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Compositio Mathematica, tome 89, no 1 (1993), p. 81-90

<http://www.numdam.org/item?id=CM_1993__89_1_81_0>
Curves of genus ten on K3 surfaces

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Received 16 July 1992; accepted in final form 11 September 1992

Introduction

Let C denote a smooth complete algebraic curve and L a line bundle on C. There is a natural map, called the Wahl or Gaussian map,

\[ \Phi_L : \wedge^2 H^0(C, L) \to H^0(C, \Omega_C^1 \otimes L^{\otimes 2}) \]

which sends \( s \wedge t \) to \( s dt - t ds \). J. Wahl made the striking observation that if C is embeddable in a K3 surface then \( \Phi_L \) is not onto for \( L = \Omega_C^1 \) (\cite{W}, Thm. 5.9); this raises the natural problem of studying the stratification of the moduli space of curves \( \mathcal{M}_g \) by the rank of the Wahl map \( \Phi(C) = \Phi_{\Omega_C^1} \). Roughly speaking, our main theorem says that the closure of the locus of curves of genus 10 which lie on a K3 is equal to the locus where \( \Phi(C) \) fails to be surjective.

In order to state the theorem precisely and explain what is special about the case of genus 10, we need to introduce some spaces. Let \( \mathcal{F}_g \) be the moduli space of K3 surfaces with a polarization of genus \( g \), \( \mathcal{P}_g \) the union, over all \( S \in \mathcal{F}_g \), of the linear series \( |O_S(1)| \). Let \( \mathcal{H} \) be the closure of the image of the natural rational map \( \mu : \mathcal{P}_g \to \mathcal{M}_g \). As the dimension of \( \mathcal{P}_g \) is \( 19 + g \) and the dimension of \( \mathcal{M}_g \) is \( 3g - 3 \), one might naively expect \( \mu \) to be dominant for \( g \leq 10 \) and finite onto its image for \( g \geq 11 \). These expectations hold for \( g \leq 9 \) (\cite{M}, Thm. 6.1) and for odd \( g \geq 11 \) and even \( g \geq 20 \) (\cite{M-M}, Thm. 1), but for \( g = 10 \), Mukai showed that \( \mu \) is not dominant (\cite{M}, Thm. 0.7). This exceptional behavior is due to the fact that the general K3 surface of genus 10 is a codimension 3 plane section of a certain 5-fold, so that when a curve lies on a general K3, it in fact lies on a 3-dimensional family of them. One of our first tasks is to show that \( \mathcal{H} \) is a divisor when \( g = 10 \).

Over the open subset \( \mathcal{M}^\circ_{10} \) of \( \mathcal{M}_{10} \) of curves without automorphisms we have the relative Wahl map; let \( \mathcal{W}^\circ \) denote its degeneracy locus and \( \mathcal{W} \) the closure of \( \mathcal{W}^\circ \) in \( \mathcal{M}_{10} \). It is a theorem of Ciliberto-Harris-Miranda \( \cite{C-H-M} \) that \( \mathcal{W} \) is a...
divisor (i.e. the Wahl map does not degenerate everywhere), and by Wahl's theorem \( K \leq \mathcal{W} \). Our result can then be stated as follows.

**THEOREM.** We have an equality of divisors

\[ \mathcal{W} = 4K. \]

Moreover, for the general curve \( C \) of genus 10 which can be embedded in a K3 surface, the codimension of the image of the Wahl map \( \Phi(C) \) is 4.

It is worth remarking that a priori not every curve of genus 10 on a K3 appears in \( \mathcal{H} \): the variety \( \mathcal{P} \) consists of pairs \((S, C)\) where \( \mathcal{O}_S(C) \) is indivisible in \( \text{Pic}(S) \). But by Wahl's theorem, every curve on a K3 has a degenerate Wahl map, so by the theorem defines a point of \( \mathcal{H} \). It would be interesting to see explicitly a family of curves polarizing K3s of genus 10 degenerating, for instance, to a plane sextic (which polarizes a K3 of genus 2).

We also note that Voisin proved ([V] Prop. 3.3) that the corank of \( \Phi(C) \) is at most 3 for a genus 10 curve satisfying certain hypothesis (3.1)(i), (ii) and (iii) (loc. cit.). These hypotheses hold for a general curve, and (i) holds for a general curve on a K3. It follows that either (ii) or (iii) fails for the general curve of genus 10 on a K3; as Voisin pointed out to us, a dimension counting argument suggests that it is (iii) which fails generically.

