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1. Introduction

(1.1) Let $K$ be a $p$-adic field, i.e. $[K:Q_p] < \infty$. Let $R$ be the valuation ring of $K$, $P$ the maximal ideal of $R$, and $\bar{K} = R/P$ the residue field of $K$. The cardinality of $\bar{K}$ is denoted by $q$, thus $K = F_q$. Let $f(x) \in K[x]$, $x = (x_1, \ldots, x_n)$, $f \in K$. Igusa's local zeta function of $f$ with respect to a character $\chi: R^\times \to \mathbb{C}^\times$ and a Schwartz–Bruhat function $\Phi: K^n \to \mathbb{C}$ is denoted by

$$Z_\Phi(s, \chi) = Z_\Phi(s, \chi, K, f),$$

see e.g. [D3, §1.1], [D2]. When $\Phi$ is the characteristic function of the residue class $a \in \bar{K}^n$, we will write $Z_a(s, \chi)$ instead of $Z_\Phi(s, \chi)$. In this note we will always assume that $\chi$ is induced by a character $\chi: \bar{K}^\times \to \mathbb{C}^\times$.

In case of good reduction, we showed in [D1] (see also [D3, §4.1]) that

$$\deg Z_a(s, \chi) \leq 0 \quad \text{and} \quad \deg Z_a(s, \chi_{\text{triv}}) = 0,$$

where $\deg$ means the degree as rational function in $q^{-s}$ and $\chi_{\text{triv}}$ is the trivial character. (We put $\deg 0 = -\infty$.) In the present note we will prove the following theorem:

(1.2) THEOREM. If $f$ is defined over a number field $F \subset \mathbb{C}$, then for almost all completions $K$ of $F$ we have the following:

If $f(0) = 0$ and no eigenvalue of the (complex) local monodromy of $f$ at $0$ has the same order as $\chi$, then $\deg Z_0(s, \chi) < 0$.

With an eigenvalue of the (complex) local monodromy of $f$ at $a \in f^{-1}(0)$ we mean an eigenvalue of the action of the counter clockwise generator of the fundamental group of $\mathbb{C}\setminus\{0\}$ on the cohomology (in some dimension) of the Milnor fiber of $f$ at $a$ (see e.g. [A] or [D3, §2.1]). It is well known that such an eigenvalue is a root of unity so that we can talk about its order. Theorem 1.2 is a direct consequence of Theorem 1.4 below, whose statement requires some more notation.
(1.3) From now on we assume that $f \in R[x]$ and $\bar{f} \neq 0$, where $\bar{f}$ denotes the reduction mod $P$ of $f$. We fix a prime $\ell \neq q$ and an embedding of $\mathbb{C}$ into an algebraic closure $\mathbb{Q}_\ell^c$ of $\mathbb{Q}_\ell$. Thus we can consider $\chi: \bar{K}^\times \rightarrow (\mathbb{Q}_\ell^c)^\times$. This $\chi$ induces a character also denoted by $\chi$, of the geometric monodromy group of $\mathbb{A}^1_{\mathbb{F}_q}$ at 0, see 2.1. Let $F_0$ be the Milnor fibre of $\bar{f}$ at 0, in the sense of etale topology. We denote by $H'(F_0, \mathbb{Q}_\ell^c)^\times$ the component of the $\ell$-adic cohomology $H'(F_0, \mathbb{Q}_\ell^c)$ on which the local geometric monodromy group acts like $\chi$ times a unipotent action, see 2.3.1.

(1.4) THEOREM. Assume that $f^{-1}(0)$ has a resolution with tame good reduction mod $P$ (see 2.2.3 or [D3, 3.2]), and that $f(0) = 0$. Then

$$\lim_{s \to -\infty} Z_0(s, \chi) = (1 - q)q^{-n} \sum (-1)^i \text{Tr}(\sigma_1, H'(F_0, \mathbb{Q}_\ell^c)^\times),$$

where $\sigma_1$ is a suitable lifting of the geometric Frobenius (see 3.2).

Theorem 1.4 is proved in 3.3 using the method of vanishing cycles which we recall in 2.1 and 2.2. A partial converse of Theorem 1.4 is given in 3.4. In Section 4 we propose a conjecture about the holomorphy of $Z_0(s, \chi)$. Finally, Section 5 contains an alternative proof of some material in [D2].

