Compositio Mathematica

Ofer Gabber

An injectivity property for étale cohomology

Compositio Mathematica, tome 86, n° 1 (1993), p. 1-14
<http://www.numdam.org/item?id=CM_1993__86_1_1_0>

Compositio Mathematica **86**: 1–14, 1993.

An injectivity property for étale cohomology

OFER GABBER

Inst. Des Hautes Etudes Scientifiques, 35, Route De Chartres, 91440 Bures Sur Yvette, France

Received 1 July 1982; accepted 1 July 1992

In this paper, we consider the following situation: Suppose $\mathcal{O}$ is a commutative local ring, $M \to \mathrm{Spec}(\mathcal{O})$ a smooth morphism, $\omega \in \mathrm{Spec}(\mathcal{O})$ the closed point, $\mathrm{U} \subset \mathrm{Spec}(\mathcal{O})$ a quasi-compact open subscheme, Φ a complex of étale sheaves [1, VII] on U whose cohomology sheaves are torsion and bounded below. Denote $Q\Phi =$ the image of Φ in the derived category [7, Ch. I] of the category of abelian sheaves on $\mathrm{U}_{\text{ét}}$.

$$\begin{array}{ccccc} \bar{M} & \xrightarrow{\ i\ } & M & \xrightarrow{\ j\ } & \pi^{-1}(\mathrm{U}) \\ \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_U} \\ \boldsymbol{\omega} & \xrightarrow{\ i_0\ } & S & \xrightarrow{\ j_0\ } & \mathrm{U}, \end{array} \qquad \begin{array}{l} S = \mathrm{Spec}(\mathcal{O}) \\ \boldsymbol{\omega} = \mathrm{Spec}(k) \\ \bar{M} = M \times_{\mathcal{O}} \boldsymbol{\omega}. \end{array}$$

Let $K = i^* Rj_*(Q(\pi_U^*\Phi)) \in D^+(\bar{M}_{\text{ét}}, \mathbb{Z})$.

If $\xi \xrightarrow[\varphi]{} M$ is a morphism where ξ is the spectrum of a field $k(\xi)$ which is separable algebraic over $k(\varphi(\xi))$, we can form $\mathcal{O}^h_{M,\xi} =$ [the unramified extension corresponding[1] to $k(\varphi(\xi)) \to k(\xi)$ of the henselization[2] of $\mathcal{O}_{M,\varphi(\xi)}$] $= \Gamma(\xi, \varphi^*_{\text{ét}}\mathcal{O}_{M_{\text{ét}}})$, and let $M^h_\xi = \mathrm{Spec}(\mathcal{O}^h_{M,\xi})$.

Consider a point $\xi \in \bar{M}$ and and the generic point $\eta \to \bar{M}^h_\xi$. [Recall that $\bar{M}^h_\xi$ is regular (and hence irreducible) as $\mathcal{O}_{\bar{M},\xi}$ is.] $\mathcal{O}^h_{M,\eta}$ is the henselization of the local ring of M^h_ξ at η, so we have a morphism $M^h_\eta \xrightarrow[\alpha]{} M^h_\xi$.

THEOREM 1. *$\forall q \in \mathbb{Z}$ the map*

$$H^q(M^h_\xi \times_{\mathcal{O}} \mathrm{U}, pr_2^*\Phi) \xrightarrow[(\alpha \times id)^*]{} H^q(M^h_\eta \times_{\mathcal{O}} \mathrm{U}, pr_2^*\Phi) \text{ is injective.}$$

Theorem 1 was conjectured by K. Kato in the case that U is the generic point in the spectrum of an unequal characteristic discrete valuation ring $\mathcal{O}$ of residue

(1)By [2, III 4.6] and taking the limit.
(2)[2, III 4.5].

characteristic $p > 0$, and $\Phi = \mathbb{Z}/(p)$. In that case, Theorem 1 helped him to define a finite filtration on $H^q(M_\xi^h \times_{\mathcal{O}} \mathrm{U}, \mathbb{Z}/(p))$ whose *gr* is described in terms of differentials. This was done in an earlier version of [3], but the final argument does not require my result. Our method was used again in [9, 1.8].

REMARK 1. Suppose $M = \mathbb{P}^1_{\mathcal{O}}$, and ξ is a closed point of $\mathbb{P}^1_k$. Then we knew (in a special case) to define a "residue" map $H^q(M_\eta^h \times_{\mathcal{O}} \mathrm{U}, pr_2^*\Phi) \to H^{q-1}(\mathrm{U}_{\text{ét}}, \Phi(-1))$, whose image contains $[k(\xi): k] \cdot H^{q-1}(\mathrm{U}, \Phi(-1))$ and which vanishes on the image of $H^q(M_\xi^h \times_{\mathcal{O}} \mathrm{U}, pr_2^*\Phi)$, and this inspired in part the present proof.

Theorem 1 can be equivalently stated as

THEOREM 1′. *The composed maps*

$$R^q\Gamma(\xi, K_{|\xi}) \xleftarrow{\sim} R^q\Gamma(\bar{M}_\xi^h, K) \to R^q\Gamma(\eta, K_{|\eta})(q \in \mathbb{Z})$$

are injective.

REMARK 2. It can be shown, on any topos, that the derived category of complexes of torsion sheaves maps by an equivalence to the derived category of complexes of abelian sheaves with torsion cohomology sheaves. So we may assume that Φ is a complex of torsion sheaves.

REMARK 3. Let $\lambda_\xi: M_\xi^h \to M$ be the canonical morphism,

$$\mu_\xi: \lambda_\xi^{-1}(\pi^{-1}(\mathrm{U})) \to \pi^{-1}(\mathrm{U})$$

its restriction, and $j_\xi: \lambda_\xi^{-1}(\pi^{-1}(\mathrm{U})) \hookrightarrow M_\xi^h$. In showing Theorem 1 $\Leftrightarrow$ Theorem 1′ one uses

$$\begin{aligned} H^q(\lambda_\xi^{-1}(\pi^{-1}(\mathrm{U})), \mu_\xi^*\pi_U^*\Phi) &\simeq H^q(M_\xi^h, Rj_{\xi*}(\mu_\xi^*\pi_U^*\Phi)) \\ &= H^q(M_\xi^h, \lambda_\xi^*(Rj_*\pi_U^*\Phi)) \xrightarrow[\sim]{} H^q(\xi, (Rj_*\pi_U^*\Phi)_{|\xi}), \end{aligned}$$

the last map being an isomorphism by ([1], VII Cor. 8.6) [extended to complexes bounded below, e.g. using the second spectral sequence [5, 0_{III} (11.4.3.2)] for hypercohomology] because $\mathcal{O}_{M,\xi}^h$ is local henselian. The equality uses that the "base change" morphism[3] $\lambda_\xi^* R^+j_* \xrightarrow{\sim} R^+j_{\xi*}\mu_\xi^*$[4] is an isomorphism by the standard description of the stalks of R^pj_* [1, VIII Th. 5.2]. Similarly for ξ replaced by η.

