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Operators on eigenmaps between spheres

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Abstract. An operator on eigenmaps between spheres is a rule that associates to a λ_p-eigenmap a λ_q-eigenmap in a natural manner. It is given by a homomorphism of an orthogonal $SO(m+1)$-module W into the tensor product $\mathscr{H}^p \otimes \mathscr{H}^q$ of spherical harmonics. Using Young tableaux, we give here an explicit description of these operators for W irreducible, in terms of the eigenmaps they operate on. Examples include the degree raising and lowering operators, infinitesimal rotations of eigenmaps and symmetrization.

1. Introduction

A spherical harmonic on S^m of order p is an eigenfunction of the (spherical) Laplacian Δ^{S^m} with eigenvalue $\lambda_p = p(p+m-1)$, or equivalently, (the restriction to S^m of) a homogeneous harmonic polynomial in $m+1$ variables. Their space is denoted by $\mathscr{H}^p = \mathscr{H}^p_{m+1}$. A map $f: S^m \to S_V$ into the unit sphere of a Euclidean vector space V is said to be a λ_p*-eigenmap* if all components of f belong to $\mathscr{H}^p$, i.e., for $\mu \in V^*$, we have $\mu \circ f \in \mathscr{H}^p$. For $p = 2$, many examples of eigenmaps are known, in fact, for $m = 3$, a full classification of the eigenmaps is given in [3]. For fixed m, the complexity of the space of all λ_p-eigenmaps increases very fast with p. It is therefore of importance to define and study operations that manufacture new eigenmaps from old ones and this is the aim of the present paper. The general construction to be given in Section 3 is motivated by the following pair of simple examples: Given a λ_p-eigenmap $f: S^m \to S^n$, let $f^{\pm}: \mathbf{R}^{m+1} \to \mathbf{R}^{m+1} \otimes \mathbf{R}^{n+1}$ be the (polynomial) map given in coordinates $i = 0, \ldots, m$ and $j = 0, \ldots, n$ by

$$(f^+)^j_i = c_p^+ H(x_i f^j) \quad \text{and} \quad (f^-)^j_i = c_p^- \frac{\partial f^j}{\partial x_i}, \tag{1}$$

where

$$c_p^+ = \sqrt{\frac{2p+m-1}{p+m-1}} \quad \text{and} \quad c_p^- = \frac{1}{\sqrt{p(2p+m-1)}}. \tag{2}$$

In these formulas we think of the components of f as harmonic homogeneous polynomials of degree p defined on $\mathbf{R}^{m+1}$. H is the harmonic projection operator (cf. Vilenkin [6]), in fact, we have

$$H(x_i f^j) = x_i f^j - \frac{\rho^2}{2p+m-1}\frac{\partial f^j}{\partial x_i}, \quad \rho^2 = x_0^2 + \cdots + x_m^2.$$

The assumption that f maps the unit sphere into the unit sphere translates into the condition

$$\sum_{j=0}^{m} (f^j)^2 = \rho^{2p}.$$

Taking the (Euclidean) Laplacian of both sides it follows easily that $f^{\pm}$ map the unit sphere into the unit sphere so that we obtain $\lambda_{p\pm1}$-eigenmaps $f^{\pm}: S^m \to S^{(m+1)(n+1)-1}$. The operator $D^{\pm}$ that associates to f the maps $f^{\pm}$ is a pair of basic examples of operators on eigenmaps between spheres. Passing from f to $f^{\pm}$ involves the extension of the range $\mathbf{R}^{n+1}$ by taking its tensor product with $\mathbf{R}^{m+1}$. We think of the latter as $\mathscr{H}^1$ and consider $D^{\pm}$ as $SO(m+1)$-module homomorphisms of $\mathscr{H}^1$ into $(\mathscr{H}^p)^* \otimes \mathscr{H}^{p\pm1}$ given by

$$D_e^+ h = \mu_p \sum_{i=0}^{m} c_i H(x_i h) \quad \text{and} \quad D_e^- h = \mu_{p-1}^{-1} \sum_{i=0}^{m} c_i \frac{\partial h}{\partial x_i}, \; h \in \mathscr{H}^p, \tag{3}$$

where we put the argument

$$e = \sum_{i=0}^{m} c_i y_i \in \mathscr{H}^1$$

of $D^{\pm}$ as a subscript and

$$\mu_p = (p+1)\frac{2p+m-1}{p+m-1}. \tag{4}$$

Comparing (1) an (3), it is now easy to describe how

$$D^{\pm} \in \hom_{SO(m+1)}(\mathscr{H}^1, (\mathscr{H}^p)^* \otimes \mathscr{H}^{p\pm1})$$

operate on a λ_p-eigenmap f to give $f^{\pm}$ (cf. also the general formulas in Section 3). One of the main results of this paper is that this situation generalizes substantially; namely, *any* nonzero $SO(m+1)$-module homomorphism D of an orthogonal $SO(m+1)$-module into $(\mathscr{H}^p)^* \otimes \mathscr{H}^q$ gives rise to an operator carrying λ_p-eigenmaps into λ_q-eigenmaps. After a review of eigenmaps and moduli spaces in Section 2, we prove this in Section 3. If W is irreducible then it is a component of $(\mathscr{H}^p)^* \otimes \mathscr{H}^q = \mathscr{H}^p \otimes \mathscr{H}^q$. The decomposition of this tensor product is known from the work of DoCarmo-Wallach [1] [7] so that, in Section 4, we give an explicit description of those operators that come from the irreducible components of the tensor product in terms of the Young tableaux. Finally, in Section 5, we give a variety of examples corresponding to the simplest Young tableaux. In particular, taking $W = SO(m+1)$ with the adjoint representation of $SO(m+1)$ induces the operator that associates to a λ_p-eigenmap its infinitesimal rotation that is again a λ_p-eigenmap.