2. Preliminaries

(2.1) Local monodromy

We choose a geometric generic point $\bar{\eta}$ of $\mathbb{A}^1_{\mathbb{F}_q}$. In particular this choice determines an algebraic closure $F_q^c$ of $F_q$. Let $S$, resp. $S_o$, be the Henselization at 0 of $\mathbb{A}^1_{\mathbb{F}_q}$ resp. $\mathbb{A}^1_{\mathbb{F}_q}$, and denote by $\eta$, resp. $\eta_o$, its generic point.

Put $G_0 = \text{Gal}(\bar{\eta}/\eta_o)$ and $I_0 = \text{Gal}(\bar{\eta}/\eta)$. The group $G_0$, resp. $I_0$, is called the arithmetical, resp. geometrical, local monodromy group of $\mathbb{A}^1_{\mathbb{F}_q}$ at 0. Via the cover

$$S_o \setminus \{0\} \rightarrow S_o \setminus \{0\}: X \mapsto X^{q-1},$$

with Galois group $F_q^c$, we consider $F_q^c$ as a quotient of $G_0$. Hence the character $\chi: F_q^c \rightarrow (\mathbb{Q}_\ell^c)^\times$ induces a homomorphism $\bar{\chi}: G_0 \rightarrow (\mathbb{Q}_\ell^c)^\times$. The restriction of this homomorphism $\bar{\chi}$ to $I_0$ will be denoted by $\chi: I_0 \rightarrow (\mathbb{Q}_\ell^c)^\times$. 

(2.2) Nearby cycles on the resolution space

(2.2.1) Let $h: Y \to X = \text{Spec} \ K[x]$ be an (embedded) resolution (of singularities) for $f^{-1}(0)$ over $K$ with good reduction mod $P$, see [D3, 1.3.1 and 3.2] or [D1]. Reduction mod $P$ is denoted by $\overline{\cdot}$, e.g. $\overline{Y}, \overline{E}_i$.

Let $E_i, i \in T$, be the irreducible components of $(f \circ h)^{-1}(0)$. Denote by $N_i$, resp. $\nu_i - 1$, the multiplicity of $E_i$ in the divisor of $f \circ h$, resp. $h^*(dx_1 \wedge \ldots \wedge dx_n)$. Put $\overline{E}_i = E_i \cup \bigcup_{j \in T} E_j$, $\overline{\overline{E}_i} = E_i \cup j \in T \overline{E}_j$, and $\overline{E}_I = \bigcap_{i \in I} \overline{E}_i$, $\overline{E}_I = \bigcup_{j \in \mathbb{N}} \overline{E}_j$ for any $I \subset T$. When $I = \emptyset$, put $\overline{E}_0 = \overline{Y}$.

(2.2.2) We denote by $R\Psi_f(C)$, resp. $R\Psi_{f \circ h}(C)$, the complex of nearby cycles on $\tilde{f}^{-1}(0) \otimes F_q^a$, resp. $(\tilde{f} \circ \tilde{h})^{-1}(0) \otimes F_q^a$, associated to a complex $C$, see [SGA 7, XIII]. To simplify notation, put $\tilde{\Psi}^i_f = R^i\Psi_f(Q^a_0)$ and $\tilde{\Psi}^i_{f \circ h} = R^i\Psi_{f \circ h}(Q^a_0)$. If $f(0) = 0$ then $(\tilde{\Psi}^i_f)_0 = H^i(F_0, Q^a_0)$, where $F_0$ denotes the Milnor fibre of $\tilde{f}$ at 0. It is well known [SGA 7, XIII 2.1.7.1] that

$$R\tilde{h}^* \circ R\Psi_{f \circ h} = R\Psi_f,$$

since $\tilde{h}$ is proper and birational. Thus, when $f(0) = 0$,

$$H^i(F_0, Q^a_0) = \pi^i((\tilde{h}^{-1}(0) \otimes F_q^a, R\Psi_{f \circ h}(Q^a_0)),$$  

and we have a spectral sequence

$$H^i(\tilde{h}^{-1}(0) \otimes F_q^a, \tilde{\Psi}^i_{f \circ h}) \Rightarrow H^{i+j}(F_0, Q^a_0).$$

(2.2.2.2)