---

(3) In the sense of [1, XVII 4.1.4].

(4) R^+j_* denotes the right derived functor of the functor $(j_{\text{ét}})_*$ between the categories of complexes bounded below (compare [7, pages 51, 87]).

1. Proof of Theorem 1

We make the following preliminary reductions:

(i) We may assume that M is affine and that there exists an étale morphism $M \xrightarrow[e]{} \mathbb{A}^m_{\mathcal{O}}$ of $\mathcal{O}$-schemes. $\forall 1 \leqslant i \leqslant m$ define $e_i = pr_i \circ e \colon M \to \mathbb{A}^1_{\mathcal{O}}$.

(ii) We may assume that if $r = \mathrm{tr.deg}_k k(\xi)$ then a transcendence basis for $k(\xi)$ over k is given by the first r coordinates of $e(\xi)$.

(iii) We replace $\mathcal{O}$ by the local ring $\mathcal{O}'$ of $\mathbb{A}^r_{\mathcal{O}}$ at the generic point of $\mathbb{A}^r_k$, and replace M over $\mathcal{O}$ by the pull-back to $\mathrm{Spec}(\mathcal{O}')$ of $M \xrightarrow[(e_1,\ldots,e_r)]{} \mathbb{A}^r_{\mathcal{O}}$, and replace (U, Φ) by their pull-backs to $\mathrm{Spec}(\mathcal{O}')$. This does not change the schemes and cohomology groups in the statement of Theorem 1. So we reduce to the case $r = 0$. Then $\xi \in M$ is a closed point, and let $m_\xi \subset \mathcal{O}(M)$ be its defining ideal.

(iv) Reduction to the case where $\mathcal{O}$ is henselian. (Replace M by $M \otimes_{\mathcal{O}} \mathcal{O}^h$, U and Φ by their pull-backs to $\mathrm{Spec}(\mathcal{O}^h)$. Then $\bar{M}$, M^h_ξ etc. will be replaced by isomorphic ones.)

(v) Let L be the separable closure of k in $k(\xi)$, and $\mathcal{O}_L$ the corresponding local ind-étale extension of $\mathcal{O}$. Let $M_L = M \otimes_{\mathcal{O}} \mathcal{O}_L$. There is a canonical isomorphic lifting ξ' of ξ to M_L, obtained from the morphism $\xi \to \mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$. So the étale morphism $M_L \to M$ induces $M^h_{L\xi'} \xrightarrow{\sim} M^h_\xi$. Thus, one reduces Theorem 1 to M_L over $\mathcal{O}_L$ and ξ', i.e. to the case where $k(\xi)$ is purely inseparable over k. Then $k(e(\xi)) \xrightarrow{\sim} k(\xi)$, because $k(\xi)$ is separable over $k(e(\xi))$ by unramifiedness of e.

(vi) By a limit argument using [1, VII Cor. 5.8], to show Theorem 1′ it suffices to show that if $\sigma \in R^q\Gamma(\bar{M}^h_\xi, K)$, $0 \neq \varphi \in \mathcal{O}^h_{\bar{M},\xi}$, and σ vanishes outside $V(\varphi) \subset \bar{M}^h_\xi$, then $\sigma = 0$. Notice, by the excision exact sequence ([1]V (6.5.3), generalized to complexes), that σ comes from an element $\sigma' \in H^q_{V(\varphi)}(\bar{M}^h_\xi, K)$. We may assume that $V(\varphi) \neq \varnothing$, which implies $m \neq 0$.

(vii) *Notations*: Let $f_1, \ldots, f_m \in \mathcal{O}(M)$ be the coordinates of e, and $\forall 0 \leqslant i \leqslant m$ let $\mathbf{e}_i \colon M \to \mathbb{A}^i_{\mathcal{O}}$ be the $\mathcal{O}$-morphism defined by $(f_1, \ldots, f_i)$. Then $\forall 0 < i \leqslant m$, $\mathbf{e}_i^{-1}(\mathbf{e}_i(\xi))$ is the zero subscheme of the "function" $P_i(f_i)$ on $\mathbf{e}_{i-1}^{-1}(\mathbf{e}_{i-1}(\xi))$, where P_i is the irreducible equation of $f_i(\xi)$ over $k(\mathbf{e}_{i-1}(\xi))$.

(viii) We shall modify e by changing successively $\forall i < m$ its i-th coordinate f_i to $f_i + g_i$ with $g_i \in m_\xi^2$ (so that the new e is still étale in a neighborhood of ξ) s.t. the new e will satisfy $\forall 0 \leqslant i < m$ the condition

$$V_i \overset{\mathrm{def}}{=} V(\varphi) \cap \lambda_\xi^{-1}(\mathbf{e}_i^{-1}(\mathbf{e}_i(\xi))) \quad \text{is (purely) } m - 1 - i \text{ dimensional.} \qquad (*_i)$$

This is known for $V_0 = V(\varphi)$. If $0 < i < m$ and $f_1, \ldots, f_{i-1}$ are already corrected, and Σ is the finite set of generic points of V_{i-1}, we choose g_i s.t. $\forall \alpha \in \lambda_\xi(\Sigma)$, $[g_i(\alpha) = 0] \Leftrightarrow [P_i(f_i)(\alpha) \neq 0]$. (Notice $\xi \notin \lambda_\xi(\Sigma)$.)

$(*_{m-1})$ means that $\lambda_\xi^{-1}(e'^{-1}(e'(\xi))) \cap V(\varphi) = \{\xi\}$, where $e' \overset{\mathrm{def}}{=} \mathbf{e}_{m-1}$. Let $B \overset{\mathrm{def}}{=} \mathbb{A}^{m-1}_{\mathcal{O}}$ be the target of e'.