2. Eigenmaps and their moduli spaces

The standard minimal immersion $f_{\lambda_p}: S^m \to S_{\mathscr{H}^p}$ defined by

$$f_{\lambda_p}(x) = \sum_{j=0}^{n(\lambda_p)} f_{\lambda_p}^j(x) f_{\lambda_p}^j,$$

with $\{f_{\lambda_p}^j\}_{j=o}^{n(\lambda_p)} \subset \mathscr{H}^p$ an orthonormal basis, is a λ_p-eigenmap. Here orthonormality is with respect to the normalized L_2-scalar product

$$\langle h_1, h_2 \rangle_p = \frac{n(\lambda_p) + 1}{\mathrm{vol}(S^m)} \int_{S^m} h_1 h_2 v_{S^m}, \tag{5}$$

where v_{S^m} is the volume form on S^m, $\mathrm{vol}(S^m) = \int_{S^m} v_{S^m}$ is the volume of S^m and

$$n(\lambda_p) + 1 = \dim \mathscr{H}^p = (2p + m - 1)\frac{(p + m - 1)!}{(p+1)!(m-1)!}.$$

f_{λ_p} does not depend on the choice of the orthonormal basis.

f_{λ_p} is universal in the sense that, for any λ_p-eigenmap $f: S^m \to S_V$, there exists a linear map $A: \mathscr{H}^p \to V$ such that $f = A \circ f_{\lambda_p}$.

A λ_p-eigenmap $f: S^m \to S_V$ is *full* if the image of f spans V. In general, restricting to span $(\mathrm{im}\, f) \cap S_V$, f gives rise to a full λ_p-eigenmap that we will denote by the same symbol. Two λ_p-eigenmaps $f_1: S^m \to S_{V_1}$ and $f_2: S^m \to S_{V_2}$ are *equivalent* if there exists an isometry $U: V_1 \to V_2$ such that $f_2 = U \circ f_1$.

We now associate to f the symmetric linear endomorphism

$$\langle f \rangle_{\lambda_p} = A^{\mathrm{T}} A - I \in S^2(\mathscr{H}^p), \qquad (I = \text{identity}).$$

This establishes a parametrization of the space of equivalence classes of full λ_p-eigenmaps $f: S^m \to S_V$ by the compact convex body

$$\mathscr{L}_{\lambda_p} = \{C \in \mathscr{E}_{\lambda_p} | C + I \geqslant 0\}$$

in the subspace

$$\mathscr{E}_{\lambda_p} = \operatorname{span}\{\operatorname{proj}[f_{\lambda_p}(x)] | x \in S^m\}^\perp \subset S^2(\mathscr{H}^p).$$

Here '$\geqslant$' stands for positive semidefinite, 'proj' is orthogonal projection onto the line spanned by the argument, and the orthogonal complement is with respect to the standard scalar product

$$\langle C, C' \rangle = \operatorname{trace} C'^{\mathrm{T}} \cdot C, \quad C, C' \in S^2(\mathscr{H}^p).$$

We call $\mathscr{L}_{\lambda_p}$ the (standard) *moduli space* of λ_p-eigenmaps. (For more details as well as for the general theory of moduli spaces, cf. [4].) The fundamental problem of 'classifying' all λ_p-eigenmaps raised in [2] as a fundamental problem in harmonic map theory is thereby equivalent to describing $\mathscr{L}_{\lambda_p}$.

By definition, the standard minimal immersion f_{λ_p} is equivariant with respect to the homomorphism $\rho_{\lambda_p}: SO(m+1) \to SO(\mathscr{H}^p)$ that is nothing but the orthogonal $SO(m+1)$-module structure on $\mathscr{H}^p$ defined by $a \cdot h = h \circ a^{-1}$, $a \in SO(m+1)$ and $h \in \mathscr{H}^p$. Equivariance means that

$$f_{\lambda_p} \circ a = \rho_{\lambda_p}(a) \circ f_{\lambda_p}, \quad a \in SO(m+1).$$

With respect to the extended module structure, $\mathscr{E}_{\lambda_p}$ is a submodule of $S^2(\mathscr{H}^p)$. Clearly, $\mathscr{L}_{\lambda_p} \subset \mathscr{E}_{\lambda_p}$ is an invariant subset, in fact, for a full λ_p-eigenmap f: $S^m \to S_V$, we have

$$a \cdot \langle f \rangle_{\lambda_p} = \langle f \circ a^{-1} \rangle_{\lambda_p}, \quad a \in SO(m+1).$$

DoCarmo-Wallach [1], [7] gave the decomposition of $\mathscr{H}^p \otimes \mathscr{H}^q$, $p \geqslant q$, into irreducible components. We have (after complexification):

$$\mathscr{H}^p \otimes \mathscr{H}^q \cong \sum_{(a,b) \in \Delta^{p,q}; a+b \equiv p+q \pmod 2} V_{m+1}^{(a,b,0,\ldots,0)}. \tag{6}$$

Here $\Delta^{p,q} \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(p-q, 0)$, $(p+q, 0)$ and (p, q) and $V_{m+1}^{(a_1,\ldots,a_l)}$, $l = [|(m+1)/2|]$, is the (complex) irreducible $SO(m+1)$-module with highest weight vector $(a_1, \ldots, a_l)$ (relative to the standard maximal torus in $SO(m+1)$). (If $m = 3$, $V_{m+1}^{(a,b,0,\ldots,0)}$ means $V_4^{(a,b)} \oplus V_4^{(a,-b)}$.)

