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Remark to a problem on 0-1 matrices

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Recently in [8] V. de Valk made a detailed analysis of the maximal and minimal values of the sum of elements of the matrix M^2 , where M is a $n \times n$ matrix with all elements from the set $\{0, 1\}$. For the sum of the elements of some matrix we use the symbol $\| \cdot \|$. Our purpose is to prove the inequality

$$n\|M\| - \|M^2\| \leq \frac{n^3 - n}{3} \quad (1)$$

by means of elementary tools. The inequality (1) can be deduced from Theorem 2 in [8], but the proof of this more precise theorem is rather complicated. As the inequality (1) is interesting by itself, it makes sense to have a short direct proof for it separately.

We can use the following problem as an example to the use of the inequality (1). Let G be some complete directed graph with n vertices and let every edge (including the loops in vertices) be painted in one of two colours, A and B . We are asking for the exact upper bound of the total number of double-coloured pairs of subsequent edges (paths of length two) in G .

It is sufficient to estimate one half of this quantity, namely the number of the pairs coloured in the sequence (A, B) . We introduce the incidence matrix $M = (M_{ij})$ of the edges painted in A ($M_{ij} = 1$ iff the oriented edge from the i th to the j th vertex is painted in A , $M_{ij} = 0$ otherwise) and we denote the row, column and total sums of elements by

$$H_i := \sum_{j=1}^n M_{ij}, \quad V_j := \sum_{i=1}^n M_{ij}, \quad K := \|M\| := \sum_{i=1}^n \sum_{j=1}^n M_{ij}. \quad (2)$$

Let the number of A -coloured edges, which terminate in the i th vertex ($i = 1, \dots, n$), be a_i and let b_i be the number of B -coloured edges, which start in this vertex. The number under consideration is then equal to $\sum_{i=1}^n a_i b_i$. Obviously $a_i = V_i$ and, as G is a complete graph, $b_i = n - H_i$.

Therefore

$$\sum_{i=1}^n a_i b_i = nK - \sum_{i=1}^n H_i V_i. \quad (3)$$

Further

$$\begin{aligned} \sum_i H_i V_i &= \sum_i \left(\sum_j M_{ij} \right) \left(\sum_k M_{ki} \right) \\ &= \sum_j \sum_k \sum_i M_{ki} M_{ij} \end{aligned}$$

and the last expression is equal to the sum of all elements of the matrix M^2 . Therefore $\sum_i a_i b_i$ is equal to the expression on the left side of inequality (1).

THEOREM. *Let M be an arbitrary $n \times n$ matrix ($n \in \mathbb{N}$) with elements M_{ij} from the set $\{0, 1\}$ and let H_i, V_i ($i = 1, \dots, n$) and K be defined by (2). Then*

$$\sum_{i=1}^n (H_i - V_i)^2 \leq nK - \sum_{i=1}^n H_i V_i \leq \frac{n^3 - n}{3}. \quad (4)$$

Proof. The first inequality follows immediately from a result by F. Matúš ([5], [6]), namely from the Khintchine-type inequality

$$\sum_{i=1}^n H_i^2 + \sum_{i=1}^n V_i^2 \leq nK + \sum_{i=1}^n H_i V_i.$$

In order to prove the second inequality, which is equivalent to (1), we proceed by induction. For $n = 1$ the result is trivial. If $n > 1$, we suppose the validity of (1) for $n - 1$. From the matrix M we remove the n th row and n th column and we denote the remainder by \bar{M} . By analogy with (2) we introduce for $(n - 1) \times (n - 1)$ matrix \bar{M} the corresponding values \bar{H}_i, \bar{V}_i and \bar{K} . One obtains

$$\begin{aligned} nK - \sum_{i=1}^n H_i V_i &= \sum_{i=1}^n H_i (n - V_i) \\ &= \sum_{i=1}^{n-1} (\bar{H}_i + M_{in})(n - 1 - \bar{V}_i + 1 - M_{ni}) + H_n (n - V_n) \\ &= \sum_{i=1}^{n-1} \bar{H}_i (n - 1 - \bar{V}_i) + \sum_{i=1}^{n-1} \bar{H}_i (1 - M_{ni}) \\ &\quad + \sum_{i=1}^{n-1} M_{in} (n - \bar{V}_i - M_{ni}) + H_n (n - V_n). \end{aligned}$$

The first sum in the last expression is equal to $(n-1)\bar{K} - \sum_{i=1}^{n-1} \bar{H}_i \bar{V}_i$ and it can be estimated using the induction hypothesis. The remaining terms will be rewritten as

$$R_n = \sum_{i=1}^{n-1} \bar{H}_i (1 - M_{ni}) + \sum_{i=1}^{n-1} M_{in} (n - \bar{V}_i - M_{ni}) \\ + \left(\sum_{i=1}^{n-1} M_{ni} + M_{nn} \right) \left(n - \sum_{j=1}^{n-1} M_{jn} - M_{nn} \right).$$

We introduce the subsets I and J of subscripts $i \in \{1, \dots, n-1\}$ such that

$$i \in I \Leftrightarrow M_{in} = 1, \quad i \in J \Leftrightarrow M_{ni} = 1.$$

Let $k := \text{card } I$ and $l := \text{card } J$, then apparently $0 \leq k \leq n-1$ and $0 \leq l \leq n-1$. Using the introduced symbols we can rearrange the above expression as follows

$$R_n = \sum_{i \notin J} \bar{H}_i + \sum_{i \in I} (n - \bar{V}_i) - \sum_{i \in I \cap J} M_{in} M_{ni} + (l + M_{nn})(n - k - M_{nn}).$$

Now we estimate the difference $\sum_{i \notin J} \bar{H}_i - \sum_{i \in I} \bar{V}_i$, which is not greater than $\sum_{i \notin J} \bar{H}'_i$, where \bar{H}'_i are the sums of rows of the matrix, which we obtain from \bar{M} turning out the k columns with subscripts in I . We have

$$\sum_{i \notin J} \bar{H}'_i \leq (n-1-l)(n-1-k).$$

Further we make use of the obvious identity

$$\sum_{i \in I \cap J} M_{in} M_{ni} = \text{card}(I \cap J).$$

Therefore we have

$$R_N \leq (n-1-l)(n-1-k) + kn - \text{card}(I \cap J) + l(n-k) \\ + M_{nn}(n-k-l-M_{nn}) \\ = (n-1)^2 + k + l - \text{card}(I \cap J) + M_{nn}(n-k-l-M_{nn}).$$

Using the well known fact, that

$$\text{card } I + \text{card } J - \text{card}(I \cap J) = \text{card}(I \cup J) \leq n-1,$$

we obtain the inequality

$$R_n \leq n(n-1),$$

which is obvious if $M_{nn} = 0$ and for $M_{nn} = 1$ it follows from

$$R_n \leq (n-1)^2 - \text{card}(I \cap J) + n - 1 \leq n(n-1).$$

Therefore finally

$$nK - \sum_{i=1}^n H_i V_i \leq \frac{(n-1)^3 - (n-1)}{3} + n(n-1) = \frac{n^3 - n}{3}. \quad \square$$

REMARK. The inequality (1) generally can not be improved, as for arbitrary n there exist matrices, for which the equality holds. It is easy to verify, that the matrix with the elements $M_{ij} = 1$ for $i \leq j$ and $M_{ij} = 0$ for $i > j$ is such an example. The matrices for which in (4) instead of the first inequality equality holds can be specified using the results in [6].

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