(ix) Then σ will vanish on the generic point Q of $N \overset{\mathrm{def}}{=} \lambda_\xi^{-1}(e'^{-1}(e'(\xi)))$, so to show $\sigma = 0$ it suffices to show that $(R^q\Gamma(\xi, K) \simeq) R^q\Gamma(N_{\text{ét}}, K) \to R^q\Gamma(Q_{\text{ét}}, K)$ is injective, which is just the problem of (vi) for the situation of

$$(\xi' \in M' \xrightarrow[\pi']{} S' \supset \mathbf{U}', \Phi'),$$

where $S' = \operatorname{Spec}(\mathcal{O}^h_{B,e'(\xi)})$, π' is the base change of e' by $S' \to \mathbb{A}^{m-1}_{\mathcal{O}}$, and $(\mathbf{U}', \Phi')$ is the pull-back of $(\mathbf{U}, \Phi)$ to S'. [Cf. reductions (iii), (iv)]. So the problem of (vi) is reduced to the case $m = 1$.

(x) Since $e\colon M \to \mathbb{A}^1_{\mathcal{O}}$ induces $M^h_\xi \xrightarrow{\sim} (\mathbb{A}^1_{\mathcal{O}})^h_{e(\xi)}$, the problem of (vi) is reduced to the case of $M = \mathbb{A}^1_{\mathcal{O}}$. Take $\hat{M} = \mathbb{P}^1_{\mathcal{O}}$. Let Σ be the ∞ section of $\mathbb{P}^1_{\mathcal{O}}$ (so $\Sigma \xrightarrow{\sim} \operatorname{Spec}(\mathcal{O})$), and similarly for $\bar{\Sigma}, \Sigma_U$. Let $\bar{K} = R\hat{j}_*(\pi^*\Phi)$. So K (for $\hat{M}$ over $\mathcal{O}$) is $\hat{\imath}^*\bar{K}$.

$$\begin{tikzcd}
\bar{\Sigma} \arrow[r, "i_\infty"'] \arrow[d] & \Sigma \arrow[r, "j_\infty"'] \arrow[d] & \Sigma_U \arrow[d] & \\
\mathbb{P}^1_k \arrow[r, "\hat{\imath}"'] & \mathbb{P}^1_{\mathcal{O}} \arrow[r, "\hat{j}"'] & \mathbb{P}^1_{\mathbf{U}} \arrow[r, "\pi"'] & \mathbf{U} \\
\mathbb{A}^1_k \arrow[r, "i"'] \arrow[u, hook] & \mathbb{A}^1_{\mathcal{O}} \arrow[r, "j"'] \arrow[u, hook] & \mathbb{A}^1_{\mathbf{U}} \arrow[u, hook] &
\end{tikzcd}$$

We have to show that $H^q_\xi(\bar{M}^h_\xi, K) \to H^q(\bar{M}^h_\xi, K)$ is the zero map. But[(5)] $H^q_\xi(\mathbb{P}^1_k, K) \xrightarrow{\sim} H^q_\xi(\bar{M}^h_\xi, K)$, so it suffices to show that if $\tau \in H^q_\xi(\mathbb{P}^1_k, K)$ then its image $\bar{\tau} \in H^q(\mathbb{P}^1_k, K)$ vanishes on $\mathbb{A}^1_k$. This follows from

LEMMA 1. $\operatorname{Ker}(H^q(\mathbb{P}^1_k, K) \to H^q(\bar{\Sigma}, K)) \subset \operatorname{Ker}(H^q(\mathbb{P}^1_k, K) \to H^q(\mathbb{A}^1_k, K))$.

[In fact, the two kernels are equal].

Proof. Since $\bar{K}$ is a complex with torsion cohomology sheaves, we have by the proper base change theorem ([1], XII Cor. 5.5, extended to complexes as before) $H^q(\mathbb{P}^1_{\mathcal{O}}, \bar{K}) \xrightarrow{\sim} H^q(\mathbb{P}^1_k, K)$. The first group is isomorphic to $H^q(\mathbb{P}^1_{\mathbf{U}}, \pi^*_U\Phi)$ by the definition of $\bar{K}$. Furthermore, if τ is an element of the first kernel (in Lemma 1), then its lifting $\tilde{\tau}$ to $H^q(\mathbb{P}^1_{\mathcal{O}}, \bar{K})$ vanishes on Σ by $H^q(\Sigma, \bar{K}) \xrightarrow{\sim} H^q(\bar{\Sigma}, K)$. Hence $\tau_1 \overset{\text{def}}{=}$ (the restriction of $\tilde{\tau}$ to $\mathbb{P}^1_{\mathbf{U}}$) vanishes on Σ_U. We want to show $\tau|_{\mathbb{A}_k}1 = 0$. For that it suffices to show $\tilde{\tau}|_{\mathbb{A}^1_{\mathcal{O}}} = 0$, and again by $H^q(\mathbb{A}^1_{\mathcal{O}}, \bar{K}) \xrightarrow{\sim} H^q(\mathbb{A}^1_{\mathbf{U}}, \pi^*_U\Phi)$ it suffices to show $\tau_1|_{\mathbb{A}^1_{\mathbf{U}}} = 0$. So, we reduce to the $S = \mathbf{U}$ case of the following

---

(5) Using the relation [1, V6.5] between $\mathscr{H}^q_\xi$ and $R^{q-1}\phi_*$, $\phi\colon \mathbb{P}^1 - \{\xi\} \to \mathbb{P}^1$.

LEMMA 2. *Let S be a scheme, Φ a complex of torsion étale sheaves on S, and $\pi\colon \mathbb{P}^1_S \to S$. Let $S_\infty \subset \mathbb{P}^1_S$ denote the ∞ section. Then*

$$\operatorname{Ker}(H^q(\mathbb{P}^1_S, \pi^*\Phi) \xrightarrow[\lambda]{} H^q(S_\infty, \Phi)) \subset \operatorname{Ker}(H^q(\mathbb{P}^1_S, \pi^*\Phi) \xrightarrow[\mu]{} H^q(\mathbb{A}^1_S, \pi^*\Phi)).$$

[*The hyper-cohomology is taken in the sense of the derived category of torsion sheaves.*]

Proof. If F is a p-torsion étale sheaf(6) on S, we define $F(n)$ for $n \neq 0$ to be the extension by zero of the usual Tate twist [1, XVIII 1.1.1] on $S - V(p)$. If F is any torsion sheaf, we define $F(n) = \oplus_p F_p(n)$, where for every prime p, F_p is the p-torsion component of F.