In a similar vein, for the symmetric square, we have ($m \geqslant 3$):

$$S^2(\mathscr{H}^p) \cong \sum_{(a,b)\in\Delta^{p,p};\, a,b \text{ even}} V_{m+1}^{(a,b,0,\ldots,0)}. \tag{7}$$

Moreover, $\mathscr{E}_{\lambda_p}$ is nontrivial iff $m \geqslant 3$ and $p \geqslant 2$. In this case,

$$\mathscr{E}_{\lambda_p}^{\perp} = \operatorname{span}\{\operatorname{proj}[f_{\lambda_p}(x)] \mid x \in S^m\}$$

is the sum of all class 1 submodules of $S^2(\mathscr{H}^p)$ with respect to the pair $(SO(m+1), SO(m))$. Since the class 1 submodules of $(SO(m+1), SO(m))$ are exactly the spherical harmonics, they correspond to $b = 0$ in (7). Hence the decomposition of $\mathscr{E}_{\lambda_p}$ is obtained by restricting the summation above to the subtriangle $\Delta^p \subset \Delta^{p,p}$ whose vertices are $(2,2)$, $(2p-2,2)$ and (p,p).

3. Operators on eigenmaps

Let W be an orthogonal $SO(m+1)$-module (i.e. a representation space for the orthogonal group $SO(m+1)$ with invariant scalar product). Let

$$D: W \to (\mathscr{H}^p)^* \otimes \mathscr{H}^q,$$

be a homomorphism of $SO(m+1)$-modules. (Using the scalar product (5), $(\mathscr{H}^p)^* = \mathscr{H}^p$ but, at this initial stage, we keep a space and its dual separate.) For each $e \in W$, $D_e: \mathscr{H}^p \to \mathscr{H}^q$ is a linear map. Equivariance of D means that, for $a \in SO(m+1)$ and $e \in W$, we have

$$D_{a\cdot e} = a \cdot D_e = \rho_{\lambda_q}(a) \circ D_e \circ \rho_{\lambda_p}(a^{-1}).$$

We also write $D \in \hom_{SO(m+1)}(W, (\mathscr{H}^p)^* \otimes \mathscr{H}^q)$.

Dually, D can be thought of as an $SO(m+1)$-module homomorphism

$$\iota = \iota^D: \mathscr{H}^q \to W \otimes \mathscr{H}^p.$$

D and ι are connected by the formulas

$$\iota^{\mathrm{T}}(e \otimes h) = D_e h, \quad h \in \mathscr{H}^p$$

and

$$\iota(h') = \sum_{i=0}^{m} e_i \otimes D_{e_i}^{\mathrm{T}} h', \quad h' \in \mathscr{H}^q,$$

where $\{e_i\}_{i=0}^m \subset W$ is an orthonormal basis. Note that ι is either zero or injective so that, in the latter case, $\mathscr{H}^q$ is a submodule of $W \otimes \mathscr{H}^p$. Moreover,

$$\iota^{\mathsf{T}} \circ \iota = c(D) \cdot I, \quad c(D) \in \mathbf{R}. \tag{8}$$

This follows by induction with respect to m using the branching $\mathscr{H}^q_{m+1}|_{SO(m)} = \mathscr{H}^0_m \oplus \cdots \oplus \mathscr{H}^q_m$.

Composition of such operators are defined naturally. In fact, given another

$$D' \in \hom_{SO(m+1)}(W', (\mathscr{H}^q)^* \otimes \mathscr{H}^r),$$

we define the composition

$$D' \circ D \in \hom_{SO(m+1)}(W' \otimes W, (\mathscr{H}^p)^* \otimes \mathscr{H}^r)$$

by setting $(D' \circ D)_{e' \otimes e} = D'_{e'} \circ D_e$, $e \in W$ and $e' \in W'$. Dually, we have

$$\iota_{D' \circ D} = (I_{W'} \otimes \iota_D) \circ \iota_{D'}.$$

The composition is clearly associative with identity element

$$I \in \hom_{SO(m+1)}(\mathscr{H}^0, (\mathscr{H}^p)^* \otimes \mathscr{H}^p).$$

The transpose $D^{\mathsf{T}} \in \hom_{SO(m+1)}(W, (\mathscr{H}^q)^* \otimes \mathscr{H}^p)$ is defined in the usual manner by setting $(D^{\mathsf{T}})_e = (D_e)^{\mathsf{T}}$, $e \in W$.

We now describe how a nonzero D acts on eigenmaps. Let $f: \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ be any polynomial map with components in $\mathscr{H}^p$. We define

$$f^D: \mathbf{R}^{m+1} \to W \otimes \mathbf{R}^{n+1}$$

to be the map with components

$$(f^D)^j_i = c(D)^{-1/2} D_{e_i} f^j,$$

where $\{e_i\}_{i=0}^m \subset W$ is an orthonormal basis. The main result of this section is contained in the following:

THEOREM 1. *Let D be a nonzero module homomorphism of an orthogonal $SO(m+1)$-module W into $(\mathscr{H}^p)^* \otimes \mathscr{H}^q$. Given a λ_p-eigenmap $f: S^m \to S^n$, f^D maps the unit sphere into the unit sphere, i.e. $f^D: S^m \to S_{W \otimes \mathbf{R}^{n+1}}$ is a λ_q-eigenmap.*

In what follows, we call a nonzero $D \in \hom_{SO(m+1)}((\mathscr{H}^p)^* \otimes \mathscr{H}^q)$ *an operator on eigenmaps.*

We precede the proof of the theorem by two lemmas. Let D be an operator on eigenmaps as above and define

$$f^D_{\lambda_p}\colon \mathbf{R}^{m+1} \to W \otimes \mathscr{H}^p$$

by

$$f^D_{\lambda_p}(x) = c(D)^{-1/2} \sum_{i=0}^{m} \sum_{j=0}^{n(\lambda_p)} (D_{e_i} f^j_{\lambda_p})(x) \cdot e_i \otimes f^j_{\lambda_p},$$

where $\{e_i\}_{i=0}^m \subset W$ is an orthonormal basis.

