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Convexity for invariant differential operators on semisimple symmetric spaces

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Introduction

Let $X = G/H$ be a homogeneous space of a Lie group G, and let D: $C^\infty(X) \to C^\infty(X)$ be a non-trivial G-invariant differential operator. One of the natural questions one can ask for the operator D is whether it is *solvable*, in the sense that $DC^\infty(X) = C^\infty(X)$. If G is the group of translations of $X = \mathbf{R}^n$ and H is trivial, then D has constant coefficients, and it is a well known result of Ehrenpreis and Malgrange that hence D is solvable.

Assume for simplicity that G/H carries an invariant measure. This measure induces a bilinear pairing of $C_c^\infty(X)$, the space of compactly supported smooth functions on X, with itself. Let D^* denote the adjoint of D with respect to this pairing. The strategy employed by Ehrenpreis and Malgrange was essentially to use the following properties of D:

(i) There exists a *fundamental solution* for D, that is, $\delta \in D\mathscr{D}'(X)^H$, where δ is the Dirac measure at the origin, and $\mathscr{D}'(X)^H$ is the space of left-H-invariant distributions on X.

(ii) For each compact set $\Omega \subset X$ there exists a compact set $\Omega' \subset X$ such that

$$\operatorname{supp} D^*f \subset \Omega \Rightarrow \operatorname{supp} f \subset \Omega'$$

for all $f \in C_c^\infty(X)$.

In fact, for $X = \mathbf{R}^n$ one can take as Ω' the convex hull of Ω. For this reason the support property (ii) has become known as the *D-convexity of X*. It follows from (i)–(ii) that D is solvable.

The strategy has been applied in other cases as well, for example by Helgason in [14], where surjectivity is established for all non-trivial invariant differential operators on a Riemannian symmetric space. In a variant of the strategy (i) is replaced by the following weaker property (*semi-global solvability*):

(i′) For each compact set $\Omega \subset X$ and each function $g \in C^\infty(X)$ there exists a function $f \in C^\infty(X)$ such that $Df = g$ on Ω.

The conjunction of (i′) and (ii) is equivalent with the solvability of D (see Theorem 1). This is used by Rauch and Wigner in [19] where it is proved that the Casimir operator on a semisimple Lie group is solvable, and more generally by Chang in [6] where the Laplace-Beltrami operator on a semisimple symmetric space is shown to be solvable.

The purpose of the present paper is to give, also for a semisimple symmetric space $X = G/H$, a sufficient condition on an invariant differential operator D to imply (ii), the D-convexity of X. When G/H has rank one, our result follows from the above mentioned result of Chang, since the algebra $\mathbb{D}(G/H)$ of all invariant differential operators in this case is generated by the Laplace-Beltrami operator. In general this is not so, and our result shows the D-convexity for a significantly larger class of operators D. In particular, when G/H is *split* (that is, it has a vectorial Cartan subspace), all non-trivial elements of $\mathbb{D}(G/H)$ satisfy our condition.

Though we do not consider the properties (i) or (i′) in this paper, we notice that in the above-mentioned references, an important step towards obtaining (i′) is to prove that D^* acts injectively on, say $C_c^\infty(X)$ (see for example [6]). In fact the injectivity of D^* is an immediate consequence of (i′). In the present case of a semisimple symmetric space, the sufficient condition that we give for (ii) is also sufficient for D^* to be injective.

We also give a condition on D, which is necessary for both the D-convexity and the injectivity. When G/H is not split, there exists a non-trivial operator in $\mathbb{D}(G/H)$, which does not satisfy this condition. In particular, we conclude that D-convexity holds for all non-trivial elements of $\mathbb{D}(G/H)$ if and only if G/H is split. This provides a large class of spaces G/H for which there exist non-solvable non-trivial invariant differential operators. Unfortunately, our necessary condition is weaker than the sufficient condition, and the complete classification of all $D \in \mathbb{D}(G/H)$, for which D-convexity holds, remains open (for non-split G/H).

In the special case where the semisimple symmetric space is Riemannian (that is, when H is compact), we have that G/H is split and thus our condition reduces to the requirement that D is non-trivial. In this case our result is part of the above-mentioned proof by Helgason that D is surjective (see [14, p. 473]). Helgason's proof is based on his inversion formula and Paley-Wiener theorem for the Fourier transform on the Riemannian symmetric space X. These results in turn rely heavily on the work of Harish-Chandra. Simplifications avoiding these strong tools were given by Chang [7] and Dadok [8]. In another special case, that of a semisimple Lie group considered as a symmetric space, our result was obtained by Duflo and Wigner [9].

All of the references mentioned above, except [14], use the uniqueness theorem of Holmgren to derive the D-convexity of X, and so do we. The main difficulty in the present generalization lies in the handling of the more complicated geometry of X. Our main tool to overcome this difficulty is the convexity theorem of [1].

In [3] (see also [4]) the result of the present paper will be applied to obtain injectivity of the *Fourier transform* on $C_c^\infty(X)$. Our reasoning will thus be the opposite of the original reasoning of Helgason in the Riemannian case: we shall *deduce* properties of the Fourier transform from the D-convexity.

Motivation

As mentioned in the introduction the main motivation for studying D-convexity is the following theorem. Here G is a Lie group (with at most countably many connected components) and H is a closed subgroup, of which we only assume that G/H carries an invariant measure (this assumption is only used for defining D^*).