To prove the theorem we first study the cohomology of a certain 5-fold \( X \), which is a homogeneous space for the exceptional Lie group \( G_2 \), using a theorem of Bott as in [M]. This allows us to show, in Section 2, that \( \mathcal{H} \) is a divisor and that for every \( C \) which is a smooth codimension 4 plane section of \( X \), the corank of \( \Phi(C) \) is 4. This establishes the inequality of divisors \( \mathcal{W} \geq 4\mathcal{H} \). In Section 3, we compute the classes of the divisors \( \mathcal{W} \) and \( \mathcal{H} \) and find that \( \mathcal{W} \) is linearly equivalent to \( 4\mathcal{H} \). The desired equality of divisors then follows.

**1. The cohomology of the 5-fold \( X \)**

One of the main tools in our analysis will be the cohomology groups of a certain homogeneous variety \( X \) used by Mukai [M] to study the moduli space of K3 surfaces of genus 10. To recall the definition, let \( g \) be the complex semisimple Lie algebra attached to the exceptional root system \( G_2 \), let \( G \) be the corresponding simply connected Lie group, and let \( \rho : G \to \text{Aut}(g) \) be the adjoint representation. If \( v \in g \) is a lowest weight vector for \( \rho \), then \( X = \rho(G)v \) is the orbit of \( v \). Equivalently, if \( P \subseteq G \) is the maximal parabolic subgroup of \( G \) associated to the longer of the two roots in a system of simple roots for \( g \), then \( X \cong G/P \). The homogeneous variety \( X \) has dimension 5 and is naturally embedded in \( \mathbb{P}(g) \) as a subvariety of degree 18; its canonical bundle is isomorphic to \( \mathcal{O}(-3) \) ([M],...
Mukai shows that the general K3 surface of genus 10 is a codimension 3 plane section of $X$ and any abstract isomorphism between two such K3s is realized by the action of $G$ on the Grassmannian of codimension 3 planes in $\mathbf{P}(\mathfrak{g})$ ([M], Thm. 0.2).

Recall that homogeneous vector bundles on $X$ are in one to one correspondence with finite dimensional linear representations of $P$. For example, if $\{\alpha_1, \alpha_2\}$ is a basis for the root system $G_2$ with $\alpha_1$ the shorter root, so that $P$ is the subgroup corresponding to the subalgebra whose roots are all of the negative roots together with $\alpha_1$, then the tangent bundle to $X = G/P$ corresponds to the (reducible) representation of $P$ with highest weight $w_1 = 3\alpha_1 + 2\alpha_2$. It has an irreducible rank 4 subbundle corresponding to the representation of $P$ with highest weight $\alpha_2 + 3\alpha_1$ and the quotient is isomorphic to $\mathcal{O}_X(1)$, corresponding to the irreducible representation of $P$ with highest weight $w_1$. Similarly $N_X$, the normal bundle of $X$ in $\mathbf{P}(\mathfrak{g})$, has a composition series with quotients of rank 1, 3 and 4 corresponding to irreducible representations with highest weights 0, $4\alpha_1 + 2\alpha_2$, and $6\alpha_1 + 3\alpha_2$ respectively.

Now a theorem of Bott ([B]; see also [M], 1.6) asserts that when $E$ is an irreducible homogeneous vector bundle on a compact homogeneous variety $X = G/P$, at most one of the cohomology groups $H^i(X, E)$ is non-zero, and when non-zero, the group is an irreducible $G$-module. Moreover, he gives a recipe for calculating the index of the non-vanishing cohomology group. Application of this result to the $X$ considered above, which we leave as a pleasant exercise for the reader (compare [M], Section 1), yields the following result.

**LEMMA 1.1**

1. We have $h^0(X, T_X(-1)) = 0$ and $H^0(X, T_X) \cong \mathfrak{g}$ as a $G$-module. Moreover, $h^i(X, T_X(-i)) = h^i(X, T_X(-i-1)) = 0$ for $i = 1, 2, 3, 4$.
2. We have $H^0(X, N_X(-1)) \cong \mathfrak{g}$ as a $G$-module and $h^i(X, N_X(-i-1)) = 0$ for $i = 1, \ldots, 4$. Also, $h^i(X, N_X(-i-2)) = 0$ for $i = 0, \ldots, 4$.

