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Degree of local zeta functions and monodromy

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1. Introduction

(1.1) Let K be a p-adic field, i.e. $[K:Q_p] < \infty$. Let R be the valuation ring of K, P the maximal ideal of R, and $\bar{K} = R/P$ the residue field of K. The cardinality of $\bar{K}$ is denoted by q, thus $\bar{K} = F_q$. Let $f(x) \in K[x]$, $x = (x_1, \ldots, x_n)$, $f \notin K$. Igusa's local zeta function of f with respect to a character $\chi: R^\times \to \mathbb{C}^\times$ and a Schwartz–Bruhat function $\Phi: K^n \to \mathbb{C}$ is denoted by

$$Z_\Phi(s, \chi) = Z_\Phi(s, \chi, K, f),$$

see e.g. [D3, §1.1], [D2]. When Φ is the characteristic function of the residue class $a \in \bar{K}^n$, we will write $Z_a(s, \chi)$ instead of $Z_\Phi(s, \chi)$. In this note we will always assume that χ is induced by a character $\chi: \bar{K}^\times \to \mathbb{C}^\times$.

In case of good reduction, we showed in [D1] (see also [D3, §4.1]) that $\deg Z_a(s, \chi) \leqslant 0$ and $\deg Z_a(s, \chi_{\text{triv}}) = 0$, where deg means the degree as rational function in q^{-s} and χ_{triv} is the trivial character. (We put $\deg 0 = -\infty$.) In the present note we will prove the following theorem:

(1.2) THEOREM. *If f is defined over a number field $F \subset \mathbb{C}$, then for almost all completions K of F we have the following*:

If $f(0) = 0$ and no eigenvalue of the (complex) local monodromy of f at 0 has the same order as χ, then $\deg Z_0(s, \chi) < 0$.

With an eigenvalue of the (complex) local monodromy of f at $a \in f^{-1}(0)$ we mean an eigenvalue of the action of the counter clockwise generator of the fundamental group of $\mathbb{C}\backslash\{0\}$ on the cohomology (in some dimension) of the Milnor fiber of f at a (see e.g. [A] or [D3, §2.1]). It is well known that such an eigenvalue is a root of unity so that we can talk about its order. Theorem 1.2 is a direct consequence of Theorem 1.4 below, whose statement requires some more notation.

(1.3) From now on we assume that $f \in R[x]$ and $\bar{f} \neq 0$, where $\bar{f}$ denotes the reduction mod P of f. We fix a prime $\ell \nmid q$ and an embedding of $\mathbb{C}$ into an algebraic closure Q_ℓ^a of Q_ℓ. Thus we can consider χ as a character $\chi: \bar{K}^\times \to (Q_\ell^a)^\times$. This χ induces a character also denoted by χ, of the geometric monodromy group of $\mathbb{A}^1_{F_q}$ at 0, see 2.1. Let F_0 be the Milnor fibre of $\bar{f}$ at 0, in the sense of etale topology. We denote by $H^i(F_0, Q_\ell^a)^\chi$ the component of the ℓ-adic cohomology $H^i(F_0, Q_\ell^a)$ on which the local geometric monodromy group acts like χ times a unipotent action, see 2.3.1.

(1.4) THEOREM. *Assume that* $f^{-1}(0)$ *has a resolution with tame good reduction* mod P (*see* 2.2.3 *or* [D3, 3.2]), *and that* $f(0) = 0$. *Then*

$$\lim_{s \to -\infty} Z_0(s, \chi) = (1 - q)q^{-n} \sum_i (-1)^i \operatorname{Tr}(\sigma_1, H^i(F_0, Q_\ell^a)^\chi),$$

where σ_1 *is a suitable lifting of the geometric Frobenius* (*see* 3.2).

Theorem 1.4 is proved in 3.3 using the method of vanishing cycles which we recall in 2.1 and 2.2. A partial converse of Theorem 1.4 is given in 3.4. In Section 4 we propose a conjecture about the holomorphy of $Z_\Phi(s, \chi)$. Finally, Section 5 contains an alternative proof of some material in [D2].