LEMMA 1. *We have*

$$\iota(f_{\lambda_q}(x)) = c(D)^{1/2} f^D_{\lambda_p}(x), \quad x \in S^m.$$

In particular, $f^D_{\lambda_p}$ maps the unit sphere into the unit sphere, i.e. it gives rise to a λ_q-eigenmap $f^D_{\lambda_p}\colon S^m \to S_{W \otimes \mathscr{H}^p}$ which, when made full, is equivalent to f_{λ_q}.

Proof. We compute

$$\begin{aligned}
\iota(f_{\lambda_q}(x)) &= \sum_{l=0}^{n(\lambda_q)} f^l_{\lambda_q}(x)\iota(f^l_{\lambda_q}) \\
&= \sum_{l=0}^{n(\lambda_q)} f^l_{\lambda_q}(x) \sum_{i=0}^{m} e_i \otimes D^{\mathrm{T}}_{e_i} f^l_{\lambda_q} \\
&= \sum_{i=0}^{m} \sum_{l=0}^{n(\lambda_q)} \sum_{j=0}^{n(\lambda_p)} f^l_{\lambda_q}(x) \langle D^{\mathrm{T}}_{e_i} f^l_{\lambda_q}, f^j_{\lambda_p} \rangle e_i \otimes f^j_{\lambda_p} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n(\lambda_q)} f^l_{\lambda_q}(x) \langle f^l_{\lambda_q}, D_{e_i} f^j_{\lambda_p} \rangle e_i \otimes f^j_{\lambda_p} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n(\lambda_p)} (D_{e_i} f^j_{\lambda_p})(x) e_i \otimes f^j_{\lambda_p} \\
&= c(D)^{1/2} f^D_{\lambda_p}(x).
\end{aligned}$$

Finally, the last statement follows from (8).

Next, given D as above, we define

$$\Phi = \Phi^D\colon S^2(\mathscr{H}^p) \to S^2(\mathscr{H}^q)$$

by

$$\Phi(C) = c(D)^{-1} \iota^{\mathrm{T}} \circ (I_W \otimes C) \circ \iota.$$

Clearly, Φ is a homomorphism of $SO(m+1)$-modules.

REMARK. Easy computation shows that we have

$$\Phi^{D'\circ D} = \Phi^{D'} \circ \Phi^{D} \tag{9}$$

Finally, given any map $f\colon \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$ with components in $\mathscr{H}^p$, we introduce the function

$$\phi(f) = \sum_{i=0}^{m} \sum_{j=0}^{n} ((f^D)_i^j)^2.$$

Clearly, $\phi(f)$ is a homogeneous polynomial of degree $2q$.

LEMMA 2. *Let $A\colon \mathscr{H}^p \to \mathbf{R}^{n+1}$ be a linear map and set $C = A^{\mathrm{T}}A - I \in S^2(\mathscr{H}^p)$. Then, for $f = Af_{\lambda_p}\colon \mathbf{R}^{m+1} \to \mathbf{R}^{n+1}$, we have*

$$\langle \Phi(C), \operatorname{proj}[f_{\lambda_q}(x)] \rangle = \phi(f)(x) - \phi(f_{\lambda_p})(x).$$

Proof. Using matrix coefficients with respect to the orthonormal bases, we have

$$\begin{aligned}
\langle \Phi(C), \operatorname{proj}[f_{\lambda_q}(x)] \rangle &= \langle \Phi(C) f_{\lambda_q}(x), f_{\lambda_q}(x) \rangle \\
&= c(D)^{-1} \langle (I \otimes C)\iota(f_{\lambda_q}(x)), \iota(f_{\lambda_q}(x)) \rangle \\
&= \langle (I \otimes C) f_{\lambda_p}^D(x), f_{\lambda_p}^D(x) \rangle \\
&= c(D)^{-1} \sum_{i=0}^{m} \sum_{j,j'=0}^{n(\lambda_p)} c_{jj'} (D_{e_i} f_{\lambda_p}^j)(x)(D_{e_i} f_{\lambda_p}^{j'})(x) \\
&= c(D)^{-1} \sum_{i=0}^{m} \sum_{l=0}^{n} (D_{e_i} f^l)(x)^2 - c(D)^{-1} \sum_{i=0}^{m} \sum_{j=0}^{n(\lambda_p)} (D_{e_i} f_{\lambda_p}^j)(x)^2 \\
&= \phi(f)(x) - \phi(f_{\lambda_p})(x).
\end{aligned}$$