Note that $G_0$ acts on all terms of this spectral sequence, by transport of structure (choice of $\eta$), and the spectral sequence is $G_0$-equivariant. We recall from [SGA 7, Exp. I Thm 3.3] the following basic facts:

For any $I \subset T$ with $I \neq \emptyset$ and any closed point $s \in \tilde{E}_I \otimes F_q^a$, there is a canonical isomorphism

$$(\tilde{\Psi}^i_{f \circ h})^\text{tame}_s \cong (\tilde{\Psi}^0_{f \circ h})^\text{tame}_s \otimes \wedge (M_I(-1)),$$

(2.2.2.3)

where $M_I$ is the dual of the kernel of the linear map $(Q^a_0)^I \to Q^a_0: (z_i)_{i \in I} \mapsto \sum_{i \in I} N_i z_i$, $M_I(-1)$ is a Tate twist of $M_I$, and the superscript \text{tame} denotes the tame part. Moreover

$$(\tilde{\Psi}^0_{f \circ h})^\text{tame}_s \cong (Q_0^a)^C_I,$$

(2.2.2.4)
with $C_i$ a finite set on which $I_0$ acts transitively, and $|C_i|$ equal to the largest common divisor of the $N_i$, $i \in I$, which is prime to $q$.

(2.2.3) Till the end of 2.2.3 we will assume now that the resolution $h$ has tame good reduction, i.e. it has good reduction and $N_i$ is prime to $q$ for each $i \in T$. Then it easily follows from [K, p. 180] that the action of $I_0$ on $\Psi_{f^*h}$ is tame.

A local calculation shows that the $\Psi_{f^*h}$ are lisse on $\tilde{E}_f \otimes F_q^a$ and that locally on $\tilde{E}_f \otimes F_q^a$ the isomorphisms 2.2.2.3 on the stalks are induced by an isomorphism of the sheaves. Since these isomorphisms are canonical they glue together to a canonical isomorphism

\[ \Psi_{f^*h}^i \cong \Psi_{f^*h}^0 \otimes^{f} (M_f(-1)) \text{ on } \tilde{E}_f \otimes F_q^a \]  

(2.2.3.1)

which is compatible with the action of $G_0$.

(2.3) Isotopic components

(2.3.1) For any constructible $Q^a_{\ell}$-sheaf $F$ (or vector space) on which $I_0$ acts, we denote by $F^\chi$ the $\chi$-unipotent part of $F$, i.e. the largest subsheaf on which $I_0$ acts like $\chi$ times a unipotent action.

(2.3.2) To the character $\chi: F^\chi \to (Q^a_{\ell})^\chi$ is associated the lisse rank one $Q^a_{\ell}$-sheaf $L_\chi$ on $A_{F_q}^1 \setminus \{0\}$, see [SGA 4^1/2, Sommes Trig.]. The action of the arithmetical monodromy group $G_0$ at 0 on $(L_\chi)^{\eta}$ is given by $\chi^{-1}$.

Let $\nu$ be the open immersion $\nu: \tilde{Y} \setminus (f^*h)^{-1}(0) \subset \subset \tilde{Y}$ and $\alpha: \tilde{Y} \setminus (f^*\tilde{h})^{-1}(0) \to A_{F_q}^1 \setminus \{0\}$ the restriction of $f^*\tilde{h}$. Put $F_\chi = \nu_* \alpha^* L_\chi$. The cohomology of this sheaf appears in the explicit formula for $Z_0(s, \chi)$, see 3.1.

(2.3.3) LEMMA. There is a canonical isomorphism

\[ F_\chi|_{(f^*h)^{-1}(0) \otimes F_q^a} \cong (\Psi_{f^*h}^0)^{\chi} \otimes (L_\chi)^{\eta}. \]

Proof. Because the action of $I_0$ on the stalks of the tame part of $\Psi_{f^*h}^0$ is semi-simple (cf. 2.2.2.4), $(\Psi_{f^*h}^0)^{\chi}$ equals the largest subsheaf of $\Psi_{f^*h}^0$ on which $I_0$ acts like $\chi$. Moreover there is a canonical isomorphism

\[ R\Psi_{f^*h}(\alpha^* L_\chi) \cong R\Psi_{f^*h}(L_\chi)^{\eta}. \]  

(2.3.3.1)
Thus it suffices to prove that there is a canonical isomorphism
\[ \mathcal{F}_X | (j \circ \tilde{h})^{-1}(0) \otimes F_q^a \cong (R^0\Psi_{j \circ \tilde{h}}(\alpha^* \mathcal{L}_X))^\delta, \] (2.3.3.2)
where the superscript $I_0$ denotes the largest subsheaf on which $I_0$ acts trivially.

