cit.]). We have

$$\mathrm{DR}_S(j_*j^*M) = Rj_*j^*\mathrm{DR}_S(M) \quad \text{(cf. [2])}, \tag{1.9.2}$$
$$\mathrm{DR}_S(j_!j^*M) = j_!j^*\mathrm{DR}_S(M) \quad \text{(cf. [1])}. \tag{1.9.3}$$

See also (1.12) below for (1.9.3).

(1.10) COROLLARY. *For $M \in M_{\mathrm{coh}}(\mathcal{D}_S)_{\mathrm{mon}}$, the restriction of M to S^* is a free $\mathcal{O}_{S^*}$-module of* rank $\Sigma_{\alpha\in\Lambda} \dim M(S)^\alpha$. *In particular, $M^{\mathrm{an}}|_{S^*}$ is a vector bundle with connection* [2], *and* $\mathrm{DR}_S(M)[-1]|_{S^*}$ *is a local system.*

Proof. It is enough to show the first assertion. We may assume M meromorphic type by (1.9). Then $M(S)$ is a free $\mathbf{C}[t, t^{-1}]$-module of rank $\Sigma_{\alpha\in\Lambda} \dim M(S)^\alpha$ by (1.2.3) and (1.4.3), and the assertion follows.

(1.11) PROPOSITION. *Let L be a local system on S^* with complex coefficients, L_∞ the group of multivalued sections of L with the monodromy T, and L_∞^α the*

$\exp(-2\pi i\alpha)$-eigenspace of L_∞ with respect to T_s, where $T = T_sT_u$ is the Jordan decomposition. Then there exists uniquely $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ of meromorphic (resp. microlocal) type with an isomorphism

$$L = \mathrm{DR}_S(M)[-1]|_{S^*}, \tag{1.11.1}$$

where DR_S *is as in* (1.1.3). *Furthermore, we have a canonical isomorphism*

$$M(S)^\alpha = L_\infty^\alpha \tag{1.11.2}$$

for $\alpha \in \Lambda$, such that $-(t\partial_t - \alpha)$ corresponds to $N := (\log T_u)/2\pi i$.

Proof. By (1.9) it is enough to show the assertion for M meromorphic type. By (1.7), there exists uniquely $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ of meromorphic type with the isomorphism (1.11.2). Then M^{an} is identified with a $\mathscr{O}_S[t^{-1}]$-submodule of $j_*(\mathscr{O}_{S^*} \otimes_{\mathbf{C}} L)$ generated by

$$t^\alpha \exp(-(\log t)N)u \tag{1.11.3}$$

for $u \in L_\infty^\alpha$ with $\alpha \in \Lambda$ (see [2]), because (1.11.3) satisfies the same relation as the element of $M(S)^\alpha$ corresponding to $u \in L_\infty^\alpha$ by definition of M (and $M^{\mathrm{an}}|_{S^*}$ and $\mathscr{O}_{S^*} \otimes_{\mathbf{C}} L$ have the same rank). In particular, $M^{\mathrm{an}}|_{S^*} = \mathscr{O}_{S^*} \otimes_{\mathbf{C}} L$, and the assertion follows.

REMARK. This proposition shows that we have an equivalence of categories between the category of monodromical $\mathscr{D}_S$-modules of meromorphic type and the category of local systems on S^*, in a compatible way with (1.7.1). Note that the isomorphism (1.11.2) depends on the choice of the branch of $\log t$ (i.e., the choice of a lift of 1 to a universal covering of S^*).