2. Preliminaries

(2.1) *Local monodromy*

We choose a geometric generic point $\bar{\eta}$ of $\mathbb{A}^1_{F_q}$. In particular this choice determines an algebraic closure F_q^a of F_q. Let S, resp. S_o, be the Henselization at 0 of $\mathbb{A}^1_{F_q^a}$ resp. $\mathbb{A}^1_{F_q}$, and denote by η, resp. η_o, its generic point.

Put $G_0 = \operatorname{Gal}(\bar{\eta}/\eta_o)$ and $I_0 = \operatorname{Gal}(\bar{\eta}/\eta)$. The group G_0, resp. I_0, is called the arithmetical, resp. geometrical, local monodromy group of $\mathbb{A}^1_{F_q}$ at 0. Via the cover

$$S_o \backslash \{0\} \to S_o \backslash \{0\} : X \mapsto X^{q-1},$$

with Galois group $F_q^\times$, we consider $F_q^\times$ as a quotient of G_0. Hence the character $\chi: F_q^\times \to (Q_\ell^a)^\times$ induces a homomorphism $\tilde{\chi}: G_0 \to (Q_\ell^a)^\times$. The restriction of this homomorphism $\tilde{\chi}$ to I_0 will be denoted by $\chi: I_0 \to (Q_\ell^a)^\times$.

(2.2) *Nearby cycles on the resolution space*

(2.2.1) Let $h: Y \to X = \operatorname{Spec} K[x]$ be an (embedded) resolution (of singularities) for $f^{-1}(0)$ over K with good reduction mod P, see [D3, 1.3.1 and 3.2] or [D1]. Reduction mod P is denoted by $\bar{\ }$, e.g. $\bar{Y}$, $\bar{E}_i$.

Let $E_i, i \in T$, be the irreducible components of $(f \circ h)^{-1}(0)$. Denote by N_i, resp. $\nu_i - 1$, the multiplicity of E_i in the divisor of $f \circ h$, resp. $h^*(dx_1 \wedge \ldots \wedge dx_n)$. Put $\overset{\circ}{E}_i = E_i \backslash \cup_{j \neq i} E_j$, $\overset{\circ}{\bar{E}}_i = \bar{E}_i \backslash \cup_{j \neq i} \bar{E}_j$ and $\bar{E}_I = \cap_{i \in I} \bar{E}_i$, $\overset{\circ}{\bar{E}}_I = \bar{E}_I \backslash \cup_{j \notin I} \bar{E}_j$ for any $I \subset T$. When $I = \emptyset$, put $\bar{E}_\emptyset = \bar{Y}$.

(2.2.2) We denote by $R\Psi_{\bar{f}}(C)$, resp. $R\Psi_{\bar{f} \circ \bar{h}}(C)$, the complex of nearby cycles on $\bar{f}^{-1}(0) \otimes F_q^a$, resp. $(\bar{f} \circ \bar{h})^{-1}(0) \otimes F_q^a$, associated to a complex C, see [SGA 7, XIII]. To simplify notation, put $\Psi_{\bar{f}}^i = R^i\Psi_{\bar{f}}(Q_\ell^a)$ and $\Psi_{\bar{f} \circ \bar{h}}^i = R^i\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a)$. If $f(0) = 0$ then $(\Psi_{\bar{f}}^i)_0 = H^i(F_0, Q_\ell^a)$, where F_0 denotes the Milnor fibre of $\bar{f}$ at 0. It is well known [SGA 7, XIII 2.1.7.1] that

$$R\bar{h}_* \circ R\Psi_{\bar{f} \circ \bar{h}} = R\Psi_{\bar{f}},$$

since $\bar{h}$ is proper and birational. Thus, when $f(0) = 0$,

$$H^i(F_0, Q_\ell^a) = \mathbb{H}^i(\bar{h}^{-1}(0) \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a)), \tag{2.2.2.1}$$

and we have a spectral sequence

$$H^i(\bar{h}^{-1}(0) \otimes F_q^a, \Psi_{\bar{f} \circ \bar{h}}^j) \Rightarrow H^{i+j}(F_0, Q_\ell^a). \tag{2.2.2.2.}$$