LEMMA 3. *We have*

$$Q\Phi \oplus Q\Phi(-1)[-2] \xrightarrow[\gamma]{\sim} \mathbb{R}_t \pi_* \pi^* \Phi^{(7)}.$$

This is shown using the proper base change theorem [1, XII, Cor. 5.2], to reduce to the case S = Spec(Ω), Ω an algebraically closed field, and the knowledge of the cohomology of $\mathbb{P}^1_\Omega$ with constant coefficients (cf. [1, IX, 3.5 and Cor. 4.7]). The map γ is defined using $\Phi \to \pi_* \pi^* \Phi$ and a Yoneda extension $(+)\colon 0 \to \pi^*\Phi \to E_1 \to E_2 \to \pi^*\Phi(-1) \to 0$ that can be constructed on $\mathbb{P} \stackrel{\text{def}}{=} \mathbb{P}^1_S$.

γ induces isomorphisms

$$\begin{aligned} H^q(S, \Phi) \oplus H^{q-2}(S, \Phi(-1)) &\xrightarrow{\ \sim\ } H^q(\mathbb{P}_{\text{ét}}, \Phi) \\ a \oplus b \quad &\longmapsto \pi^*(a) + h \cdot \pi^*(b), \end{aligned}$$

where $h = c_1(\mathcal{O}(1)) \in \operatorname{Ext}^2(\hat{\mathbb{Z}}(-1)_{\mathbb{P}}, \hat{\mathbb{Z}}_{\mathbb{P}})$. (The Ext^2 is taken in the exact category $\mathscr{C}_{\mathbb{P}}$ of pro-(étale sheaves) on $\mathbb{P}$ *of the form* "$\varprojlim$" X_n (the index category is $\mathbb{N}_+$ ordered by reverse divisibility) s.t. each X_n is a flat $\mathbb{Z}/(n)$ – Module, and $\forall m | n$ the transition map induces $X_n/mX_n \xrightarrow{\sim} X_m$.)

Then if $x = \pi^*(a) + h \cdot \pi^*(b) \in \operatorname{Ker}(\lambda)$ [see the statement of Lemma 2] we get by the triviality of $\mathcal{O}(1)|_{S_\infty}$ that $h|_{S_\infty} = 0$ so $0 = x|_{S_\infty} = (\pi^*a)|_{S_\infty} = a$. As $\mathcal{O}(1)$ is trivial also on $\mathbb{A}^1_S$, so is $x = h \cdot (\pi^* b)$.

REMARK 4. One can clearly reduce Theorem 1 by Remark 2 and a limit argument to the case when Φ is a bounded complex of $\mathbb{Z}/(N)$ – Modules for some $N > 0$. Also if $\Psi \in D(\mathbb{P}, \mathbb{Z}/(N))$ where for simplicity N is a prime power $p^a (a > 0)$, then the operation $H^q(\mathbb{P}_{\text{ét}}, \Psi(-1)) \xrightarrow[b \mapsto hb]{} H^{q+2}(\mathbb{P}_{\text{ét}}, \Psi)$ in the proof of

(6) In the sense of [1, IX Def. 1.1].

(7) Compare SGA5, VII Cor. 2.2.4.

Lemma 3 is the usual cup-product with

$$c_1(\mathcal{O}(1)) = \delta_{\text{Kummer}}(\text{cl}(\mathcal{O}(1))) \in H^2_{\text{ét}}(\mathbb{P}^1_S - V(p), \mu_N).$$

(Recall (compare SGA $4\frac{1}{2}$ [Cycle] p. 7) that if $\mathscr{V} \xrightarrow[i]{} X$ is an open immersion of schemes (or topoi...) and F, G are complexes of abelian sheaves on $\mathscr{V}_{\text{ét}}$ or $\mathscr{V}_{\text{Zar}}$, there is a cup-product operation

$$\mathbb{H}^p(\mathscr{V}, F) \times \mathbb{H}^q(X, i_! G) \to \mathbb{H}^{p+q}(X, i_!(F \otimes G)).)$$

REMARK 5. Theorem 1 and its proof extend to the case that $\mathbf{U} \to \text{Spec}(\mathcal{O})$ is any coherent morphism.

REMARK 6. Theorem 1 can be slightly generalized (as Bloch also noted) by taking $\xi = \text{Spec}(k(\xi))$ where $k(\xi)$ is a separable algebraic extension of the residue field $k(\xi_1)$ of a point $\xi_1 \in \bar{M}$. To prove this "generalized" form one can adapt the reduction steps (i)–(x), e.g. in reduction (v) one should speak about a canonical lifting $\xi \to M_L$ of $\xi \to M$. After performing reduction (v) one will have $k(e(\xi_1)) \xrightarrow{\sim} k(\xi_1) \xrightarrow{\sim} k(\xi)$. (Alternatively, one can reduce by a limit argument to Theorem 1 as stated, using that M^h_ξ is a filtered projective limit of schemes of the form N^h_η, where (N, η) are certain pointed étale M-schemes.)

REMARK 7. One may consider the injectivity propery for various cases of non smooth morphisms. Suppose for example that f is flat and that the special fibre is a curve with an ordimary double point at ξ. Then the injectivity property holds in the case of rational tangents (one maps to the product of cohomologies at the two η's), but not in the case of conjugate tangents (one η).

2. An application to cohomological purity for the Brauer group

Let R be a regular strictly henselian local ring of dimension $n \geqslant 2$, with maximal ideal m and residue field k. Set $U = \text{Spec}(R) - \{m\}$, $W = \text{Spec}(R)$. In ([6], Section 6) Grothendieck conjectured that

$$H^2_{\text{ét}}(U, \mathbb{G}_m) = 0. \tag{2.1}$$

In (ibid.), (2.1) is proven when $\dim(R) = 2$. In ([4], Ch. I, Theorem 2′) we proved (2.1) when $\dim(R) = 3$. Using Theorem 1 (and results stated below (2.5) in the equicharacteristic case), we shall show

THEOREM 2.2. (2.1) *holds if R is a strict henselization*[8] *of a local ring $\mathcal{O}_{M,\xi}$ (of*

---

[8] i.e., we choose a separable closure $k(\bar{\xi})$ of $k(\xi)$ and let $R = \mathcal{O}^{sh}_{M,\bar{\xi}}$.

dimension $d \geqslant 2$) of a smooth scheme M over a field or a discrete valuation ring Λ. (This was known for Λ a field.)