Now suppose that $S$ is a smooth codimension 3 plane section of $X$ and that $C$ is a smooth hyperplane section of $S$; then $S$ is a K3 surface and $C$ is a canonically embedded curve of genus 10. Using Koszul resolutions of $\mathcal{O}_S$ and $\mathcal{O}_C$ as $\mathcal{O}_X$-modules, one easily checks the following assertions.

**LEMMA 1.2**

1. $h^0(S, N_S(-1)) = 14$.
2. $h^0(C, T_X(-1)|_C) = 0$ and $h^0(C, T_X|_C) = 14$.
3. $h^0(C, N_C(-2)) = 0$ and $h^0(C, N_C(-1)) = 14$.

(Here $N_C$ and $N_S$ are the normal bundles to $C$ and $S$ in the projective spaces they span in $\mathbf{P}(\mathfrak{g})$; the last part also uses the standard isomorphism $N_X|_C \cong N_C$.)
2. The corank of the Wahl map

We retain the notations of the introduction.

**Proposition 2.1.** Suppose $S$ is a general K3 surface of genus 10. Then $h^1(S, T_S(-1)) = 3$ and $h^2(S, T_S(-1)) = 1$.

*Proof.* Consider the exact sequence

$$0 \rightarrow T_S(-1) \rightarrow T_P(-1)|_S \rightarrow N_S(-1) \rightarrow 0$$

where $S \subseteq P = P^{10}$ is the given embedding. The long exact sequence of cohomology yields

$$0 \rightarrow H^0(S, T_P(-1)|_S) \rightarrow H^0(S, N_S(-1)) \rightarrow H^1(S, T_S(-1)) \rightarrow H^1(S, T_P(-1)|_S).$$

But the Euler sequence for $T_P|_S$ implies that $h^0(T_P(-1)|_S) = 11$ and $h^1(T_P(-1)|_S) = 0$. Indeed, we have

$$0 \rightarrow H^0(S, \mathcal{O}_S)^{11} \rightarrow H^0(S, T_P(-1)|_S) \rightarrow H^1(S, \mathcal{O}_S(-1))$$

$$\rightarrow H^1(S, T_P(-1)|_S) \rightarrow H^2(S, \mathcal{O}_S(-1)) \rightarrow H^2(S, \mathcal{O}_S)^{11}$$

with $H^1(S, \mathcal{O}_S) = 0$ ($S$ is a K3) and $H^1(S, \mathcal{O}_S(-1)) = 0$ ([K], Thm. 2.5); moreover, the map $H^2(S, \mathcal{O}_S(-1)) \rightarrow H^2(S, \mathcal{O}_S)^{11}$ is injective by duality and the projective normality of $S$ ([Ma], Prop. 2). By Lemma 1.2, $h^0(S, N_S(-1)) = 14$, so $h^1(S, T_S(-1)) = 3$. As $h^0(S, T_S(-1)) = 0$, Riemann-Roch implies $h^2(S, T_S(-1)) = 1$.

**Proposition 2.2.** The locus $\mathcal{X} \subseteq \mathcal{M}_{10}$ is a divisor.

*Proof.* First we need some deformation theory. Generally, given a smooth complete curve $C$ in a smooth complete surface $S$, we have the tangent sheaf $T_S$ of $S$, the tangent sheaf $T_C$ of $C$ and the restriction $T_S|_{C} = T_S \otimes \mathcal{O}_C$. Extending the latter two sheaves by 0 on $S$, we can define a coherent sheaf $F$ on $S$ as the fiber product

$$F \rightarrow T_C$$

$$\downarrow \quad \downarrow$$

$$T_S \rightarrow T_C|_S.$$ 

The sheaf $F$ is locally free of rank 2 and sits in exact sequences

$$0 \rightarrow T_S(-C) \rightarrow F \rightarrow T_C \rightarrow 0$$  \hspace{1cm} (2.3)

and

$$0 \rightarrow F \rightarrow T_S \rightarrow N_{C|S} \rightarrow 0.$$  \hspace{1cm} (2.4)
It is easy to check that the space of first order deformations of the pair $C \subseteq S$ is isomorphic to $H^1(S, F)$.