Note that G_0 acts on all terms of this spectral sequence, by transport of structure (choice of $\bar{\eta}$), and the spectral sequence is G_0-equivariant. We recall from [SGA 7, Exp. I Thm 3.3] the following basic facts:

For any $I \subset T$ with $I \neq \emptyset$ and any closed point $s \in \overset{\circ}{\bar{E}}_I \otimes F_q^a$, there is a canonical isomorphism

$$(\Psi_{\bar{f} \circ \bar{h}}^j)_s^{\text{tame}} \cong (\Psi_{\bar{f} \circ \bar{h}}^0)_s^{\text{tame}} \otimes \bigwedge^j (M_I(-1)), \tag{2.2.2.3}$$

where M_I is the dual of the kernel of the linear map $(Q_\ell^a)^I \to Q_\ell^a : (z_i)_{i \in I} \mapsto \Sigma_{i \in I} N_i z_i$, $M_I(-1)$ is a Tate twist of M_I, and the superscript *tame* denotes the tame part. Moreover

$$(\Psi_{\bar{f} \circ \bar{h}}^0)_s^{\text{tame}} \cong (Q_\ell^a)^{C_I}, \tag{2.2.2.4}$$

with C_I a fiinite set on which I_0 acts transitively, and $|C_I|$ equal to the largest common divisor of the N_i, $i \in I$, which is prime to q.

(2.2.3) Till the end of 2.2.3 we will assume now that the resolution h has *tame* good reduction, i.e. it has good reduction and N_i is prime to q for each $i \in T$. Then it easily follows from [K, p. 180] that the action of I_0 on $\Psi^j_{\bar f \circ \bar h}$ is tame.

A local calculation shows that the $\Psi^j_{\bar f \circ \bar h}$ are lisse on $\mathring{\bar E}_I \otimes F^a_q$ and that locally on $\mathring{\bar E}_I \otimes F^a_q$ the isomorphisms 2.2.2.3 on the stalks are induced by an isomorphism of the sheaves. Since these isomorphisms are canonical they glue together to a *canonical* isomorphism

$$\Psi^j_{\bar f \circ \bar h} \cong \Psi^0_{\bar f \circ \bar h} \otimes \bigwedge^j (M_I(-1)) \qquad \text{on } \mathring{\bar E}_I \otimes F^a_q \tag{2.2.3.1}$$

which is compatible with the action of G_0.

(2.3) *Isotopic components*

(2.3.1) For any constructible Q^a_ℓ-sheaf $\mathscr{F}$ (or vector space) on which I_0 acts, we denote by $\mathscr{F}^\chi$ the χ-unipotent part of $\mathscr{F}$, i.e. the largest subsheaf on which I_0 acts like χ times a unipotent action.

(2.3.2) To the character $\chi : F^\times_q \to (Q^a_\ell)^\times$ is associated the lisse rank one Q^a_ℓ-sheaf $\mathscr{L}_\chi$ on $\mathbb{A}^1_{F_q}\backslash\{0\}$, see [SGA $4\frac{1}{2}$, Sommes Trig.]. The action of the arithmetical monodromy group G_0 at 0 on $(\mathscr{L}_\chi)_{\bar\eta}$ is given by χ^{-1}.

Let ν be the open immersion $\nu : \bar Y \backslash (\bar f \circ \bar h)^{-1}(0) \hookrightarrow \bar Y$ and $\alpha : \bar Y \backslash (\bar f \circ \bar h)^{-1}(0) \to \mathbb{A}^1_{F_q}\backslash\{0\}$ the restriction of $\bar f \circ \bar h$. Put $\mathscr{F}_\chi = \nu_* \alpha^* \mathscr{L}_\chi$. The cohomology of this sheaf appears in the explicit formula for $Z_0(s, \chi)$, see 3.1.