(1.12) PROPOSITION. *If $M \in M_{\mathrm{coh}}(\mathscr{D}_S)_{\mathrm{mon}}$ is microlocal type, we have*

$$(\mathrm{DR}_S(M))_0 = 0, \tag{1.12.1}$$

$$R\Gamma(S^{\mathrm{an}}, \mathrm{DR}_S(M)) = 0. \tag{1.12.2}$$

Proof. By (1.10), $\mathrm{DR}_S(M)[-1]|_{S^*}$ is a local system, and it is enough to show (1.12.1). Using a filtration defined in the category of monodromical $\mathscr{D}_S$-modules of microlocal type, we may assume $\Sigma_{\alpha\in\Lambda} \dim M(S)^\alpha = 1$. Then it is isomorphic to $M(\alpha) = \mathscr{D}_S/\mathscr{D}_S(t\partial_t - \alpha)$ if $M(S)^\alpha = \mathbf{C}$ for $\alpha \in \Lambda\backslash\{0\}$ (see the proof of (1.8)). In the other case, we can check that M is isomorphic to $\mathscr{D}_S/\mathscr{D}_S(t\partial_t)$. Then we can check the assertion (see also [8]).

2. Algebraic Gauss-Manin system

(2.1) Let $f: X \to Y$ be a morphism of smooth complex algebraic varieties. The direct image $\int_f M$ of a $\mathscr{D}_X$-module M is defined by

$$\int_f M = Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} M) \quad \text{with} \quad \mathscr{D}_{Y\leftarrow X} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathscr{D}_Y \otimes_{\mathcal{O}_Y} (\omega_Y)^\vee), \tag{2.1.1}$$

where ω_X is the dualizing sheaf, and $^\vee$ denotes the dual line bundle. See [1], [6], [10], etc. Note that, if f is an open embedding, $\int_f M$ is defined by the sheaf theoretic direct image. In the case Y is the affine line S, the direct image $\int_f M$ will be more explicitly expressed later (see (2.6)).

We denote the cohomological direct image $\mathscr{H}^p \int_f M$ by $\int_f^p M$. For $M = \mathcal{O}_X$, the direct image $\int_f \mathcal{O}_X$ (or $\int_f^p \mathcal{O}_X$) is called the *Gauss-Manin system* of f [7].

For a $\mathscr{D}_X$-module M, let

$$\mathrm{DR}_X(M) = (\Omega^{\cdot}_X(M))^{\mathrm{an}}[\dim X], \tag{2.1.2}$$

where $\Omega^{\cdot}_X(M)$ denotes the de Rham complex as in [1], [2], and an is defined as in (1.1.2). By [1] we have:

(2.2) PROPOSITION. *If M is regular holonomic, the cohomological direct images $\int_f^p M$ are regular holonomic, and we have a natural isomorphism*

$$\mathrm{DR}_Y\left(\int_f M\right) = Rf_*(\mathrm{DR}_X(M)). \tag{2.2.1}$$

(2.3) COROLLARY. *Assume f smooth with relative dimension r, and $\int_f^p \mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of finite rank* (*i.e., a vector bundle with integrable connection* [2]). *Let L^p be the local system defined by the horizontal sections of $(\int_f^p \mathcal{O}_X)^{\mathrm{an}}$. Then we have natural isomorphisms*

$$\mathrm{DR}_Y\left(\int_f^p \mathcal{O}_X\right) = L^p[\dim Y], \tag{2.3.1}$$

$$L^p = R^{p+r} f_* \mathbf{C}_{X^{\mathrm{an}}}. \tag{2.3.2}$$

Proof. The first assertion follows from the Poincaré lemma. Then the second follows from (2.2.1) using $\mathrm{DR}_X(\mathcal{O}_X) = \mathbf{C}_{X^{\mathrm{an}}}[\dim X]$, because $\mathcal{O}_X$ is regular holonomic by definition [1].

REMARK. If we assume that $R^{p+r}f_*\mathbf{C}_{X^{an}}$ is a local system, the corollary follows also from the relative version of [4] using a desingularization of the divisor at infinity of a compactification of f, because we may replace Y by its Zariski-open subset.