We first review known facts. U is regular, so by ([2], IV 1.8) $H^2(U_{\text{ét}}, \mathbb{G}_m)$ injects into $H^2(\eta, \mathbb{G}_m)$, (η = the generic point of U), and thus it is torsion. Also $H^1_{\text{ét}}(U, \mathbb{G}_m) = \operatorname{Pic} U = 0$ (by the proof that R is a U.F.D. [EGA IV, 21.11]), so the Kummer sequence gives

$$H^2_{\text{ét}}(U, \mathbb{G}_m)_n \xleftarrow{\sim} H^2(U_{\text{ét}}, \mu_n) \forall n > 0 \text{ prime to } \operatorname{char}(k). \tag{2.3}$$

(Here $A_n \stackrel{\text{def}}{=} \operatorname{Ker}(A \xrightarrow[n]{} A)$, for any abelian group A.)

Now, to show Theorem 2.2 it suffices to show $H^2_{\text{ét}}(U, \mathbb{G}_m)_l = 0$ for every prime l. Suppose Λ is a field.

In the case $l \neq \operatorname{char}(\Lambda) = \operatorname{char}(k)$, one knows the stronger statement

$$H^i_{\text{ét}}(U, \mu_l) = 0 \quad \forall i \neq 0, 2d-1, \tag{*}$$

which is deduced from the absolute cohomological purity theorem ([1], XVI, Th. 3.7, Cor. 3.9). [$H^i(U_{\text{ét}}, \mu_l)$ appears as the stalk at $\bar{\xi}$ of $R^i j_* \mu_l$, where j denotes the inclusion $M - \overline{\{\xi\}} \hookrightarrow M$. To make [1, XVI] applicable, we first replace Λ by its perfect closure Λ_{perf}, M by $M \otimes_\Lambda \Lambda_{\text{perf}}$, and ξ by the unique point above it in the new M. Using [1, VIII 1.2], this replacement "does not change" the group $H^i(U_{\text{ét}}, \mu_l)$; and it makes $Y \stackrel{\text{def}}{=} \overline{\{\xi\}}$ generically smooth over $S \stackrel{\text{def}}{=} \operatorname{Spec}(\Lambda)$.]

The case $d > 2$, $l = \operatorname{char}(k) > 0$ was done by Hoobler [8], using the exact sequence

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \longrightarrow Z^1 \xrightarrow[1-C]{} \Omega^1 \longrightarrow 0 \tag{2.4}$$

on $\operatorname{Spec}(R)_{\text{ét}}$.

Before proving Theorem (2.2), we indicate our results in the equicharacteristic case.

THEOREM 2.5. (2.1) *holds for p-torsion if R is of characteristic $p > 0$.*

Proof (*sketch*). One can show that the Cartier isomorphism

$$\Omega^q \xrightarrow[C^{-1}]{\sim} \mathscr{H}^q_{\text{DR}} = Z^q/B^q$$

holds on any regular $\mathbb{F}_p$-scheme X, and that the sequence (2.4) is exact. (The main point is proving exactness at Z^1.) Also, Z^q and Ω^q can be shown to be flat $\mathcal{O}_{X(p)}$-(resp. $\mathcal{O}_X$-) Modules, and hence for X affine they are filtered direct limits of free $\mathcal{O}_{X(p)}$-(resp. $\mathcal{O}_X$-) modules (D. Lazard). Now, it is known (for any

Cohen-Macaulay local ring) that

$$H^i_{\{m\}}(W_{\text{ét}}, \mathcal{O}_W) = H^i_{\{m\}}(W_{\text{Zar}}, \mathcal{O}_W) = 0 \quad \text{for } i \neq d,$$

so the same vanishing holds for the cohomologies $H^i_{\{m\}}(W_{\text{ét}}, \Omega^q)$ and $H^i_{\{m\}}(W, Z^q)$ in the set-up of Theorem 2.5. So (following [8]) if $d > 2$, the long exact local cohomology sequence associated to

$$0 \longrightarrow \mathbb{G}_m/\mathbb{G}_m^p \longrightarrow Z^1 \longrightarrow \Omega^1 \longrightarrow 0$$

on $W_{\text{ét}}$ gives

$$0 = H^2_{\{m\}}(W, \mathbb{G}_m/\mathbb{G}_m^p) \xleftarrow{\sim} H^1_{\text{ét}}(U, \mathbb{G}_m/\mathbb{G}_m^p) \xrightarrow{\sim} H^2_{\text{ét}}(U, \mathbb{G}_m)_p.$$

2.6. *Proof of Theorem 2.2 for Λ a discrete valuation ring.* Let $K = \text{fract}(\Lambda)$. By the above, it remains to prove (2.1) for l-torsion, $l \neq \text{char}(K)$, and we may assume that $\mathcal{O} = \mathcal{O}^{\text{st}}_{M,\xi}$ where ξ is a point in the closed fibre of $M \to \text{Spec}(\Lambda)$. Let $k_0 =$ the residue field of Λ. Define $Y = U \otimes_\Lambda K$ and $Z = U \otimes_\Lambda k_0$. Let ζ be the generic point of Z, $V = \text{Spec}(\mathcal{O}^h_{U,\zeta})$ the henselization of U at ζ, and $V' = V \otimes_\Lambda K = V \times_U Y$. Consider the commutative diagram of "restriction" maps

$$\begin{tikzcd}
H^2(Z, \mathbb{G}_m) \arrow[d] & H^2(U, \mathbb{G}_m) \arrow[l] \arrow[r, "\beta"'] \arrow[d] & H^2(Y, \mathbb{G}_m) \arrow[r, "\delta"'] \arrow[d, "\gamma"] & H^2(\eta, \mathbb{G}_m) \\
H^2(\zeta, \mathbb{G}_m) & H^2(V, \mathbb{G}_m) \arrow[l, "\alpha"'] \arrow[r] & H^2(V', \mathbb{G}_m) &
\end{tikzcd} \tag{2.7}$$

By [6] the theorem holds when $\dim(R) = 2$. So we may assume $\dim(R) \geqslant 3$. Then $\dim(R \otimes_\Lambda k_0) = \dim(R) - 1 \geqslant 2$, so by the case $\Lambda =$ (a field) we have $H^2(Z, \mathbb{G}_m) = 0$. Also α is bijective by ([6], Theorem 11.7 (2)). Hence the diagram shows $\gamma\beta = 0$.