Proof of Theorem 1. By Lemma 1 and (8), we have $\phi(f_{\lambda_p})(x) = |f_{\lambda_q}(x)|^2 = \rho^{2q}$, $\rho^2 = x_0^2 + \cdots + x_m^2$, where the latter equality is because f_{λ_q} maps the unit sphere into the unit sphere. According to Section 1, $\mathscr{E}_{\lambda_p}$ is the sum of those irreducible submodules of $S^2(\mathscr{H}^p)$ that are not class 1 with respect to $(SO(m+1), SO(m))$. Since $\Phi\colon S^2(\mathscr{H}^p) \to S^2(\mathscr{H}^q)$ is a homomorphism of modules, it follows that $\Phi(\mathscr{E}_{\lambda_p}) \subset \mathscr{E}_{\lambda_q}$. In particular, if $C \in \mathscr{E}_{\lambda_p}$ then $\Phi(C)$ is orthogonal to $\operatorname{proj}[f_{\lambda_q}(x)]$ for all $x \in S^m$. For $C = A^{\mathrm{T}}A - I$, by Lemma 2, we obtain $\phi(f) = \phi(f_{\lambda_p}) = \rho^{2q}$ and we are done.

Equivalence is preserved under D so that it is natural to consider the map of

$\mathscr{L}_{\lambda_p}$ into $\mathscr{L}_{\lambda_q}$ which, for any λ_p-eigenmap $f: S^m \to S^n$, associates to $\langle f \rangle$ the element $\langle f^D \rangle$. In the next result we show that this map is nothing but Φ.

PROPOSITION 1. *For any λ_p-eigenmap $f: S^m \to S^n$, we have*

$$\Phi(\langle f \rangle) = \langle f^D \rangle,$$

in particular, $\Phi(\mathscr{L}_{\lambda_p}) \subset \mathscr{L}_{\lambda_q}$.

Proof. As usual, let $f = Af_{\lambda_p}$ with $A: \mathscr{H}^p \to \mathbf{R}^{n+1}$ linear. Then, we have

$$f^D = (I \otimes A) f^D_{\lambda_p} = c(D)^{-1/2}(I \otimes A)\iota(f_{\lambda_q})$$

so that

$$\langle f^D \rangle = c(D)^{-1}\iota^{\mathrm{T}}(I \otimes A)^{\mathrm{T}}(I \otimes A)\iota - I = c(D)^{-1}\iota^{\mathrm{T}}(I \otimes (A^{\mathrm{T}}A - I))\iota = \Phi(\langle f \rangle).$$

REMARK. f^D has at least as many symmetries as f. More precisely, define the symmetry group of f as

$$SO(m+1)_f = \{a \in SO(m+1) \mid \text{there exists } A \in SO(n+1) \text{ such that } f \circ a = A \circ f\}.$$

Since this is just the isotropy at $\langle f \rangle$, we obtain

$$SO(m+1)_f \subset SO(m+1)_{f^D}.$$

EXAMPLE 1. We now return to the two operators

$$D^{\pm} \in \hom_{SO(m+1)}(\mathscr{H}^1, (\mathscr{H}^p)^* \otimes \mathscr{H}^{p\pm 1})$$

mentioned in the introduction. (Note that they also occur in [5] as a technical tool to establish an equivariant imbedding of $\mathscr{L}_{\lambda_p}$ into $\mathscr{L}_{\lambda_{p+1}}$).

Setting $W = \mathscr{H}^1$ we first define D^+ by its dual $\iota^+: \mathscr{H}^{p+1} \to \mathscr{H}^1 \otimes \mathscr{H}^p$ by

$$\iota^+(h') = \sum_{i=0}^{m} y_i \otimes \frac{\partial h'}{\partial x_i}, \quad h' \in \mathscr{H}^{p+1},$$

where $\{y_i\}_{i=0}^m \in \mathscr{H}^1$ is the standard orthonormal basis in the space of linear polynomials in the variables $y_0, \ldots, y_m$. (In what follows, we write $\iota^{D^+} = \iota^+$, etc.) Equivarince of ι^+ is easily checked. The transpose of ι^+ is easily computed using

$$\left\langle h, \frac{\partial h'}{\partial x_i} \right\rangle = \mu_p \langle H(x_i h), h' \rangle, \quad h \in \mathscr{H}^p,\ h' \in \mathscr{H}^{p+1}, \tag{10}$$

where μ_p is given in (4). We obtain that

$$(\iota^+)^{\mathrm{T}}(y_i \otimes h) = \mu_p H(x_i h), \quad h \in \mathscr{H}^p,$$

so that, in (8), we have

$$c(D^+) = \left(\frac{\mu_p}{c_p^+}\right)^2,$$

where c_p^+ is given in (2). Formula (3) for D^+ then follows.

One of the main results of [5] (cf. Theorem 3) is that $\Phi^+ : S^2(\mathscr{H}^p) \to S^2(\mathscr{H}^{p+1})$ is injective.

In a similar vein, we define $D^- \in \hom_{SO(m+1)}(\mathscr{H}^1, (\mathscr{H}^p)^* \otimes \mathscr{H}^{p-1})$ by its dual

$$\iota^-(h'') = \sum_{i=0}^{m} y_i \otimes H(x_i h''), \quad h'' \in \mathscr{H}^{p-1}.$$

We then have

$$(\iota^-)^{\mathrm{T}}(y_i \otimes h) = \frac{1}{\mu_{p-1}} \frac{\partial h}{\partial x_i}, \quad h \in \mathscr{H}^p,$$

and so

$$c(D^-) = \frac{1}{(\mu_{p-1} c_p^-)^2}$$

with the correct formula (3) for D^-. Finally, using (10), we also get $(D^-)^{\mathrm{T}} = \mu_{p-1}^{-1} D^+$.