(2.4) LEMMA. *Let* $f: X \to Y$ *be as in* (2.1), *and assume* Y *is the affine line* S *with coordinate* t *as in Section* 1 *so that* f *is identified with a function on* X. *Let* $\theta = \partial_t - \mathrm{id}$, *and*

$$K_f = \mathrm{Cone}\left(\theta\colon \int_f \mathcal{O}_X \to \int_f \mathcal{O}_X\right). \tag{2.4.1}$$

Then we have a natural isomorphism

$$(K_f)^{\mathrm{an}} = \mathrm{DR}_S\left(\int_f \mathcal{O}_X\right), \tag{2.4.2}$$

where an *is defined as in* (1.1.2).

Proof. This follows from $\partial_t - 1 = \mathrm{e}^t \partial_t \mathrm{e}^{-t}$ in $\mathcal{D}_S^{\mathrm{an}}$.

(2.5) PROPOSITION. *For* $f: X \to S$ *as above, assume* X^{an} *contractible and purely* n*-dimensional. Then we have a natural isomorphism*

$$R\Gamma(S^{\mathrm{an}}, (K_f)^{\mathrm{an}}) = \mathbf{C}[n]. \tag{2.5.1}$$

Proof. This follows from (2.2) and (2.4).

REMARK. We can apply (2.2) also to the direct image of $\mathcal{O}_X$ by $X \to pt$ and the direct image of $\int_f^p \mathcal{O}_X$ by $S \to pt$. In this case, (2.2) means the commutativity of the direct image with the functor An in (1.1.2), and follows also from [4] and [2] (see also (1.9.2) above) respectively.

(2.6) Let $f: X \to S$ be as in (2.4), and M a $\mathcal{D}_X$-module. We define a structure of $\mathcal{D}_X$-module on $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$ by

$$g(u \otimes \partial_t^i) = gu \otimes \partial_t^i, \qquad \zeta(u \otimes \partial_t^i) = \zeta u \otimes \partial_t^i - (\zeta f)u \otimes \partial_t^{i+1} \tag{2.6.1}$$

for $g \in \mathcal{O}_X$, $\zeta \in \Theta_X$ and $u \in M$. It has also the action of $R = \mathbf{C}[t, \partial_t]$ (see (1.1)) by

$$\partial_t(u \otimes \partial_t^i) = u \otimes \partial_t^{i+1}, \qquad t(u \otimes \partial_t^i) = fu \otimes \partial_t^i - iu \otimes \partial_t^{i-1}, \tag{2.6.2}$$

which commutes with the action of $\mathcal{D}_X$. Then $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$ is identified with the direct image of M by the embedding i_f by the graph of f, and $u \otimes \partial_t^i$ is identified with $\partial_t^i\delta(t-f)\otimes u$. Here $\delta(t-f)$ is the delta function with support $\{f=t\}$, and satisfies the relation

$$t\delta(t-f) = f\delta(t-f), \qquad \zeta\delta(t-f) = -(\zeta f)\partial_t\delta(t-f), \tag{2.6.3}$$

which gives (2.6.1–2).

Since the direct image of a $\mathscr{D}$-module by a smooth projection with fiber X is given by the sheaf theoretic direct image of the relative de Rham complex shifted by $\dim X$ (see for example [1]), the direct image $\int_f M$ is expressed as

$$\int_f M = Rf_*(\Omega_X^{\cdot}(M\otimes_{\mathbf{C}}\mathbf{C}[\partial_t])[\dim X], \tag{2.6.4}$$

factorizing f into the closed embedding i_f and the projection. This can be also obtained by using induced $\mathscr{D}$-modules [9]. Note that, if f is an affine morphism, the derived direct image Rf_* can be replaced by f_*.