Recall that $\delta\beta$ is injective, so β is injective, and it remains to show that γ is injective on l-torsion. (In fact, γ is injective.) As Pic $Y =$ Pic $V = 0$, the map γ_l is "isomorphic" to the map

$$H^2(Y, \mu_l) \xrightarrow[pr_2^*]{} H^2(V', \mu_l);$$

pr_2^* is injective by Theorem 1 (and Remark 6) [with $\mathcal{O}$ there taken to be Λ, and

$\mathrm{U} = \mathrm{Spec}(K)]$, as the sheaves μ_l on $Y_{\text{ét}}$ and $V'_{\text{ét}}$ are canonically the pull-backs of $\Phi \stackrel{\text{def}}{=} (\mu_l$ on $\mathrm{Spec}(K)_{\text{ét}})$.

We state additional results concerning (2.1).

THEOREM 2.8. *Let R be a noetherian ring, henselian with respect to an ideal I, $U \subset \mathrm{Spec}(R)$ and open subscheme containing $V(I)^c$, $\hat{R}$ the I-adic completion of R, and $\hat{U}$ = the inverse image of U in* $\mathrm{Spec}(\hat{R})$. *We have $\hat{U} \xrightarrow[\alpha]{} U$. Then*

(i) *α^*: $H^2(U_{\text{ét}}, \mathbb{G}_m) \to H^2(\hat{U}_{\text{ét}}, \mathbb{G}_m)$ is injective, and defines an isomorphism on the torsion subgroups.*

(ii) *$\forall q \geqslant 2$, $\forall n \geqslant 1$, α^*: $H^q(U_{\text{fppf}}, \mu_n) \to H^q(\hat{U}_{\text{fppf}}, \mu_n)$ is bijective.*

COROLLARY. *The truth of* (2.1) *depends only on the completion $\hat{\mathcal{O}}$.*

PROPOSITION 2.9. *Suppose $\mathcal{O}_1 \to \mathcal{O}_2$ is a local homomorphism of noetherian local rings, which is formally smooth for the m-adic topologies (in the sense of EGA 0_{IV}, 19.3). Then there is a direct system B_i of $\mathcal{O}_1$-algebras, which are local rings of smooth $\mathcal{O}_1$-algebras,* s.t. *the transition maps $B_i \to B_j$ are flat and $m_j = m_i B_j$ (where $m_i \subset B_i$ is the maximal ideal) $\forall i \leqslant j$, and* s.t. $\hat{\mathcal{O}}_2 \simeq (\varinjlim B_i)^\wedge$ *as $\hat{\mathcal{O}}_1$-algebras.*

COROLLARY 2.10. *If $\Lambda \to R$ is a formally smooth local homorphism of noetherian local rings, with Λ a field or a D.V.R., and R of dimension $\geqslant 2$, then R satisfies* (2.1). *This applies in particular when R is an unramified regular local ring (of* dim. $\geqslant 2$).

Indeed, Corollary 2.8 allows us to replace R by any R' having the same completion, and by Proposition 2.9 R' can be chosen s.t. (2.1) holds for it by Theorem 2.2 and a standard passage to the limit [1, VII Corollary 5.9].

THEOREM 2.11. *If R is a strictly henselian regular local ring with maximal ideal m, and $N \geqslant 1$ is an integer prime to* $\mathrm{char}(R/m)$, *then*

$$H^q_{\text{ét}}(\mathrm{Spec}(R) - \{m\}, \mathbb{Z}/(N)) = 0 \quad \textit{for every } 0 < q < \dim(R) - 1.$$

In particular, we get that conjecture (2.1) holds in general for torsion prime to the residue characteristic.

3. Appendix

We recall some technical facts which are needed for understanding Lemmas 2, 3 of Section 1 as stated.

A. Let $X \xrightarrow[\pi]{} Y$ be a coherent morphism of locally coherent topoi [1, VI]. Let $\mathbf{F}(X, \mathbb{Z}_t)$ (resp. $\mathbf{F}(X, \mathbb{Z})$) denote the abelian category of torsion (resp. all) abelian sheaves on X and $D(X, \mathbb{Z}_t)$ (resp. $D(X, \mathbb{Z})$) the associated derived category. We have a commutative diagram of categories

$$\begin{tikzcd} \mathbf{F}(X, \mathbb{Z}_t) \arrow[r, "\pi_*"'] \arrow[d, hook] & \mathbf{F}(Y, \mathbb{Z}_t) \arrow[d, hook] \\ \mathbf{F}(X, \mathbb{Z}) \arrow[r, "\pi_*"'] & \mathbf{F}(Y, \mathbb{Z}) \end{tikzcd},$$

and similarly for complexes of sheaves. Taking right derived functors, we obtain

$$\begin{tikzcd} D(X, \mathbb{Z}_t) \arrow[r, "R_t\pi_*"] \arrow[d, hook, "i_X"'] & D(Y, \mathbb{Z}_t) \arrow[d, hook, "i_Y"] \arrow[dl, Rightarrow, "\alpha"] \\ D(X, \mathbb{Z}) \arrow[r, "R\pi_*"'] & D(Y, \mathbb{Z}) \end{tikzcd}, \tag{3.1}$$

where $\alpha: i_Y \circ R_t\pi_* \to R\pi_* \circ i_X$ is a natural transformation.

PROPOSITION 3.2. *If $K \in D(X, \mathbb{Z}_t)$ and either K is bounded below or π_* has finite cohomological dimension on $\mathbf{F}(X, \mathbb{Z})$, then $\alpha(K)$ is an isomorphism.*

PROPOSITION 3.3. *If $X \to Y$ is a proper morphism of schemes and $\pi: \tilde{X}_{\text{ét}} \to \tilde{Y}_{\text{ét}}$ the associated morphism of étale topoi, then for any $K \in D(X, \mathbb{Z}_t)$ there is a convergent spectral sequence*

$$E_2^{pq} = R^p\pi_*(\mathscr{H}^q(K)) \Rightarrow \mathscr{H}^{p+q}(R_t\pi_*K).$$

In particular, if the dimensions of the fibers of f are all $\leqslant d$ and $K \in D^{\leqslant N}$, then $R_t\pi_*K \in D^{\leqslant N+2d}$.