Returning to the case where $S$ is a general $K3$ of genus 10 and $C$ is a smooth plane section of $C$, the long exact cohomology sequence of (2.3) gives

$$0 \to H^1(S, T_S(-C)) \to H^1(S, F) \to H^1(C, T_C) \to H^2(S, T_S(-C))$$

$$\to H^2(S, F) \to 0$$

and by Proposition 2.1, $h^2(S, T_S(-C)) = 1$. But $H^1(S, F) \to H^1(C, T_C)$ cannot be surjective as the locus of curves on $K3$s has codimension at least one in $\mathcal{M}_{10}$. Thus $h^2(S, F) = 0$, $h^1(S, F) = 29$ and the codimension of the image of $H^1(S, F) \to H^1(C, T_C)$ is exactly 1. But this last map is the differential of the map $\mu$ of the Introduction, so the image of $\mu$ actually fills out a divisor.

REMARK 2.5. Let $\mu : \mathcal{P} \to \mathcal{M}_{10}$ be the rational moduli map as in the Introduction. If $\mathcal{X}$ is the closure of the image of $\mu$ and $N$ is the normal bundle of $\mathcal{X}$ in $\mathcal{M}_{10}$ then it follows from the long exact cohomology sequence of (2.4) and the analysis above that the fiber at $(C, S) \in \mathcal{P}$ (for $C$ a curve in the $K3$ surface $S$) of the bundle $\mu^*(N)$ is the one dimensional vector space $H^2(S, T_S(-C))$.

PROPOSITION 2.6. If $C$ is a smooth codimension 4 plane section of $X$, then $\text{Corank } \Phi(C) = 4$. For every $C$ in $\mathcal{X}$, $\text{Corank } \Phi(C) \geq 4$.

Proof. By [B-E-L] (2.11), $\text{Corank } \Phi(C) = h^0(C, N_C(-1)) - g$ where $N_C$ is the normal bundle to $C$ in its canonical embedding. But by Lemma 1.2, $h^0(C, N_C(-1)) = 14$ for a smooth codimension 4 plane section of $X$. The second assertion follows by semi-continuity.

REMARKS 2.7. (a) If $C$ is any smooth codimension 4 plane section of $X$ then the Clifford index of $C$ is at least 3: if $\text{Cliff}(C) \leq 2$, $C$ is either hyperelliptic, trigonal, or a degeneration of a smooth plane sextic and in all these cases, the corank of $\Phi(C)$ is strictly greater than 4.

(b) It is possible to give (at least) two other proofs of the inequality $\text{Corank } \Phi(C) \geq 4$: if $C$ has $\text{Cliff}(C) \geq 3$, it follows from results in [B-E-L] that $h^0(N_C(-2)) = 0$ where $N_C$ is the normal bundle to $C$ in its canonical embedding. On the other hand, a smooth codimension 4 plane section $C$ of $X$ is clearly 4-extendable, so applying a theorem of Zak (described in [B-E-L] and [B-E-L], 2.11, we find $\text{Corank } \Phi(C) \geq 4$.

(c) For a third proof, let $C$ be a smooth codimension 4 plane section of $X$ and consider the commutative diagram

$$\begin{array}{ccc}
\bigwedge^2 H^0(X, \mathcal{O}_X(1)) & \xrightarrow{e} & H^0(X, \Omega^1_X(2))
\downarrow h & & \downarrow c
\bigwedge^2 H^0(C, \mathcal{O}_C(1)) & \xrightarrow{f} & H^0(C, \Omega^2_C(2)).
\end{array}$$
Here the horizontal maps are the Wahl maps for \( \mathcal{O}(1) \) and the other maps are the natural restrictions. Now \( b \) is clearly surjective, so the image of \( d = \Phi(C) \) is contained in the image of \( f \). We claim that \( f \) has corank 4: the exact sequence of cohomology of \( 0 \to N^*_C(2) \to \Omega^1_C(2) \to \Omega^1_C(2) \) gives

\[
H^0(C, \Omega^1_C(2)) = f(\mathcal{O}_C(-1)^\otimes 4) = 4
\]

and the claim follows by observing that \( h^1(N^*_C(2)) = h^1(\mathcal{O}_C(-1)^\otimes 4(2)) = 4 \) and that \( H^1(\Omega^1_C(2)) = H^0(T_X(-1)^\otimes 0) \) (Lemma 1.2).