(2.7) PROPOSITION. *For $f: X\to S$ as above, assume $X=\mathbf{C}^n$. Then we have a natural quasi-isomorphism* (0.4) *in the introduction.*

Proof. By (2.6.4), $\int_f \mathscr{O}_X$ is expressed by $f_*(\Omega_X^{\cdot}\otimes_{\mathbf{C}}\mathbf{C}[\partial_t])[n]$, where the differential of $\Omega_X^{\cdot}\otimes_{\mathbf{C}}\mathbf{C}[\partial_t]$ is given by

$$\omega\otimes\partial_t^i\to d\omega\otimes\partial_t^i - df\wedge\omega\otimes\partial_t^{i+1}. \tag{2.7.1}$$

See (2.6.1). Then we have a short exact sequence of complexes

$$0\to\Omega_X^{\cdot}\otimes_{\mathbf{C}}\mathbf{C}[\partial_t]\xrightarrow{\theta}\Omega_X^{\cdot}\otimes_{\mathbf{C}}\mathbf{C}[\partial_t]\to\Omega_X^{\cdot}\to 0, \tag{2.7.2}$$

where the differential of $\Omega_X^{\cdot}$ is D_f. Then we get the assertion taking the exact functors f_* and $\Gamma(S,*)$.

(2.8) PROPOSITION. *Let $\bar{S}=\mathbf{P}^1$ with a natural inclusion $j': S\to\bar{S}$. Let M be a regular holonomic $\mathscr{D}_S$-module, and $K=\mathrm{Cone}(\theta: M\to M)$ for θ as above. Let U be a Zariski-open subset of S on which M is locally free over $\mathscr{O}_U$, and denote the local system $\mathrm{DR}_S(M)[-1]|_{U^{\mathrm{an}}}$ by L. Then we have a canonical isomorphism*

$$H^0_{\{\infty\}}((j'_*K)^{\mathrm{an}}) = L_t \tag{2.8.1}$$

for $t\in U^{\mathrm{an}}(\subset\mathbf{C})$ such that $\mathrm{Im}\,t=0$ and $\mathrm{Re}\,t\gg 0$. Furthermore,

$$H^i_{\{\infty\}}((j'_*K)^{\mathrm{an}}) = 0 \quad \text{for } i\neq 0. \tag{2.8.2}$$

Proof. Let $V=\{t\in\mathbf{C}: |t|>R\}\subset U^{\mathrm{an}}$ for R sufficiently large, and $V'=$

$V \cup \{\infty\}$. Then $(j'_*M)^{an}|_{V'}$ is an extension of $M^{an}|_V$ as regular holonomic $\mathscr{D}_{V'}$-module such that the action of the local coordinate $s(=t^{-1})$ is bijective, and such an extension is unique by [2]. So $H^i_{\{\infty\}}((j'_*K)^{an})$ is uniquely determined by $M^{an}|_V$, or equivalently, by $L|_V$ (see [loc. cit.]). Since V is homotopy equivalent to $(S^*)^{an}$, we may assume $U = S^*$, and M is a monodromical $\mathscr{D}_S$-module of microlocal type by (1.11). Then

$$H^i(S^{an}, K^{an}) = H^i(S^{an}, \mathrm{DR}_S(M)) \tag{2.8.3}$$

by the same argument as the proof of (2.4). So it is zero for any i by (1.12.2), and we may replace $H^i_{\{\infty\}}((j'_*K)^{an})$ in (2.8.1–2) by $H^i(\bar{S}^{an}, (j'_*K)^{an})$ using the long exact sequence:

$$\to H^i_{\{\infty\}}((j'_*K)^{an}) \to H^i(\bar{S}^{an}, (j'_*K)^{an}) \to H^i(S^{an}, K^{an}) \to \tag{2.8.4}$$

By GAGA, we have

$$H^i(\bar{S}^{an}, (j'_*K)^{an}) = H^i(\bar{S}, j'_*K) = H^i(S, K), \tag{2.8.5}$$

where the last isomorphism follows from the exactness of j'_*. Moreover,

$$R\Gamma(S, K) = \mathrm{Cone}(\theta: M(S) \to M(S)) = \bigoplus_{\alpha \in \Lambda} M(S)^\alpha \tag{2.8.6}$$

using (1.2.3) and (1.4.3). So the assertion follows from the isomorphism (1.11.2). Here we identify L_t with L_∞ by taking a lift of t to a universal covering of S^*, at which $\log t$ is real valued.