We will denote by an index $S$ the base change $S \to \mathbb{A}_F^1$; for example $\tilde{Y}_S = \tilde{Y} \otimes \mathbb{A}_F^1 S$. Consider the following diagram of natural maps
\[
\begin{array}{ccccccccc}
\vdots & \downarrow & \tilde{Y}_S & \leftarrow & \nu_S & \downarrow & (\tilde{Y} \setminus (j \circ \tilde{h})^{-1}(0))_S & \leftarrow & (\tilde{Y}_S)_{\tilde{\eta}} \\
\{0\} & \downarrow & \tilde{S} & \leftarrow & \alpha_S & \downarrow & S \setminus \{0\} & \leftarrow & \eta & \tilde{\eta}
\end{array}
\]

Consider also the natural map $e : S \setminus \{0\} \to \mathbb{A}_F^1 \setminus \{0\}$. By [SGA 41, Th. finitude 1.9] we have
\[ \alpha_S^\delta \gamma_S(\gamma^* e^* \mathcal{L}_X) \equiv j_*(j^* \alpha_S^\delta e^* \mathcal{L}_X). \]

Hence
\[ i^*(\nu_S)_* \alpha_S^\delta \gamma_S(\gamma^* e^* \mathcal{L}_X) \equiv i^*(\nu_S)_* j_*(j^* \alpha_S^\delta e^* \mathcal{L}_X) = R^0\Psi_{j \circ \tilde{h}}(\alpha^* \mathcal{L}_X). \]

Taking $I_0$-invariants we get
\[ i^*(\nu_S)_* \alpha_S^\delta (\gamma^* e^* \mathcal{L}_X)^{I_0} \equiv (R^0\Psi_{j \circ \tilde{h}}(\alpha^* \mathcal{L}_X))^{I_0}. \]

But $e^* \mathcal{L}_X \cong (\gamma^* e^* \mathcal{L}_X)^{I_0}$, hence
\[ \mathcal{F}_X | (j \circ \tilde{h})^{-1}(0) \otimes F_q^a \cong i^*(\nu_S)_* \alpha_S^\delta e^* \mathcal{L}_X \cong (R^0\Psi_{j \circ \tilde{h}}(\alpha^* \mathcal{L}_X))^{I_0}. \]

\[ \square \]

3. Cohomological interpretation of $\operatorname{lim}_{s \to -\infty} Z_0(s, \chi)$

(3.1) Let $F \in \operatorname{Gal}(F_q^a/F_q)$ be the geometric Frobenius. We recall from [D2] that
\[ Z_0(s, \chi) = q^{-n} \sum_{t \in T} c_{t, X, 0} \prod_{i \in I} \frac{q - 1}{q^{N_i + \nu_i} - 1}, \] (3.1.1)
where
\[ c_{l,x,0} = \sum_{i} (-1)^i \text{Tr}(F, H'_c((\bar{\tilde{E}}_{l} \cap \tilde{h}^{-1}(0)) \otimes F_q^a, \tilde{\mathcal{F}}_x)). \] (3.1.2)

Hence
\[ \lim_{s \to -\infty} Z_0(s, \chi) = q^{-n} \sum_{l \in \mathcal{T}} c_{l,x,0}(1 - q)^{l_i}. \] (3.1.3)

(3.2) With a suitable lifting of the geometric Frobenius (mentioned in the statement of Theorem 1.4) we mean any element \( \sigma_1 \in G_0 \) which induces the geometric Frobenius on \( F_q^a \) and which acts trivially on \( (\mathcal{L}_x, \eta) \) (see 2.3.2).