**Corollary 2.8.** We have an inequality of divisors \( \mathcal{W} \geq 4\mathcal{K} \).

**Proof.** Let \( \mathcal{M} = \mathcal{M}_0^{10} \) denote the moduli space of smooth automorphism-free genus 10 curves over the complex numbers, \( \pi: \mathcal{C} \to \mathcal{M} \) the universal curve, \( \omega = \Omega^1_{\mathcal{C}/\mathcal{M}} \) the sheaf of relative differentials and \( \lambda = \det(\pi_* (\omega)) \in \text{Pic}(\mathcal{M}). \) We have the relative Wahl map

\[
\Phi: \bigwedge^2 \pi_*(\omega) \to \pi_*(\omega^\otimes 3)
\]

which is a map of bundles of rank 45; let \( \mathcal{W} \) denote its degeneracy locus. By [C-H-M] the support of \( \mathcal{W} \) is a proper subvariety of \( \mathcal{M} \) and hence \( \mathcal{W} \) is a divisor.

By Proposition 2.6, the universal Wahl map \( \Phi \) has corank at least 4 at each point of \( \mathcal{X} \). It follows that \( \det(\Phi) \) vanishes to order at least 4 along \( \mathcal{X} \). Indeed, take a small arc \( \{C_t\} \) crossing \( \mathcal{X} \) transversally at a general point \( C_0 \in \mathcal{X} \) and apply the following observation: if \( \{M_t\} \) is a one parameter family of square matrices then \( \text{ord}_{t=0} \det(M_t) \geq \dim \ker(M_0) \); this is easily seen by diagonalizing the matrix \( \{M_t\} \) over the discrete valuation ring of convergent power series in \( t \).

### 3. The classes of \( \mathcal{W} \) and \( \mathcal{X} \)

We continue to use the notations of the Introduction and Section 2. For divisors \( D \) and \( E \), linear equivalence will be denoted \( D \sim E \). If \( L \) is a line bundle, we write \( D \sim L \) to mean that the line bundles \( \mathcal{O}(D) \) and \( L \) are isomorphic. We will show that \( \mathcal{W} \sim 28\lambda \) and that \( \mathcal{X} \sim 7\lambda \). The divisor \( \mathcal{W} - 4\mathcal{X} \) is then linearly equivalent to zero and by Corollary 2.8 it is effective. But in the variety \( \mathcal{M} = \mathcal{M}_0^{10} \) the only effective divisor \( D \) linearly equivalent to zero is \( D = 0 \): since \( \mathcal{M} \) has a projective compactification with boundary of codimension 2, if \( D \) were not zero, there would exist a complete curve \( T \subset \mathcal{M} \) not contained in \( D \) and intersecting \( D \); since \( D \sim 0 \) we have \( D_T = \deg(\mathcal{O}(D)|_T) = 0 \), a contradiction. It follows that \( \mathcal{W} = 4\mathcal{X} \).

**Proposition 3.1.** \( \mathcal{W} \sim 28\lambda \).

**Proof.** Since \( \mathcal{W} \) is the divisor of zeros of the section \( \det(\Phi) \), \( \mathcal{W} \) belongs to the
class \( c_1(\pi_* (\omega^\otimes 3)) - c_1(\wedge^2 \pi_* (\omega)) \). From [Mu], 5.10, \( c_1(\pi_* (\omega^\otimes 3)) \sim 3\lambda \). By the splitting principle if \( E \) is a bundle of rank \( r \) then \( c_1(\wedge^2 E) = (r - 1)c_1(E) \), so \( c_1(\wedge^2 \pi_* (\omega)) \sim 9\lambda \) and the result follows.

Computing the class of \( \mathcal{K} \) will require some more preparation. We start with some enumerative formulas. If \( f : X \to B \) is a flat family of curves, where \( X \) and \( B \) are smooth complete and \( \dim(B) = 1 \), it follows from the Leray spectral sequence that \( \chi(X, \mathcal{O}_X) = \chi(B, \mathcal{O}_B) - \chi(B, R^1 f_* \mathcal{O}_X) \). Applying Riemann-Roch and duality to \( E = R^1 f_* \mathcal{O}_X \), we obtain \( \chi(E) = \deg(E) + \text{rk}(E)\chi(B) \) and \( R^1 f_* \mathcal{O}_X = (f_* \omega_X|_B)^* \) so

\[
\deg(\lambda|_B) = \chi(X, \mathcal{O}_X) - \chi(B, \mathcal{O}_B)\chi(C, \mathcal{O}_C)
\]

where we write \( \lambda|_B \) for \( \det(f_* \omega_X|_B) \) and where \( C \) is a general fiber of \( f \).

