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On secant spaces to projective curves

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In this note we give an affirmative answer (in a stronger form) to a question raised in a recent paper ([1], problem 1.4). The main tool for the proof will be [1], th. 1.2. To state the question, we need to introduce a notation.

Let r, d, e, n be integers with $2 \leqslant n + 1 \leqslant r$ and $e \geqslant n + 1$, C a smooth complete connected curve and T a g^r_d on C; let $C^{(e)}$ be the symmetric product; set (as in [1]) $V^n_e(T) := \{D \in C^{(e)} : D$ imposes at most $n + 1$ conditions to $T\}$. Set

$$t(r, n, e) := (n + 1 - e)(r - n) + e \tag{1}$$

By a standard determinantal description, every irreducible component of $V^n_e(T)$ has dimension at least $t(r, n, e)$ (see e.g. [1], §1).

The question will be answered proving (over any algebraically closed base field) the following theorem 0.1.

THEOREM 0.1. *Fix integers r, d, e, n, with $e \geqslant n + 1$, $r > n \geqslant 2$, $d \geqslant 2e - 1$. Let C be a smooth complete connected curve and Γ a g^r_d on C which is not a complete linear system. Assume*

$$(n + 1 - e)(r - n) + e \geqslant 0 \tag{2}$$

Then $V^n_e(\Gamma)$ is not empty.

Note that the bound "$t(r, n, e) \geqslant 0$" required in (2) for non-complete linear systems is weaker than the bound required in [1], th. 1.2, in the case of complete linear systems, and that it is "sharp". The condition "$d \geqslant 2e - 1$" will be used only to apply the statement of [1], th. 1.2; hence any improvement of [1], th. 1.2, related to this condition should give a corresponding improvement of 0.1 (see the related discussion in [1], 1.3). The fact that the bound (2) in 0.1 is better than the one in [1], th. 1.2, occurs essentially for numerical reasons. Indeed theorem 0.1 will be proven by a reduction to the case proven in [1], th. 1.2.

A few words on the case of positive characteristic. Theorem 1.2 of [1] was claimed only in characteristic 0. However that proof (and in particular [1], lemma 1.2.1) works without changes in positive characteristic if all the references used there are justified (mainly [2]). The key tool for [1], lemma 1.2.1, and for [2] is a section of [3] which works in any characteristic (and this was explicitly remarked in [3], Remark 2.8).

Proof of 0.1. We may easily reduce to the case (which will be assumed from now on) that Γ has no base point. Since Γ is not complete, it corresponds to a hyperplane of a g_d^{r+1}, Φ (or, in geometric language, "the $\mathbf{P}^r$ corresponding to Γ is seen as the projection from a point $\mathbf{u}$ of the $\mathbf{P}^{r+1}$ corresponding to Φ"). If $V_e^n(\Phi) \neq \emptyset$, then it is obvious that $V_e^n(\Gamma) \neq \emptyset$. Hence we may assume that $V_e^n(\Phi) = \emptyset$. This implies that if we take any degree e effective divisor Z of C, there is at most one $(n + 1)$-dimensional subspace of Φ containing it. Set $k := r - n$. Note that by (2) we have:

$$t(n + 1, r + 1, e) = t(n, r, e) + k \geqslant k \tag{3}$$

If Φ is complete, by (2) we may apply [1], th. 1.2, and find $V_e^{n+1}(\Phi) \neq \emptyset$. If Φ is not complete, we may work by induction on the codimension of Γ in the complete linear system $|\Gamma|$; in both cases we may assume $V_e^{n+1}(\Phi) \neq \emptyset$. As remarked in [1], by the determinental description of $V_e^{n+1}(\Phi)$ every irreducible component, T, of $V_e^{n+1}(\Phi)_{\mathrm{red}}$ has dimension at least $t(n + 1, r + 1, e) \geqslant k$ by (2). Fix any such T and let S be the "complete integral subvariety of $\mathbf{P}^{r+1}$ which is the union of all $(n + 1)$-dimensional linear spaces parametrized by T"; S is complete because T is complete. It is sufficient to check that $\mathbf{u} \in S$. Hence we may assume by contradiction $S \neq \mathbf{P}^{r+1}$, i.e. $\dim(S) \leqslant n + k$. Since $\dim(S) < n + k$, using a suitable incidence variety and counting dimensions we see that for a general $x \in S$ there is at least a 1-dimensional family, $T(x)$, of elements of T. Take as x a smooth point of S. Every $L \in T(x)$ is contained in the Zariski tangent space $T_x S$, i.e. in a fixed hyperplane. Since by definition of linear system the image of C by the map corresponding to Φ spans $\mathbf{P}^{r+1}$, we see that the union of the effective divisors contained in these linear spaces is supported by at most d points of C. Since a set with d elements has finitely many subsets, the contradiction comes from the assumption "$V_e^n(\Phi) = \emptyset$", i.e. by the fact that every $L \in T$ is uniquely determined by the degree e effective divisor of C contained in L. $\square$

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