(3.3) Proof of Theorem 1.4. We will prove that
\[ \frac{q^n}{1 - q} \lim_{s \to -\infty} Z_0(s, \chi) = \sum_{l} (-1)^l \text{Tr}(\sigma, H'(F_0, Q^a) \otimes (\mathcal{L}_x, \eta)), \] (3.3.1)

for any \( \sigma \in G_0 \) which induces the geometric Frobenius \( F \) on \( F_q^a \). This yields the theorem when we take for \( \sigma \) a suitable lifting \( \sigma_1 \) as in 3.2. The right-hand-side of (3.3.1) equals
\[
\sum_{i,j} (-1)^{i+j} \text{Tr}(\sigma, H'_i(\tilde{h}^{-1}(0) \otimes F_q^a, \Psi^i \otimes (\mathcal{L}_x, \eta)), \quad \text{by (2.2.2.2)},
\]
\[
= \sum_{i} \sum_{j} (-1)^{i+j} \text{Tr}(\sigma, H'_c((\bar{\tilde{E}}_{i} \cap \tilde{h}^{-1}(0)) \otimes F_q^a, (\Psi^i \otimes (\mathcal{L}_x, \eta)), \quad \text{by (2.2.3.1)},
\]
\[
= \sum_{i} \sum_{j} (-1)^{i+j} \text{Tr}(\sigma, H'_c((\bar{\tilde{E}}_{i} \cap \tilde{h}^{-1}(0)) \otimes F_q^a, (\Psi^0 \otimes (\mathcal{L}_x, \eta)), \quad \text{by (2.3.3)},
\]
\[
= \sum_{i} \sum_{j} (-1)^{i+j} \text{Tr}(F, H'_c((\bar{\tilde{E}}_{i} \cap \tilde{h}^{-1}(0)) \otimes F_q^a, \tilde{\mathcal{F}}_x)) \cdot \text{Tr}(F, \otimes (\mathcal{L}_x, \eta), \quad \text{by (3.1.2)},
\]
\[
= \sum_{l} c_{l,x,0}(1 - q)^{l_i-1}, \quad \text{by (3.1.2)}.\]
Combining this with 3.1.3 proves 3.3.1 and finishes the proof of Theorem 1.4. □

We now turn to a partial converse of Theorem 1.4. For any finite extension $L$ of the field $K$, the norm from $L$ to $K$ is denoted by $N_{L/K}$.

(3.4) PROPOSITION. If $f$ is defined over a number field $F \subset \mathbb{C}$, then for almost all completions $K$ of $F$ we have the following: Assume the order of $\chi$ equals the order of some eigenvalue of the (complex) local monodromy of $f$ at some complex point of $f^{-1}(0)$. Then there are infinitely many unramified extensions $L$ of $K$ such that $\deg Z_{\lambda}(s, \chi \circ N_{L/K}, L, f) = 0$ for some integral $a \in L^n$ with $f(a) = 0$.

Proof. It is well known [B] that $R\Psi_{f}(Q_{a})[n - 1]$ is a perverse sheaf. Let $C := (R\Psi_{f}(Q_{a})[n - 1])^{X}$ be the maximal subobject (in the category of perverse sheaves) on which $I_{0}$ acts like $\chi$ times a unipotent action. We have $C \neq 0$ (for almost all completions $K$ of $F$). Since $C$ is perverse, there exists a geometric point $\bar{a}$ of $f^{-1}(0)$ such that $(H^{i}(C))_{\bar{a}} \neq 0$ for exactly one $i$. The proposition follows now easily from 1.4. □

(3.5) Example. Let $f(x_1, x_2) = x_2^2 - x_1^3$. Then the orders of the eigenvalues of the local monodromy are 1 and 6. Thus, for almost all completions, $\deg Z_{\lambda}(s, \chi) < 0$ if $\chi$ has order 2 and 3. (Compare with Proposition 4.5).

4. Holomorphy of $Z_{\Phi}(s, \chi)$

(4.1) We call a Schwartz–Bruhat function $\Phi$ on $K^n$ residual if $\Phi$ is zero outside $R^n$ and $\Phi(x)$ only depends on $x \mod P$.