For example, if \( C \subset S \) is a smooth curve on a smooth surface which moves in a pencil, consider \( f : \tilde{S} \to \mathbb{P}^1 \) where \( \tilde{S} \) is the blow-up of \( S \) at the base locus of the pencil. Then \( \deg(\lambda_f) = \chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) - 1 + g_C = \chi(S, \mathcal{O}_S) - 1 + g_C \) since \( \chi \) is a birational invariant. In particular, if \( S \) is a K3 surface,

\[
\deg(\lambda_f) = 1 + g_C.
\]

If \( C \) is a very ample smooth curve on a smooth complete surface \( S \), let \( \mathcal{D} \subset |C| \) denote the discriminant hypersurface, consisting of singular members of the complete linear system \( |C| \). If we consider a general (Lefschetz) pencil in \( |C| \) and apply the Leray spectral sequence to the constant sheaf \( C \) this time, we may count the number of singular fibers and obtain (see [G-H], pp. 508–510 for details) \( \deg(\mathcal{D}) = 4(g_C - 1) + C^2 + \chi_{\text{top}}(S) \). In particular, if \( S \) is a K3 surface,

\[
\deg(\mathcal{D}) = 6(g_C + 3).
\]

**Lemma 3.4.** If \( S \) is a general K3 surface of genus 10, then

(a) only finitely many smooth curves \( C \) in the linear series \( |\mathcal{O}_S(1)| \) have automorphisms.

(b) The linear series \( |\mathcal{O}_S(1)| \) contains at most a 2 dimensional family of curves with a single node and with automorphisms.

(c) \( S \) carries a Lefschetz pencil consisting entirely of curves without automorphisms.

**Proof.** (a) Consider a 19 dimensional family \( \mathcal{F} \) of K3 surfaces of genus 10 in \( \mathbb{P}^{10} \) which dominates \( \mathcal{F}_{10} \) (see, e.g., [M] for a construction) and let \( \mathcal{P} \) be the canonical \( \mathbb{P}^{10} \) bundle over \( \mathcal{F} \) (whose fiber at \( S \) is \( |\mathcal{O}_S(1)| \)). Let \( k \) be the dimension, for a general \( S \) in \( \mathcal{F} \), of the subset of \( |\mathcal{O}_S(1)| \) representing smooth curves with nontrivial automorphisms. We want to show that \( k \leq 0 \). By the definition of \( k \)
there exists a subvariety $\mathcal{A} \subset \mathcal{P}$ of dimension $19 + k$ consisting of smooth curves with automorphisms, such that $\mathcal{A}$ dominates $\mathcal{F}$. Let $\mu: \mathcal{A} \to \mathcal{M}_{10}$ be the moduli map.

As $S$ is general, its Picard group is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_S(C)$. It then follows immediately from the main theorem of [G-L] that $S$ contains no $n$-gonal curves for $n \leq 5$. But the largest component of curves with automorphisms in $\mathcal{M}_{10}$ which are not of this type has dimension 16 and consists of curves with an involution such that the quotient has genus 3. Thus the fibers of $\mu$ are at least $k + 3$-dimensional.

On the other hand the dimension of the fibers of $\mu$ is constant in a linear series $|\mathcal{O}_S(1)|$ and generically this dimension is 3 (as follows from the proof of Proposition 2.2). Thus $k \leq 0$ as was to be shown.

(b) The argument in this case is similar, except that we work in $\Delta_0 \subseteq \mathcal{M}_{10}$, the boundary component of $\mathcal{M}_{10}$ representing curves of arithmetic genus 10 with one node. Here the locus of curves with non-trivial automorphisms has dimension 17, consisting of hyperelliptic curves of (geometric) genus 9 with two points conjugate under the involution identified. We find $k \leq 2$. (Perhaps a more refined analysis would improve this estimate.)

(c) This is an immediate consequence of (a) and (b).

PROPOSITION 3.5. $\mathcal{K} \sim 7\lambda$.

Proof. Fix a general $S \in \mathcal{F}_{10}$, and let $C \subset S$ be a smooth genus 10 curve. Consider a general Lefschetz pencil $l \subset |C|$. By Lemma 3.4 $\mathcal{O}_S(l) \subset \mathcal{M}$, where $\mathcal{M}$ is the moduli space of stable genus 10 curves without automorphisms. The Picard group of the smooth variety $\mathcal{M}$ is freely generated by $\lambda$ and the classes of the divisors $\Delta_0, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ where for $i > 0, \Delta_i$ consists of stable curves with a node that separates the curve into components of genus $i$ and $10-i$, and $\Delta_0$ is the divisor of stable curves with a singular irreducible component (as follows from [A-C] Section 4 and [C] Section 1.3).