(2.3.3) LEMMA. *There is a canonical isomorphism*

$$\mathscr{F}_\chi|_{(\bar f \circ \bar h)^{-1}(0) \otimes F^a_q} \cong (\Psi^0_{\bar f \circ \bar h})^\chi \otimes (\mathscr{L}_\chi)_{\bar\eta}.$$

Proof. Because the action of I_0 on the stalks of the tame part of $\Psi^0_{\bar f \circ \bar h}$ is semi-simple (cf. 2.2.2.4), $(\Psi^0_{\bar f \circ \bar h})^\chi$ equals the largest subsheaf of $\Psi^0_{\bar f \circ \bar h}$ on which I_0 acts like χ. Moreover there is a canonical isomorphism

$$R\Psi_{\bar f \circ \bar h}(\alpha^* \mathscr{L}_\chi) \cong R\Psi_{\bar f \circ \bar h}(Q^a_\ell) \otimes (\mathscr{L}_\chi)_{\bar\eta}. \tag{2.3.3.1}$$

Thus it suffices to prove that there is a canonical isomorphism

$$\mathscr{F}_\chi|_{(\bar{f}\circ\bar{h})^{-1}(0)\otimes F_q^a} \cong (R^0\Psi_{\bar{f}\circ\bar{h}}(\alpha^*\mathscr{L}_\chi))^{I_0}, \tag{2.3.3.2}$$

where the superscript I_0 denotes the largest subsheaf on which I_0 acts trivially. We will denote by an index S the base change $S \to \mathbb{A}^1_{F_q}$; for example $\bar{Y}_S = \bar{Y} \otimes_{\mathbb{A}^1_{F_q}} S$. Consider the following diagram of natural maps

$$\begin{tikzcd}
(\bar{f}\circ\bar{h})^{-1}(0)\otimes F_q^a \arrow[r, "i"] \arrow[d] & \bar{Y}_S \arrow[d, "(\bar{f}\circ\bar{h})_S"'] & (\bar{Y}\backslash(\bar{f}\circ\bar{h})^{-1}(0))_S \arrow[l, "\nu_S"'] \arrow[d, "\alpha_S"'] & (\bar{Y}_S)_{\bar{\eta}} \arrow[l, "j"'] \arrow[d] \\
\{0\} \arrow[r] & S & S\backslash\{0\} = \eta \arrow[l] & \bar{\eta} \arrow[l, "\gamma"']
\end{tikzcd}$$

Consider also the natural map $\epsilon: S\backslash\{0\} \to \mathbb{A}^1_{F_q}\backslash\{0\}$. By [SGA $4\frac{1}{2}$, Th. finitude 1.9] we have

$$\alpha_S^* \gamma_*(\gamma^*\epsilon^*\mathscr{L}_\chi) \cong j_*(j^*\alpha_S^*\epsilon^*\mathscr{L}_\chi).$$

Hence

$$i^*(\nu_S)_*\alpha_S^*\gamma_*(\gamma^*\epsilon^*\mathscr{L}_\chi) \cong i^*(\nu_S)_*j_*(j^*\alpha_S^*\epsilon^*\mathscr{L}_\chi) = R^0\Psi_{\bar{f}\circ\bar{h}}(\alpha^*\mathscr{L}_\chi).$$

Taking I_0-invariants we get

$$i^*(\nu_S)_*\alpha_S^*(\gamma_*\gamma^*\epsilon^*\mathscr{L}_\chi)^{I_0} \cong (R^0\Psi_{\bar{f}\circ\bar{h}}(\alpha^*\mathscr{L}_\chi))^{I_0}.$$

But $\epsilon^*\mathscr{L}_\chi \cong (\gamma_*\gamma^*\epsilon^*\mathscr{L}_\chi)^{I_0}$, hence

$$\mathscr{F}_\chi|_{(\bar{f}\circ\bar{h})^{-1}(0)\otimes F_q^a} \cong i^*(\nu_S)_*\alpha_S^*\epsilon^*\mathscr{L}_\chi \cong (R^0\Psi_{\bar{f}\circ\bar{h}}(\alpha^*\mathscr{L}_\chi))^{I_0}. \qquad \square$$