B. A basic Yoneda extension on $\mathbb{P}^1$

Suppose X is a scheme, $F \hookrightarrow_i X$ a closed subscheme, $U \hookrightarrow_j X$ an open subscheme, L a line bundle on X, and $\zeta_1: L_{|U} \xrightarrow{\sim} \mathcal{O}_U$, $\zeta_2: L_{|F} \xrightarrow{\sim} \mathcal{O}_F$ are given trivializations. Assume $F \cap U = \emptyset$. Denote $\Omega = F^c$, $Z = U^c$. We shall construct an extension

$$0 \to \hat{\mathbb{Z}}_\Omega \to K \to L \to \hat{\mathbb{Z}}(-1)_Z \to 0 \tag{3.4}$$

in $\mathscr{C}_X$.

The sequence (3.4) is an inverse system of exact sequences

$$0 \to (\mathbb{Z}/(n))_\Omega \to K_n \to L_n \to (\mathbb{Z}/(n))_Z(-1) \to 0 \qquad (3.5)_n$$

$(n \in \mathbb{N}_{>0})$.

We recall that in general if $0 \to A \to B \to C \to D \to 0$ is an exact sequence in an abelian category A, then we get morphisms of complexes

$$A^{(0)} \xrightarrow{\varepsilon} (B \to C \to D) \leftarrow D^{(2)}, \qquad (3.6)$$

ε a quasi-isomorphism, (here $D^{(2)} = D[-2]$ is D placed in degree 2) which define an arrow $QD[-2] \to QA$ in $D(\mathsf{A})$.

One can show

PROPOSITION 3.7. *Suppose $n > 0$ is invertible on X and $F = \varnothing$. Recall that the isomorphism class of (L, ζ_1) is classified by*

$$\mathrm{cl}(\mathrm{L}, \zeta_1) \in \check{H}^1_Z(X, \mathbb{G}_m) = H^1_Z(X, \mathbb{G}_m)^{(9)}.$$

Then the class c defined by $(3.5)_n$ *and* (3.6),

$$c \in \mathrm{Hom}_{D(X_{\text{ét}}, \mathbb{Z}/n}((\mathbb{Z}/n)_Z(-1)[-2], (\mathbb{Z}/n)_X)$$
$$\simeq \mathrm{Ext}^2_{(\mathbb{Z}/n)_X}((\mathbb{Z}/n)_Z, (\mathbb{Z}/n)_X(1)) = H^2_Z(X, \mu_n),$$

is $-\delta(\mathrm{cl}(L, \zeta_1))$, where δ is the connecting homomorphism $H^1_Z(X, \mathbb{G}_m) \to H^2_Z(X, \mu_n)$ deduced from the Kummer sequence.

The sequence $(+)$ in the proof of Lemma 3 is gotten from (3.4) for $L = \mathcal{O}_{\mathbb{P}}(-1)$, $F = U = \varnothing$, by tensoring with $\pi^*\Phi$, (using the bi-functor $\otimes: \mathscr{C} \times (\text{torsion sheaves}) \to (\text{torsion sheaves})$). The morphism $Q\pi^*\Phi(-1)[-2] \to Q\pi^*\Phi$ in $D(\mathbb{P}, \mathbb{Z}_t)$ associated to $(+)$ is used in the definition of (the second component of) γ.

CONSTRUCTION OF $(3.5)_n$. The sequences $(3.5)_n$ will be constructed together with transition maps $(3.5)_n \to (3.5)_m$ when $m|n$. If n, m are relatively prime integers we should have $(3.5)_{nm} \xrightarrow{\sim} (3.5)_n \times (3.5)_m$, so it suffices to define $(3.5)_n$ (and the transition maps) when n is a power p^a of a fixed prime p. Let $Y \stackrel{\text{def}}{=} X - V(p)$, and denote by α the immersion $Y \hookrightarrow X$. We shall first construct $(3.5)_n$ on Y, and

(9) This cohomology group is the same on X_{Zar}, $X_{\text{ét}}$, X_{fppf}, etc.

product of

$$\begin{array}{ccc} \alpha_!((\mathbb{Z}/(n))_{Y\cap\Omega}) & \longrightarrow & \alpha_!(K_n|_Y) \\ \downarrow & & \\ (\mathbb{Z}/(n))_\Omega & & \end{array}$$

3.8. CONSTRUCTION OF $(3.5)_n$ FOR n INVERTIBLE ON X

If L is a line bundle on a scheme X, one associates to L the $\mathbb{G}_m$-torseur $L^* \overset{\text{def}}{=}$ the sheaf of invertible sections of L (here $\mathbb{G}_m$, L^* and all sheaves below are taken as sheaves on the étale site), and one associates to L^* an extension

$$e(L)\colon 0 \to \mathbb{G}_m \to E(L) \xrightarrow[\pi]{} \mathbb{Z} \to 0$$

equipped with a $\mathbb{G}_m$-isomorphism $\pi^{-1}(1) \xrightarrow{\sim} L^*$.

A trivialization $\zeta\colon L \xrightarrow{\sim} \mathcal{O}$ defines a splitting of $e(L)$, and thus a morphism $\phi_\zeta\colon (0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0) \hookrightarrow e(L)$. In particular, in our situation we get subcomplexes

$$\mathbf{Im}(\phi_{\zeta_1}) \subset e(L)_{|_U} = e(L_{|U})$$

and

$$\mathbf{Im}(\phi_{\zeta_2}) \subset e(L_{|F}).$$

Define a subcomplex $e(L, \zeta_2)$ of $e(L)$ by the cartesian square

$$\begin{array}{ccc} e(L, \zeta_2) & \longrightarrow & i_* \mathbf{Im}(\phi_{\zeta_2}) \\ \big\downarrow & & \big\downarrow \\ e(L) & \longrightarrow & i_* e(L_{|F}) \end{array} .$$

We have $j_! \mathbf{Im}(\phi_{\zeta_1}) \subset e(L, \zeta_2)$, and the quotient complex $e(L, \zeta_2)/j_! \mathbf{Im}(\phi_{\zeta_1})$ is an extension

$$(e^*)\colon 0 \to \tilde{\mathbb{G}}_m \to E^* \to \mathbb{Z}_Z \to 0,$$

where $\tilde{\mathbb{G}}_m \overset{\text{def}}{=} \mathbf{Ker}(\mathbb{G}_{mX} \to i_* \mathbb{G}_{m_F})$.