(4.2) It is well known (see [11] or [D3, 1.3.2]) that $Z_{\Phi}(s, \chi)$ is holomorphic on $\mathbb{C}$ when the order of $\chi$ divides no $N_{i}$. The $N_{i}$ are not intrinsic, but the order of any eigenvalue of the local monodromy on $f^{-1}(0)$ divides some $N_{i}$ (this follows from 2.2.2.2, 2.2.2.3 and 2.2.2.4). Being very optimistic, we propose the following conjecture:

(4.3) CONJECTURE. If $f$ is defined over a number field $F \subset \mathbb{C}$, then for almost all completions $K$ of $F$ we have the following: when $\Phi$ is residual, $Z_{\Phi}(s, \chi)$ is holomorphic unless the order of $\chi$ divides the order of some eigenvalue of the (complex) local monodromy of $f$ at some complex point of $f^{-1}(0)$.

In fact, this might be true for all $p$-adic completions $K$ of $F$ and for any $\Phi$. Veys [V2] verified this when $f$ has only two variables. Moreover the author showed that the conjecture is true for the relative invariants of a few pre-
homogeneous vector spaces (using Theorem 2 of [12] and the orbital decomposition).

(4.4) Remark. Suppose $f$ is homogeneous. Then for almost all completions $K$ of $F$ we have the following: If $Z(s, \chi)$ is holomorphic, then $Z(s, \chi) = 0$, since $\deg Z(s, \chi) < 0$ (see [D3, 4.1]). For $s = +\infty$ this yields that

$$S := \sum_{x \in (F_0)^n, \tilde{f}(x) \neq 0} \chi(\tilde{f}(x))$$

is zero when $Z(s, \chi)$ is holomorphic. Thus conjecture 4.3 implies a relation between the vanishing of the character sum $S$ and monodromy. However this relation follows directly from the formula

$$S = (q - 1)q^{n-1} \sum_i (-1)^i \text{Tr}(\sigma_i^{-1}, H^i(F_0, Q_\omega^a)^{x^{-1}}),$$

which is easily proved by standard methods.

The following proposition is a partial converse of Conjecture 4.3.

(4.5) PROPOSITION. If $f$ is defined over a number field $F \subset \mathbb{C}$, then for almost all completions $K$ of $F$ we have the following: If the order of $\chi$ divides the order of some eigenvalue of the (complex) local monodromy of $f$ at some complex point of $f^{-1}(0)$, then for infinitely many unramified extensions $L$ of $K$, $Z_\Phi(s, \chi^N_{L/K}, L, f)$ is not holomorphic on $\mathbb{C}$ for some residual $\Phi$.

I first proved this proposition in the isolated singularity case, see [D3, prop. 4.4.3]. However that proof generalizes directly to the general case, because of Lemma 4.6 below. Indeed by 4.6 and the hypothesis of 4.5 there exists $a \in f^{-1}(0)$ such that the order $d$ of $\chi$ divides the order $k$ of some reciprocal zero or reciprocal pole of the monodromy zeta function of $f$ at $a$. Hence by A'Campo [A, Thm 3], we have $\sum_{k|N_i} \chi(E_i \cap h^{-1}(a)) \neq 0$. Proceeding now as in my proof of Proposition 4.4.3 of [D3], with $Z_\Phi$ replaced by $Z_\alpha$, we obtain that $Z_\alpha(s, \chi^N_{L/K}, L, f)$ is not holomorphic for infinitely many $L$.

(4.6) LEMMA. Let $f(x) \in \mathbb{C}[x]$, $x = (x_1, \ldots, x_n)$, $f \in \mathbb{C}$. If $\lambda$ is an eigenvalue of the (complex) local monodromy of $f$ at $b \in f^{-1}(0)$, then there exists $a \in f^{-1}(0)$ such that $\lambda$ is a reciprocal zero or reciprocal pole of the monodromy zeta function of $f$ at $a$ (in the sense of [A, p. 233]).

Proof. It is well known [B] that $R\Psi f \mathbb{C}[n - 1]$ is a perverse sheaf. Let $C$ be the maximal subobject (in the category of perverse sheaves) on which the (complex) local monodromy acts like $\lambda$ times a unipotent endomorphism. The hypothesis of the Lemma implies that $C \neq 0$. Since $C$ is perverse, there
exists \( a \in \mathcal{V}^{-1}(0) \) such that \((H^i(C))_a \neq 0\) for exactly one \( i \). The lemma follows now easily. \( \square \)

5. An alternative proof of some material in [D2]

In [D2, Thm. 1.1] we proved that certain \( E_i \) do not contribute to poles of \( Z(s, \chi) \), see also [D3, 4.6]. The proof was based on the following key Lemma 5.1, for which we will now give an alternative proof.