3. Cohomological interpretation of $\lim_{s\to-\infty} Z_0(s,\chi)$

(3.1) Let $F \in \mathrm{Gal}(F_q^a/F_q)$ be the geometric Frobenius. We recall from [D2] that

$$Z_0(s,\chi) = q^{-n} \sum_{I\subset T} c_{I,\chi,0} \prod_{i\in I} \frac{q-1}{q^{N_i s+\nu_i}-1}, \tag{3.1.1}$$

where

$$c_{I,\chi,0} = \sum_i (-1)^i \operatorname{Tr}(F, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathscr{F}_\chi)). \tag{3.1.2}$$

Hence

$$\lim_{s\to-\infty} Z_0(s,\chi) = q^{-n} \sum_{I\subset T} c_{I,\chi,0}(1-q)^{|I|}. \tag{3.1.3}$$

(3.2) With a *suitable lifting* of the geometric Frobenius (mentioned in the statement of Theorem 1.4) we mean any element $\sigma_1 \in G_0$ which induces the geometric Frobenius on F_q^a and which acts trivially on $(\mathscr{L}_\chi)_{\bar{\eta}}$ (see 2.3.2).

(3.3) *Proof of Theorem* 1.4. We will prove that

$$\frac{q^n}{1-q} \lim_{s\to-\infty} Z_0(s,\chi) = \sum_i (-1)^i \operatorname{Tr}(\sigma, H^i(F_0, Q_\ell^a)^\chi \otimes (\mathscr{L}_\chi)_{\bar{\eta}}), \tag{3.3.1}$$

for *any* $\sigma \in G_0$ which induces the geometric Frobenius F on F_q^a. This yields the theorem when we take for σ a suitable lifting σ_1 as in 3.2. The right-hand-side of (3.3.1) equals

$$\sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H^i(\bar{h}^{-1}(0) \otimes F_q^a, \Psi^j_{\bar{f}\circ\bar{h}})^\chi \otimes (\mathscr{L}_\chi)_{\bar{\eta}}), \qquad \text{by (2.2.2.2)},$$

$$= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, (\Psi^j_{\bar{f}\circ\bar{h}})^\chi \otimes (\mathscr{L}_\chi)_{\bar{\eta}})),$$

$$= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, (\Psi^0_{\bar{f}\circ\bar{h}})^\chi \otimes$$

$$\otimes (\mathscr{L}_\chi)_{\bar{\eta}}) \otimes \overset{j}{\wedge} (M_I(-1))), \qquad \text{by (2.2.3.1)},$$

$$= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathscr{F}_\chi) \otimes$$

$$\otimes \overset{j}{\wedge} (M_I(-1))), \qquad \text{by (2.3.3)},$$

$$= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(F, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathscr{F}_\chi)) \operatorname{Tr}(F, \overset{j}{\wedge} (M_I(-1))),$$

$$= \sum_I c_{I,\chi,0}(1-q)^{|I|-1}, \qquad \text{by (3.1.2)}.$$

Combining this with 3.1.3 proves 3.3.1 and finishes the proof of Theorem 1.4. □

We now turn to a partial converse of Theorem 1.4. For any finite extension L of the field K, the norm from L to K is denoted by $N_{L/K}$.

(3.4) PROPOSITION. *If f is defined over a number field $F \subset \mathbb{C}$, then for almost all completions K of F we have the following: Assume the order of χ equals the order of some eigenvalue of the (complex) local monodromy of f at some complex point of $f^{-1}(0)$. Then there are infinitely many unramified extensions L of K such that* $\deg Z_{\bar{a}}(s, \chi \circ N_{L/K}, L, f) = 0$ *for some integral $a \in L^n$ with $f(a) = 0$.*