We have a short exact sequence

$$0 \to (\mu_n)_\Omega \to \tilde{\mathbb{G}}_m \xrightarrow{n} \tilde{\mathbb{G}}_m \to 0. \qquad (3.9)_n$$

(Those form an inverse system w.r.t. divisibility of n's.)

On any topos, the forgetful functor (abelian sheaves) $\rightarrow$ (sheaves of sets) has a left adjoint $\mathbf{F} \mapsto \mathbb{Z}^{(\mathbf{F})}$. There is an adjunction counit $\mathbb{Z}^{(\mathbf{F})} \xrightarrow[\varepsilon(\mathbf{F})]{} \mathbf{F}$ for $\mathbf{F}$ abelian.

From (e^*) and $(3.9)_n$, one obtains by taking fibred products new short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \tilde{\mathbb{G}}_m \arrow[r] & E^* \arrow[r] & \mathbb{Z}_Z \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u] \arrow[ur, phantom, "①"] & \mathbb{Z}^{(E^*)} \arrow[r] \arrow[u] & \mathbb{Z}_Z \arrow[r] \arrow[u, "id"'] & 0 \end{tikzcd}, \qquad \begin{matrix} (e^*) \\ (3.10) \end{matrix}$$

$$\begin{tikzcd} 0 \arrow[r] & (\mu_n)_\Omega \arrow[r] & \tilde{\mathbb{G}}_m \arrow[r, "n"] & \tilde{\mathbb{G}}_m \arrow[r] & 0 \\ 0 \arrow[r] & (\mu_n)_\Omega \arrow[r] \arrow[u, "id"'] & B \arrow[r] \arrow[u] \arrow[ur, phantom, "②"] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}, \qquad \begin{matrix} (3.9)_n \\ (3.11)_n \end{matrix}$$

in which ① and ② are cartesian by definition.

In other words, we replaced by a canonical procedure the Yoneda extension $0 \rightarrow (\mu_n)_\Omega \rightarrow \tilde{\mathbb{G}}_m \rightarrow E^* \rightarrow \mathbb{Z}_Z \rightarrow 0$ coming from (e^*) and $(3.9)_n$ by a Yoneda extension

$$0 \rightarrow (\mu_n)_\Omega \rightarrow B \rightarrow \mathbb{Z}^{(E^*)} \rightarrow \mathbb{Z}_Z \rightarrow 0, \qquad (3.12)_n$$

(defined by splicing (3.10) and $(3.11)_n$) whose third term is $\mathbb{Z}$-flat.

The complexes $(3.12)_n$ form an inverse system w.r.t. divisibility of n's. The sheaves $\mathbb{Z}_Z$ and A are torsion free, i.e. $\mathbb{Z}$-flat, so the sequences $(\mathbb{Z}/(n)) \otimes_{\mathbb{Z}} (3.10)$ and $(\mathbb{Z}/(n)) \otimes_{\mathbb{Z}} (3.11)_n$ are exact, and thus

$$(\mathbb{Z}/(n)) \otimes_{\mathbb{Z}} (3.12)_n = (0 \rightarrow (\mu_n)_\Omega \rightarrow B/nB \rightarrow (\mathbb{Z}/(n))^{(E^*)} \rightarrow (\mathbb{Z}/(n))_Z \rightarrow 0) \qquad (3.13)_n$$

is also exact.

The sequence $(3.5)_n$ is defined to be the Tate twist $(3.13)_n(-1)$.

REMARK 3.14. Suppose S is a site, $\mathcal{O}_1 \rightarrow \mathcal{O}_2$ a morphism of sheaves of rings on S, M a flat $\mathcal{O}_1$-Module, N an $\mathcal{O}_2$-Module. The "trivial duality theorem" [1, XVII 4.1.4] for the morphisms of ringed sites $(S, \mathcal{O}_2) \rightarrow (S, \mathcal{O}_1)$ and the fact that $\mathcal{O}_2 \otimes^{\mathbb{L}}_{\mathcal{O}_1} M \xrightarrow{\sim} Q(\mathcal{O}_2 \otimes_{\mathcal{O}_1} M)$ give an isomorphism

$$\mathrm{Ext}^n_{\mathcal{O}_1}(M, N) \xrightarrow[\zeta]{\sim} \mathrm{Ext}^n_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} M, N) \quad \forall n \in \mathbb{N}.$$

The above procedure of defining (3.13), slightly generalized, is in fact a way to construct ζ on the level of Yoneda extensions. (Note that the construction of ζ^{-1} is easier.)

References

[1] M. Artin, A. Grothendieck, and J. L. Verdier: Théorie des Topos et Cohomologie Étale des Schémas, *Lecture Notes in Math.* 269, 270, 305, Springer-Verlag (1972–73).

[2] M. Artin: *Grothendieck Topologies.* Harvard University (1962).

[3] S. Bloch and K. Kato: p-adic Étale Cohomology, *Publ. Math. IHES 63* (1986), 107–152.

[4] O. Gabber: Some Theorems on Azumaya Algebras. In: *Lecture Notes in Math.* 844, Springer Verlag (1981).

[5] A. Grothendieck: Eléments de Géométrie Algébrique III (première partie), *Publ. Math. IHES* 11 (1961); EGA IV, *Publ. Math. IHES* 20, 24, 28, 32 (1964–67).

[6] A. Grothendieck: Le Groupe de Brauer III. In: *Dix Exposés sur la Cohomologie des Schémas.* North Holland Pub. Co. (1968).

[7] R. Hartshorne: Residues and Duality, *Lecture Notes in Math.* 20, Springer-Verlag (1966).

[8] R. Hoobler: A Cohomological Interpretation of Brauer Groups of Rings, *Pacific Journal of Math.* 86 (1980) 89–92.

[9] K. Kato, Swan conductors for characters of degree one in the imperfect residue field case, *Contemporary Math.* Vol. 83 (1989), 101–131.