(2.9) *Proof of Theorem* (0.3). Let K_f be as in (2.4), and $j': S \to \bar{S}$ as above. By GAGA and (2.7), we have

$$H^i(\bar{S}^{an}, (j'_*K_f)^{an}) = H^i(\bar{S}, j'_*K_f) = H^i(S, K_f) = H^{i+n}(\Omega^\cdot, D_f). \tag{2.9.1}$$

We have the long exact sequence (2.8.4) with K replaced by K_f. Let $F = f^{-1}(t)$ for t as in (2.8.1). Then it is enough to show a canonical isomorphism

$$H^i_{\{\infty\}}((j'_*K_f)^{an}) = H^{i+n-1}(F, \mathbf{C}) \tag{2.9.2}$$

by (2.5.1), because we can check $H^0(\Omega^\cdot, D_f) = 0$ so that the morphism

$$\mathbf{C} = H^{-n}(S^{an}, (K_f)^{an}) \to H^{1-n}_{\{\infty\}}((j'_*K_f)^{an}) \tag{2.9.3.}$$

is injective. Then, applying (2.8) to $M = \int_f^p \mathscr{O}_X$, the assertion (2.9.2) follows from (2.3).

(2.10) REMARKS. (i) If f is weighted homogeneous, $\int_f^p \mathcal{O}_X$ $(p \neq 1-n)$ and $\int_f^{1-n} \mathcal{O}_X/\mathcal{O}_S$ are monodromical $\mathcal{D}_S$-modules of microlocal type, and the decomposition (1.4.1) by the action of $t\partial_t$ is induced by the grading of $\Omega^{\cdot}$ compatible with f as in [3]. This implies that the isomorphism (0.2) is compatible with the action of monodromy as in [loc. cit.].

(ii) In Theorem A of [3], δ does not induce an isomorphism for $k=0$. The definition of D_f should be replaced by (0.1) in this paper, which is denoted by $\bar{D}_f$ in [loc. cit.].

(iii) In the proof of (1.8) of [3], it is better to use Coim Δ instead of $\bar{\Omega} = \operatorname{Ker} \Delta$.

(2.11) EXAMPLE: $f = x^2y + x$. This is a generalized weighted homogeneous polynomial admitting *negative* weights, and the spectral sequence as in [3] does not converge. In fact, the E_0-complex is isomorphic to the Koszul complex $(\Omega^{\cdot}, \mathrm{d}f \wedge)$, and is acyclic, but the general fiber $F = f^{-1}(t)$ is isomorphic to $\mathbf{C}^*$ so that $\tilde{H}^0(F, \mathbf{C}) = 0$, $\tilde{H}^1(F, \mathbf{C}) = \mathbf{C}$. We can check $H^2(\Omega^{\cdot}, D_f) = \mathbf{C}$ as follows.

We have $f_x = 2xy + 1$, $f_y = x^2$, and

$$D_f(-x^iy^j\,\mathrm{d}x) = jx^iy^{j-1}\,\mathrm{d}x\,\mathrm{d}y - x^{i+2}y^j\,\mathrm{d}x\,\mathrm{d}y$$

$$D_f(x^iy^j\,\mathrm{d}y) = ix^{i-1}y^j\,\mathrm{d}x\,\mathrm{d}y - 2x^{i+1}y^{j+1}\,\mathrm{d}x\,\mathrm{d}y - x^iy^j\,\mathrm{d}x\,\mathrm{d}y.$$

Let $\phi\colon \Omega^2 \to \mathbf{C}$ be a map defined by $\phi(x^iy^j\,\mathrm{d}x\,\mathrm{d}y) = (-1)^i j!/(2j-i+1)!$ for $2j+1 \geqslant i$, and 0 otherwise. Then ϕ induces the isomorphism $H^2(\Omega^{\cdot}, D_f) = \mathbf{C}$.

Added in the proof. We are informed that a similar result is obtained by B. Malgrange and P. Deligne independently using the theory of Fourier transformation.

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