Proof. It is well known [B] that $R\Psi_{\bar{f}}(Q_\ell^a)[n-1]$ is a perverse sheaf. Let $C := (R\Psi_{\bar{f}}(Q_\ell^a)[n-1])^\chi$ be the maximal subobject (in the category of perverse sheaves) on which I_0 acts like χ times a unipotent action. We have $C \neq 0$ (for almost all completions K of F). Since C is perverse, there exists a geometric point $\bar{a}$ of $\bar{f}^{-1}(0)$ such that $(H^i(C))_{\bar{a}} \neq 0$ for *exactly* one i. The proposition follows now easily from 1.4. □

(3.5) *Example.* Let $f(x_1, x_2) = x_2^2 - x_1^3$. Then the orders of the eigenvalues of the local monodromy are 1 and 6. Thus, for almost all completions, $\deg Z_0(s, \chi) < 0$ if χ has order 2 and 3. (Compare with Proposition 4.5).

4. Holomorphy of $Z_\Phi(s, \chi)$

(4.1) We call a Schwartz–Bruhat function Φ on K^n residual if Φ is zero outside R^n and $\Phi(x)$ only depends on $x \bmod P$.

(4.2) It is well known (see [I1] or [D3, 1.3.2]) that $Z_\Phi(s, \chi)$ is holomorphic on $\mathbb{C}$ when the order of χ divides no N_i. The N_i are not intrinsic, but the order of any eigenvalue of the local monodromy on $f^{-1}(0)$ divides some N_i (this follows from 2.2.2.2, 2.2.2.3 and 2.2.2.4). Being very optimistic, we propose the following conjecture:

(4.3) CONJECTURE. *If f is defined over a number field $F \subset \mathbb{C}$, then for almost all completions K of F we have the following: when Φ is residual, $Z_\Phi(s, \chi)$ is holomorphic unless the order of χ divides the order of some eigenvalue of the (complex) local monodromy of f at some complex point of $f^{-1}(0)$.*

In fact, this might be true for all p-adic completions K of F and for any Φ. Veys [V2] verified this when f has only two variables. Moreover the author showed that the conjecture is true for the relative invariants of a few prehomogeneous vector spaces (using Theorem 2 of [I2] and the orbital decomposition).

(4.4) *Remark.* Suppose f is homogeneous. Then for almost all completions K of F we have the following: If $Z(s, \chi)$ is holomorphic, then $Z(s, \chi) = 0$, since $\deg Z(s, \chi) < 0$ (see [D3, 4.1]). For $s = +\infty$ this yields that

$$S := \sum_{x \in (F_q)^n, \bar{f}(x) \neq 0} \chi(\bar{f}(x))$$

is zero when $Z(s, \chi)$ is holomorphic. Thus conjecture 4.3 implies a relation between the vanishing of the character sum S and monodromy. However this relation follows directly from the formula

$$S = (q-1)q^{n-1} \sum_i (-1)^i \operatorname{Tr}(\sigma_1^{-1}, H^i(F_0, Q_\ell^a)^{\chi^{-1}}),$$

which is easily proved by standard methods.

The following proposition is a partial converse of Conjecture 4.3.

(4.5) PROPOSITION. *If f is defined over a number field $F \subset \mathbb{C}$, then for almost all completions K of F we have the following: If the order of χ divides the order of some eigenvalue of the (complex) local monodromy of f at some complex point of $f^{-1}(0)$, then for infinitely many unramified extensions L of K, $Z_\Phi(s, \chi \circ N_{L/K}, L, f)$ is not holomorphic on $\mathbb{C}$ for some residual Φ.*

I first proved this proposition in the isolated singularity case, see [D3, prop. 4.4.3]. However that proof generalizes directly to the general case, because of Lemma 4.6 below. Indeed by 4.6 and the hypothesis of 4.5 there exists $a \in f^{-1}(0)$ such that the order d of χ divides the order k of some reciprocal zero or reciprocal pole of the monodromy zeta function of f at a. Hence by A'Campo [A, Thm 3], we have $\Sigma_{k|N_i} \chi(\mathring{E}_i \cap h^{-1}(a)) \neq 0$. Proceeding now as in my proof of Proposition 4.4.3 of [D3], with Z_0 replaced by Z_a, we obtain that $Z_a(s, \chi \circ N_{L/K}, L, f)$ is not holomorphic for infinitely many L.