(5.1) LEMMA. [D2, 4.1]. Assume the notation of 2.2.1. and 2.3.2. Let \( \chi \) be a character of \( \overline{K}^\times \) of order \( d \), and \( i_0 \in T \). Suppose \( E_{i_0} \) is proper, \( d|N_{i_0} \), and \( E_{i_0} \) intersects no \( E_j \) with \( d|N_j, j \neq i_0 \). Then

\[
H^i_c(\mathcal{E}_{i_0} \otimes F^a_q, \mathcal{F}_\chi) = 0 \quad \text{for all } i \neq n - 1.
\]

(5.2) An alternative proof for Lemma 5.1. A local calculation, using the hypothesis of the Lemma, shows that for every closed point \( s \in \mathcal{E}_{i_0}\backslash \hat{\mathcal{E}}_{i_0} \), the local monodromy of \( \mathcal{F}_\chi|_{\mathcal{E}_{i_0}} \) at \( s \) has no invariants. Hence by [SGA 4\_2, Sommes Trig. 1.19.1] and tame ramification, we have

\[
H^i_c(\mathcal{E}_{i_0} \otimes F^a_q, \mathcal{F}_\chi) = H^i(\mathcal{E}_{i_0} \otimes F^a_q, \mathcal{F}_\chi), \quad \text{for all } i.
\]

Thus by Poincaré duality we only have to prove the Lemma for \( i > n - 1 \). Because \( \mathcal{E}_{i_0} \) is proper, \( \mathcal{h}(\mathcal{E}_{i_0}) \) is finite. Hence we may assume that \( \mathcal{h}(\mathcal{E}_{i_0}) = \{0\} \). We claim that

\[
H^i_c(\mathcal{E}_{i_0} \otimes F^a_q, \mathcal{F}_\chi) \subset (\Psi^i_{\mathcal{h}})^0 \otimes (\mathcal{L}_\chi)^{\mathcal{h}}, \quad \text{for all } i. \tag{5.2.1}
\]

This claim proves the Lemma since it is well known that \( \Psi^i_{\mathcal{h}} = 0 \) when \( i > n - 1 \), see [SGA 7, Exp. I Th. 4.2].

From 2.2.2.3, 2.2.2.4 and the hypothesis of the Lemma, it follows that

\[
(\Psi^i_{\mathcal{h}})^0 = 0 \tag{5.2.2}
\]

for any closed point \( s \in \mathcal{E}_{i_0}\backslash \hat{\mathcal{E}}_{i_0} \) and \( i \geq 0 \), and also for any closed point \( s \in \hat{\mathcal{E}}_{i_0} \) and \( i \geq 1 \). (Indeed the \( \chi \)-unipotent part is contained in the tame part.) Thus applying the Mayer–Vietoris sequence for \( \mathcal{h}^{-1}(0) = \mathcal{E}_{i_0} \cup (\mathcal{h}^{-1}(0)\backslash \hat{\mathcal{E}}_{i_0}) \) and the spectral sequence of hypercohomology we obtain

\[
\mathbb{H}^i(\mathcal{h}^{-1}(0) \otimes F^a_q, R\Psi_{f^*\mathcal{h}}(Q^a)) = \mathbb{H}^i(\mathcal{E}_{i_0} \otimes F^a_q, R\Psi_{f^*\mathcal{h}}(Q^a)) \oplus
\]
Together with 2.2.2.1 this yields
\begin{equation}
\oplus H^i((\tilde{h}^{-1}(0)\backslash \hat{E}_q) \otimes F_q^a, R\psi_{f^+\hat{h}}(Q^a)) \chi.
\end{equation}

Again by 5.2.2 we have
\begin{equation}
H^i(\hat{E}_q \otimes F_q^a, R\psi_{f^+\hat{h}}(Q^a)) \chi \subset (\psi_i^a)^0, \quad \text{for all } i. \quad (5.2.3)
\end{equation}

by degeneration of (the $\chi$-unipotent part of) the spectral sequence of hypercohomology. The claim 5.2.1 follows now from 5.2.3, 5.2.4 and Lemma 2.3.3. This terminates the proof of Lemma 5.1. \qed

References