EXAMPLE 2. Taking the q-th power of D^+, we obtain the operator

$$(D^+)^q \in \hom_{SO(m+1)}(\mathscr{P}^q[y_0, \ldots, y_m], (\mathscr{H}^p)^* \otimes \mathscr{H}^{p+q}),$$

where $\mathscr{P}^q[y_0, \ldots, y_m] = \mathscr{H}^1 \otimes \cdots \otimes \mathscr{H}^1$($q$ times) denotes the space of homogeneous polynomials of degree q in the variables $y_0, \ldots, y_m$. Explicitly, we have

$$\begin{aligned}(D^+)^q_{y_{i_1}, \ldots, y_{i_q}} h &= \mu_p \mu_{p+1} \cdots \mu_{p+q-1} H(x_{i_1} H(x_{i_2} \cdots H(x_{i_q} h) \cdots)) \\ &= \mu_p \mu_{p+1} \cdots \mu_{p+q-1} H(H(x_{i_1} \cdots x_{i_q}) h), \quad h \in \mathscr{H}^p.\end{aligned}$$

By the very definition of the harmonic projection H it is zero on

$\rho^2\mathcal{P}^{q-2}[y_0,\ldots,y_m]$, where $\rho^2 = y_0^2 + \cdots + y_m^2$. Hence, using the decomposition

$$\mathcal{P}^q[y_0,\ldots,y_m] = \mathcal{H}^q \otimes \rho^2\mathcal{P}^{q-2},$$

it follows that (up to a constant multiple) $(D^+)^q|\mathcal{H}^q$ is given by

$$D^{+,q} \in \hom_{SO(m+1)}(\mathcal{H}^q, (\mathcal{H}^p)^* \otimes \mathcal{H}^{p+q}),$$

where

$$D^{+,q}_{f^j_{\lambda_q}} h = H(f^j_{\lambda_q} h), \quad h \in \mathcal{H}^p.$$

Note that, by (9), the induced module homomorphism

$$\Phi^{+,q}: S^2(\mathcal{H}^p) \to S^2(\mathcal{H}^{p+q})$$

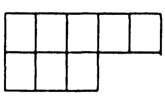
is injective.

A similar description applies to $(D^-)^q$.

4. Operators in terms of the Young tableau

The purpose of this section is to give an explicit description of the operator $D^{a,b}$ that corresponds to the component $V^{(a,b,0,\ldots,0)}$ in $(\mathcal{H}^p)^* \otimes \mathcal{H}^q = \mathcal{H}^p \otimes \mathcal{H}^q$ (cf. (6)). Since the representations that occur here are absolutely irreducible, we work over **C** and adjust the notation accordingly. (Note that $D^{a,b}$ is unique up to a constant multiple.)

Let $p \geqslant q$. According to the proof of (6) by DoCarmo and Wallach [1] $\mathcal{H}^q \otimes \mathcal{H}^p/\mathcal{H}^{q-1} \otimes \mathcal{H}^{p-1}$ has a multiplicity 1 decomposition and each component is given by a Young tableau $\Sigma_{a,b}$:



with row lengths a and b, where $a \geqslant b \geqslant 0$, $b \leqslant q$ and $a + b = p + q$. We now briefly describe how the component $V^{(a,b,0,\ldots,0)}_{m+1}$ is determined by $\Sigma_{a,b}$. First let $\mathcal{P}^p[x_0,\ldots,x_m]$ and $\mathcal{P}^q[y_0,\ldots,y_m]$ be the space of homogeneous polynomials of degree p and q in the variables $x_0,\ldots,x_m$ and $y_0,\ldots,y_m$, respectively. We realize

$$\mathcal{P}^{p,q} = \mathcal{P}^q[y_0,\ldots,y_m] \otimes \mathcal{P}^p[x_0,\ldots,y_m]$$

as an $SO(m+1)$-submodule of the Weyl space $\otimes^{p+q}\mathbf{C}^{m+1}$ of tensors of rank

$p + q$. Let $\mathscr{P}_0^{p,q} \subset \mathscr{P}^{p,q}$ be the traceless part, and denote by $T^{p,q} \subset \mathscr{P}_0^{p,q}$ be the submodule consisting of those tensors which have the property that contraction with respect to any two indices is zero. This means that

$$T^{p,q} = \left\{ g \in \mathscr{H}^q \otimes \mathscr{H}^p \mid \sum_{i=0}^{m} \partial^2 g/\partial x_i \partial y_i = 0 \right\}.$$

Let $\varepsilon(\Sigma_{a,b})$ be the Young symmetrizer, i.e. if $R(\Sigma_{a,b})$ and $C(\Sigma_{a,b})$ denote those permutations in the symmetric group $\mathscr{S}_{a+b} = \mathscr{S}_{p+q}$ that leave the rows and columns invariant, respectively, then

$$\varepsilon(\Sigma_{a,b}) = \sum_{r \in R(\Sigma_{a,b}), s \in C(\Sigma_{a,b})} \operatorname{sgn}(s) sr,$$

and it is an element of the group ring $\mathbf{Z}_{\mathscr{S}_{p+q}}$. Now $\varepsilon(\Sigma_{a,b}) T^{p,q} \neq 0$ and is (a multiple of) $V_{m+1}^{(a,b,0,\ldots,0)}$.