(4.6) LEMMA. *Let $f(x) \in \mathbb{C}[x]$, $x = (x_1, \ldots, x_n)$, $f \notin \mathbb{C}$. If λ is an eigenvalue of the (complex) local monodromy of f at $b \in f^{-1}(0)$, then there exists $a \in f^{-1}(0)$ such that λ is a reciprocal zero or reciprocal pole of the monodromy zeta function of f at a (in the sense of [A, p. 233]).*

Proof. It is well known [B] that $R\Psi_f\mathbb{C}[n-1]$ is a perverse sheaf. Let C be the maximal subobject (in the category of perverse sheaves) on which the (complex) local monodromy acts like λ times a unipotent endomorphism. The hypothesis of the Lemma implies that $C \neq 0$. Since C is perverse, there exists $a \in f^{-1}(0)$ such that $(H^i(C))_a \neq 0$ for *exactly* one i. The lemma follows now easily. □

5. An alternative proof of some material in [D2]

In [D2, Thm. 1.1] we proved that certain E_i do not contribute to poles of $Z(s, \chi)$, see also [D3, 4.6]. The proof was based on the following key Lemma 5.1, for which we will now give an alternative proof.

(5.1) LEMMA. [D2, 4.1]. *Assume the notation of* 2.2.1. *and* 2.3.2. *Let χ be a character of $\bar{K}^\times$ of order d, and $i_0 \in T$. Suppose E_{i_0} is proper*, $d|N_{i_0}$, and *E_{i_0} intersects no E_j with $d|N_j$, $j \neq i_0$. Then*

$$H^i_c(\mathring{\bar{E}}_{i_0} \otimes F^a_q, \mathscr{F}_\chi) = 0 \qquad \textit{for all } i \neq n-1.$$

(5.2) *An alternative proof for Lemma* 5.1. A local calculation, using the hypothesis of the Lemma, shows that for every closed point $s \in \bar{E}_{i_0} \backslash \mathring{\bar{E}}_{i_0}$ the local monodromy of $\mathscr{F}_\chi|_{\mathring{\bar{E}}_{i_0}}$ at s has no invariants. Hence by [SGA $4\frac{1}{2}$, Sommes Trig. 1.19.1] and tame ramification, we have

$$H^i_c(\mathring{\bar{E}}_{i_0} \otimes F^a_q, \mathscr{F}_\chi) = H^i(\mathring{\bar{E}}_{i_0} \otimes F^a_q, \mathscr{F}_\chi), \qquad \text{for all } i.$$

Thus by Poincaré duality we only have to prove the Lemma for $i > n-1$. Because $\bar{E}_{i_0}$ is proper, $\bar{h}(\bar{E}_{i_0})$ is finite. Hence we may assume that $\bar{h}(\bar{E}_{i_0}) = \{0\}$. We claim that

$$H^i_c(\mathring{\bar{E}}_{i_0} \otimes F^a_q, \mathscr{F}_\chi) \subset (\Psi^i_{\bar{f}})^\chi_{\bar{0}} \otimes (\mathscr{L}_\chi)_{\bar{\eta}}, \qquad \text{for all } i. \tag{5.2.1}$$

This claim proves the Lemma since it is well known that $\Psi^i_{\bar{f}} = 0$ when $i > n-1$, see [SGA 7, Exp. I Th. 4.2].

From 2.2.2.3, 2.2.2.4 and the hypothesis of the Lemma, it follows that

$$(\Psi^i_{\bar{f} \circ \bar{h}})^\chi_{\bar{s}} = 0 \tag{5.2.2}$$

for any closed point $s \in \bar{E}_{i_0} \backslash \mathring{\bar{E}}_{i_0}$ and $i \geqslant 0$, and also for any closed point $s \in \mathring{\bar{E}}_{i_0}$ and $i \geqslant 1$. (Indeed the χ-unipotent part is contained in the tame part.) Thus applying the Mayer–Vietoris sequence for $\bar{h}^{-1}(0) = \bar{E}_{i_0} \cup (\bar{h}^{-1}(0) \backslash \mathring{\bar{E}}_{i_0})$ and the spectral sequence of hypercohomology we obtain