Denote $\overline{\mathcal{K}}$ the closure of $\mathcal{K}$ in $\mathcal{M}$. Then we have a relation

$$\overline{\mathcal{K}} \sim a \cdot \lambda - b_0 \cdot \Delta_0 - b_2 \cdot \Delta_2 - b_3 \cdot \Delta_3 - b_4 \cdot \Delta_4 - b_5 \cdot \Delta_5$$

(3.6)

with $a, b_i \in \mathbb{Z}$. Now we pull-back (3.6) to $l$ in order to determine $a$. Since the surface $S$ is general, its Picard group is generated by the class of $C$ and then there are no reducible curves in $|C|$. This implies that $\Delta_i \cdot l = 0$ for $i > 0$ (notice that since $l$ is general its singular members have only nodes as singularities). From (3.3), $\Delta_0 \cdot l = 78$ (notice that $\mathcal{S}$, the blow-up of $S$ along the base locus of the pencil $l$, is smooth and hence $\mu(l)$ is transverse to $\Delta_0$) and from (3.2) we obtain $\lambda \cdot l = 11$.

To find $\overline{\mathcal{K}} \cdot l = \deg \mu^*(N_{\overline{\mathcal{K}}|l})|l$, we need to compute the degree of the line bundle over $l$ with fiber $H^2(S, T_S(-C))$ for $C \in l$ (Remark 2.5). More precisely,
suppose $l$ is spanned by $C_0 = \{s_0 = 0\}$ and $C_1 = \{s_1 = 0\}$ for $s_0, s_1 \in H^0(S, L)$ (we write $L = \mathcal{O}_S(C)$). We have a diagram

\[
\begin{array}{ccc}
\tilde{S} \subset S \times \mathbb{P}^1 & \xrightarrow{g} & S \\
\downarrow f & & \\
\mathbb{P}^1 & & \\
\end{array}
\]

and $\tilde{S} = \{(x, t_0, t_1)|t_0 \cdot s_0(x) + t_1 \cdot s_1(x) = 0\} \subset S \times \mathbb{P}^1$ is the zero set of a section of $f^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes g^*L$. Then

\[
\mathcal{X}.l = \deg R^2f_*(T_{S \times \mathbb{P}^1}|_{\tilde{S}}(-\tilde{S}))
\]

\[
= \deg R^2f_*(g^*T_S \otimes g^*(L^*) \otimes f^*\mathcal{O}_{\mathbb{P}^1}(-1))
\]

\[
= \deg R^2f_*(g^*T_S \otimes L^*)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)
\]

which equals (by base change and cohomology) $\deg H^2(S, T_S \otimes L^*) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = -1$.

Combining these results we obtain the relation

\[
-1 = 11a - 78b_0.
\]

(3.7)

The integral solutions to this equation are $a = 7 + 7k, b_0 = 1 + 11k$ for $k \in \mathbb{Z}$.

We know (2.8) that $W > 4X$ and (3.1) that $W \sim 28\lambda$. Hence $0 \leq a \leq 7$ and so $k = 0, a = 7$, as desired.

As explained at the beginning of this section, the linear equivalence $W \sim 4X$ together with the inequality $W > 4X$ implies $W = 4X$; this completes the proof of the main theorem.

REMARK 3.8. Note that our computation of the class of $X$ in $\text{Pic}(\mathcal{M})$ uses the inequality $a \leq 7$ (coming from Corollary 2.8 and Proposition 3.1) and the equality 3.7, together with the fact that the coefficients $a$ and $b_0$ in 3.7 are integral. This integrality is why we work in the smooth variety $\mathcal{M}_{10}$. A more traditional approach, which we were unable to carry out, would proceed by writing down several pencils of genus 10 curves, computing their intersections with $X, \lambda$, and the $\Lambda_i$, and then solving the resulting system of linear equations over $\mathbb{Q}$.

Acknowledgements

It is a pleasure to thank J. Harris, who suggested the problem, as well as R. Lazarsfeld and S. Mukai for their valuable help.
References