We now define

$$\Psi : \mathscr{P}^{p,q} \to (\mathscr{H}^p)^* \otimes \mathscr{H}^q$$

as follows: If $e \in \mathscr{P}^{p,q}$ then $\Psi(e)$: $\mathscr{H}^p \to \mathscr{H}^q$ is given by

$$\Psi(e) = e(\delta_0, \ldots, \delta_m; \partial_0, \ldots, \partial_m),$$

where

$$e = e(y_0, \ldots, y_m; x_0, \ldots, x_m)$$

and, for $i = 0, \ldots, m$, we put $\partial_i = \partial/\partial x_i$ and $\delta_i = H(x_i \cdot)$. (Note that $[\partial_i, \partial_k] = [\delta_i, \delta_k] = 0$ so that this definition makes sense.) Observe also that Ψ is a homomorphism of $SO(m+1)$-modules, where the module structure on the space of polynomials is given by precomposition with the inverse.

PROPOSITION 2. *Ψ is surjective.*

Proof. Let L: $\mathscr{H}^p \to \mathscr{H}^q$ be linear and assume that L is orthogonal to the image of Ψ. Fix $0 \leqslant i_1, \ldots, i_p; k_1, \ldots, k_q \leqslant m$. Setting $e = y_{k_1} \cdots y_{k_q} \otimes x_{i_1} \cdots x_{i_p}$, we have

$$\langle \Psi(e), L \rangle = \sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n(\lambda_q)} \langle \Psi(e) f_{\lambda_p}^j, f_{\lambda_q}^l \rangle \langle L f_{\lambda_p}^j, f_{\lambda_q}^l \rangle = 0.$$

Since the transpose of δ_i is, up to a constant multiple, ∂_i, we obtain

$$\sum_{j=0}^{n(\lambda_p)} \sum_{l=0}^{n(\lambda_q)} \langle L f_{\lambda_p}^j, f_{\lambda_q}^l \rangle \frac{\partial^p f_{\lambda_p}^j}{\partial x_{i_1} \cdots \partial x_{i_p}} \frac{\partial^q f_{\lambda_q}^l}{\partial x_{k_1} \cdots \partial x_{k_q}} = 0.$$

By (10), $\partial^p f^j_{\lambda_p}/\partial x_{i_1} \cdots \partial x_{i_p}$ is a constant multiple of $\langle f^j_{\lambda_p}, H(x_{i_1} \cdots x_{i_p})\rangle$ and so we arrive at

$$\langle L(H(x_{i_1} \cdots x_{i_p})), \quad H(x_{k_1} \cdots x_{k_q})\rangle = 0.$$

Finally, the $H(x_{i_1} \cdots x_{i_p})$'s span $\mathscr{H}^p$ so that $L = 0$ follows.

We now use the decomposition

$$\mathscr{P}^q[y_0, \ldots, y_m] = \mathscr{H}^q \oplus \rho^2 \mathscr{P}^{q-2}[y_0, \ldots, y_m], \quad \rho^2 = y_0^2 + \cdots y_m^2.$$

As the operators in the image of Ψ act on harmonic polynomials, $\rho^2 \mathscr{P}^{q-2}[y_0, \ldots, y_m] \oplus \mathscr{P}^p[x_0, \ldots, x_m]$ is in the kernel of Ψ. The same holds when the variables x and y are interchanged. Factoring out with these, we obtain that

$$\Psi: \mathscr{H}^q \otimes \mathscr{H}^p \to (\mathscr{H}^p)^* \otimes \mathscr{H}^q$$

is an isomorphism of $SO(m+1)$-modules.

Finally we define

$$D^{a,b} = \Psi|_{V_{m+1}^{(a,b,0,\ldots,0)}} \in \hom_{SO(m+1)}(V_{m+1}^{(a,b,0,\ldots,0)}, (\mathscr{H}^p)^* \otimes \mathscr{H}^q).$$

It follows that, for each $e \in V_{m+1}^{(a,b,0,\ldots,0)}$, D_e is a polynomial in the operators δ_i and ∂_i, $i = 0, \ldots, m$, homogeneous of degree q in the δ_i's and degree p in ∂_i's.

Now let $a + b < p + q$ and $a + b \equiv p + q \pmod 2$ so that $V_{m+1}^{(a,b,0,\ldots,0)}$ is a component of $\mathscr{H}^p \otimes \mathscr{H}^q$. Setting $p + q - a - b = 2t$, we define $D^{a,b}$, on $V_{m+1}^{(a,b,0,\ldots,0)} \subset \mathscr{H}^p \otimes \mathscr{H}^q$, by

$$D_e^{a,b} = \sum_{i_1, \ldots, i_t = 0}^{m} \delta_{i_1} \cdots \delta_{i_t} D_e^{a,b} \partial_{i_1} \cdots \partial_{i_t}, \tag{11}$$

where, on the right hand side, $D_e^{a,b}$ acts on $V_{m+1}^{(a,b,0,\ldots,0)}$ as a component of $\mathscr{H}^a \otimes \mathscr{H}^b$. By the proof of the decomposition theorem for the tensor product [1] (cf. also [4]) $D^{a,b}$ is nonzero and hence injective on $V_{m+1}^{(a,b,0,\ldots,0)}$. (In fact, using the notation of [1], DoCarmo and Wallach showed that the differential operator $D: \mathscr{H}^{p+1} \otimes \mathscr{H}^{q+1} \to \mathscr{H}^p \otimes \mathscr{H}^q$ defined by

$$D(h' \otimes h'') = \sum_{i=0}^{m} \partial h'/\partial y_i \otimes \partial h''/\partial x_i, \; h' \in \mathscr{H}^{p+1}, \; h'' \in \mathscr{H}^{q+1}$$

is surjective (cf. also [4], pp. 102–106). Computation shows that, under the isomorphisms $(\mathscr{H}^{p+1})^* = \mathscr{H}^{p+1}$ and $(\mathscr{H}^p)^* = \mathscr{H}^p$, up to a constant multiple, D corresponds to the map

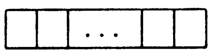
$$(\mathscr{H}^{p+1})^* \otimes \mathscr{H}^{q+1} \to (\mathscr{H}^p)^* \otimes \mathscr{H}^q,$$

given by $C \mapsto \Sigma_{i=0}^m \partial_i \circ C \circ \delta_i$, and this is just the transpose of the right hand side of (11) for $t = 1$. Now the statement follows by induction with respect to t.)