$$\mathbb{H}^i(\bar{h}^{-1}(0) \otimes F^a_q, R\Psi_{\bar{f} \circ \bar{h}}(Q^a_\ell))^\chi = \mathbb{H}^i(\bar{E}_{i_0} \otimes F^a_q, R\Psi_{\bar{f} \circ \bar{h}}(Q^a_\ell))^\chi \oplus$$

$$\oplus\, \mathbb{H}^i((\bar{h}^{-1}(0)\backslash \mathring{\bar{E}}_{i_0}) \otimes F_q^a, R\Psi_{\bar{f}\circ\bar{h}}(Q_\ell^a))^\chi.$$

Together with 2.2.2.1 this yields

$$\mathbb{H}^i(\bar{E}_{i_0} \otimes F_q^a, R\Psi_{\bar{f}\circ\bar{h}}(Q_\ell^a))^\chi \subset (\Psi_{\bar{f}}^i)_0^\chi, \qquad \text{for all } i. \tag{5.2.3}$$

Again by 5.2.2 we have

$$\begin{aligned}\mathbb{H}^i(\bar{E}_{i_0} \otimes F_q^a, R\Psi_{\bar{f}\circ\bar{h}}(Q_\ell^a))^\chi &= H^i(\bar{E}_{i_0} \otimes F_q^a, (\Psi_{\bar{f}\circ\bar{h}}^0)^\chi) \\ &= H_c^i(\mathring{\bar{E}}_{i_0} \otimes F_q^a, (\Psi_{\bar{f}\circ\bar{h}}^0)^\chi)\end{aligned} \tag{5.2.4}$$

by degeneration of (the χ-unipotent part of) the spectral sequence of hypercohomology. The claim 5.2.1 follows now from 5.2.3, 5.2.4 and Lemma 2.3.3. This terminates the proof of Lemma 5.1. □

References

[A] N. A'Campo: La fonction zêta d'une monodromie, *Comment. Math. Helv.* **50** (1975) 233–248.

[B] J.-L. Brylinski: (Co)-Homologie d'Intersection et Faisceaux Pervers, *Séminaire Bourbaki*, nr. 585 (1981/82), *Astérisque* 92–93 (1982).

[D1] J. Denef: On the degree of Igusa's local zeta function, *Amer. Journ. Math.* **109** (1987) 991–1008.

[D2] J. Denef: Local zeta functions and Euler characteristics, *Duke Math. Journ.* **63** (1991) 713–721.

[D3] J. Denef, Report on Igusa's local zeta function, *Séminaire Bourbaki*, nr. 741 (1990/91), *Astérisque* 201/202/203 (1991), 359–386.

[DL] J. Denef et F. Loeser: Caractéristiques d'Euler–Poincaré, Fonctions zêta locales et Modifications analytiques, *J. Amer. Math. Soc.* **5** (1992) 705–720.

[I1] J. Igusa: Lectures on forms of higher degree, Tata Inst. Fund. Research, Bombay (1978).

[I2] J. Igusa: Some results on p-adic complex powers, *Am. J. Math.* **106** (1984) 1013–1032.

[K] N. Katz: Sommes exponentielles, *Astérisque* **79** (1980).

[SGA 4 $\frac{1}{2}$] P. Deligne: Cohomologie étale, *Lecture Notes in Math.* 569, Springer, Berlin, 1977

[SGA 7] A. Grothendieck, P. Deligne et N. Katz: Groupes de monodromie en géométrie algébrique, Vol. I, II, *Lecture Notes in Math.* 288, 340, Springer, Berlin, 1972, 1973.

[V1] W. Veys: Poles of Igusa's local zeta function and monodromy, *Bull. Soc. Math. France* (to appear).

[V2] W. Veys: Holomorphy of local zeta functions for curves, *Math. Ann.* **295** (1993) 635–641.