5. Examples

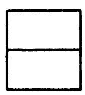
EXAMPLE 1. Let $p \geqslant 1, q = p - 1$ and $a = 1, b = 0$. It is clear that the Young tableau consisting of a single box corresponds to the operator D^- discussed in Example 1 of the previous section.

EXAMPLE 2. Let $p, q \geqslant 1$ and $a = q, b = 0$. It is equally clear that the Young tableau



consisting of a single row of length q corresponds to $D^{-,q}$ treated in Example 2.

EXAMPLE 3. Let $p = q = 1$ and $a = b = 1$ so that the Young tableau $\Sigma_{1,1}$ is



The Young symmetrizer in $\mathbf{Z}_{\mathscr{S}_2}$ is $\varepsilon(\Sigma_{1,1}) = (1)(2) - (12)$. We write an element of

$$\mathscr{P}^1[y_0, \ldots, y_m] \otimes \mathscr{P}^1[x_0, \ldots, x_m]$$

as $\Sigma_{i,k=0}^m c_{ik} y_k \otimes x_i$. This belongs to $T^{1,1}$ iff $\Sigma_{i=0}^m c_{ii} = 0$. Applying the Young symmetrizer we obtain

$$e = \sum_{i,k}^m c_{ik}(y_k \otimes x_i - y_i \otimes x_k)$$

as a typical element of $\varepsilon(\Sigma_{1,1})T^{1,1}$. According to our construction, the corresponding operator is

$$D_e^{1,0} = \sum_{i,k=0}^m c_{ik}(\delta_k \circ \partial_i - \delta_i \circ \partial_k).$$

Hence $D_e^{1,0}$ acts on $h \in \mathcal{H}^1$ as

$$D_e^{1,0}h = \sum_{i,k=0}^{m} c_{ik}\left(H\left(x_k \frac{\partial h}{\partial x_i}\right) - H\left(x_i \frac{\partial h}{\partial x_k}\right)\right)$$

$$= \sum_{i,k=0}^{m} c_{ik}\left(x_k \frac{\partial h}{\partial x_i} - x_i \frac{\partial h}{\partial x_k}\right)$$

$$= \sum_{i,k=0}^{m} c_{ik} A_{ik} h,$$

where $A_{ik} = x_k \partial/\partial x_i - x_i \partial/\partial x_k$ is the operator of infinitesimal rotation on the $x_i x_k$-plane. We now let $p = q \geqslant 1$ (keeping $a = b = 1$). We claim that the operator $D_e^{1,0}$ acting on $\mathcal{H}^p$ given by (11) is

$$D_e^{1,0} = (p-1)! \sum_{i,k=0}^{m} c_{ik} A_{ik}.$$

The shortest proof is by induction with respect to p. For the general step $p \Rightarrow p+1$, we have, for $h \in \mathcal{H}^{p+1}$,

$$D_e^{1,0}h = \sum_{r=0}^{m} \delta_r D_e^{1,0} \partial_r h$$

$$= (p-1)! \sum_{r=0}^{m} \sum_{i,k=0}^{m} c_{ik} H\left(x_r A_{ik} \frac{\partial h}{\partial x_r}\right)$$

$$= p! \sum_{i,k=0}^{m} c_{ik} A_{ik} h.$$

The $SO(m+1)$-module corresponding to the Young tableau $\Sigma_{1,1}$ is clearly $SO(m+1)$ with the adjoint representation. Given a λ_p-eigenmap $f: S^m \to S^n$, the associated λ_p-eigenmap $f^D: S^m \to S_{\mathbf{R}^{n+1} \otimes SO(m+1)}$ has coordinates that are obtained from that of f by rotating them (infinitesimally) on each coordinate plane in $\mathbf{R}^{m+1}$. We call f^D the eigenmap of infinitesimal rotations of f.

EXAMPLE 4. Let $p = q = 1$, $a = 2$ and $b = 0$ so that the Young tableau $\Sigma_{2,0}$ is

The Young symmetrizer is $\varepsilon(\Sigma_{2,0}) = (1)(2) + (12)$. Using the notation of Example 3, we have

$$e = \sum_{i,k=0}^{m} c_{ik}(y_k \otimes x_i + y_i \otimes x_k)$$

with corresponding operator

$$D_e^{2,0} = \sum_{i,k=0}^{m} c_{i,k}(\delta_k \circ \partial_i + \delta_i \partial_k).$$

$D_e^{2,0}$ acts on $h \in \mathscr{H}^1$ as

$$D_e^{2,0} h = \sum_{i,k=0}^{m} c_{ik} H\left(x_k \frac{\partial h}{\partial x_i} + x_i \frac{\partial h}{\partial x_k}\right).$$

For $D_e^{2,0}$ acting on $h \in \mathscr{H}^p$, we get

$$D_e h = (p-1)! \sum_{i,k=0}^{m} c_{ik} H\left(x_k \frac{\partial h}{\partial x_i} + x_i \frac{\partial h}{\partial x_k}\right).$$

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