THEOREM 1. *Let $D \in \mathbb{D}(G/H)$ be an invariant differential operator. Then D is solvable if and only if* (i′) *and* (ii) *hold.*

Proof. This follows from [22, Ch. I, Thm. 3.3], using regularization by $C_c^\infty(G)$ to prove the equivalence of our definition of D-convexity with that of [22, Ch. I, Def. 3.1]. Note also the final remark of that section in loc. cit. $\square$

Notation

From now on, let G be a real reductive Lie group of Harish-Chandra's class, τ an involution of G, and H an open subgroup of the fixed point group G^τ. Then $X = G/H$ is a reductive symmetric space of Harish-Chandra's class (see [2]). Let K be a τ-stable maximal compact subgroup of G, and let θ be the associated Cartan involution. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the eigen-decompositions of the Lie algebra $\mathfrak{g}$ induced by τ and θ, then $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of H and K, respectively. Let B be a non-degenerate, G- and τ-invariant bilinear form on $\mathfrak{g}$ which extends the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, and which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Then the above-mentioned eigen-decompositions are orthogonal with respect to B.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$, and a maximal abelian subspace (a *Cartan subspace*) $\mathfrak{a}_1$ of $\mathfrak{q}$, containing $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{p}$. Let $\mathfrak{m}$ be the orthocomplement (with respect to B) of $\mathfrak{a}$ in its centralizer $\mathfrak{g}^\mathfrak{a}$, and let $\mathfrak{a}_m = \mathfrak{a}_1 \cap \mathfrak{m}$. Via the orthogonal decomposition $\mathfrak{a}_1 = \mathfrak{a}_m + \mathfrak{a}$ we view $\mathfrak{a}_{mc}^*$ and

$\mathfrak{a}_c^*$ as subspaces of $\mathfrak{a}_{1c}^*$. Let Σ and Σ_1 denote the root systems of $\mathfrak{a}$ and $\mathfrak{a}_1$ in $\mathfrak{g}_c$, respectively, then Σ consists of the non-trivial restrictions to $\mathfrak{a}$ of the elements of Σ_1. Denote by W and W_1 the Weyl groups of these two roots systems, then W is naturally isomorphic to $N_{W_1}(\mathfrak{a})/Z_{W_1}(\mathfrak{a})$, the normalizer modulo the centralizer of $\mathfrak{a}$ in W_1, and to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, the normalizer modulo the centralizer of $\mathfrak{a}$ in K. Let $W_{K\cap H}$ be the canonical image of $N_{K\cap H}(\mathfrak{a})$ in W.

Recall that $G=KAH$, and that if $g=kah$ according to this decomposition, then the orbit $W_{K\cap H}\log a$ is uniquely determined by g. For a $W_{K\cap H}$-invariant set $S\subset\mathfrak{a}$, we denote the subset $K\exp(S)H$ of X by X_S. Then $S=\{\log a\,|\,aH\in X_S\}$, and every K-invariant subset of X is of the form X_S.

Invariant differential operators

Let $\mathbb{D}(G/H)$ be the algebra of invariant differential operators on G/H. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_c$ and $U(\mathfrak{g})^H$ the subalgebra of H-invariant elements, then there is a natural isomorphism of the quotient $U(\mathfrak{g})^H/(U(\mathfrak{g})^H\cap U(\mathfrak{g})\mathfrak{h}_c)$ with $\mathbb{D}(G/H)$, induced by the right action R of $U(\mathfrak{g})$ on $C^\infty(G)$ (see [15, p. 285]).

Let Σ_1^+ be a positive system for Σ_1, and let $\mathfrak{n}_1$ be the sum of the corresponding positive root spaces $\mathfrak{g}_c^\alpha$ ($\alpha\in\Sigma_1^+$). We have the following direct sum decomposition

$$\mathfrak{g}_c=\mathfrak{n}_1+\mathfrak{a}_{1c}+\mathfrak{h}_c. \tag{1}$$

Using this decomposition and Poincare-Birkhoff-Witt, a map ${}'\gamma: U(\mathfrak{g})\to U(\mathfrak{a}_1)$ is defined by $u\equiv{}'\gamma(u)$ modulo $\mathfrak{n}_1U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{h}_c$. From this map an algebra isomorphism γ of $\mathbb{D}(G/H)\simeq U(\mathfrak{g})^H/(U(\mathfrak{g})^H\cap U(\mathfrak{g})\mathfrak{h}_c)$ onto $S(\mathfrak{a}_1)^{W_1}$, the set of W_1-invariant elements in the symmetric algebra of $\mathfrak{a}_{1c}$ (which is isomorphic to $U(\mathfrak{a}_1)$ because $\mathfrak{a}_1$ is abelian), is obtained by letting $\gamma(u)(\lambda)={}'\gamma(u)(\lambda+\rho_1)$ for $u\in U(\mathfrak{g})^H$, $\lambda\in\mathfrak{a}_{1c}^*$ (see [11, p. 15, Thm. 3]). Here $\rho_1\in\mathfrak{a}_{1c}^*$ is given by half the trace of the adjoint action on $\mathfrak{n}_1$. Thus $\mathbb{D}(G/H)$ is identified as a polynomial algebra with $\dim\mathfrak{a}_1$ independent generators.

Assume that Σ_1^+ is chosen to be compatible with $\mathfrak{a}$, that is, the set of nonzero restrictions to $\mathfrak{a}$ of elements from Σ_1^+ is a positive system Σ^+ for Σ. Let $\mathfrak{n}$ be the sum of the corresponding positive root spaces $\mathfrak{g}^\alpha$ ($\alpha\in\Sigma^+$), then we also have the following direct sum decomposition

$$\mathfrak{g}=\mathfrak{n}+\mathfrak{m}+\mathfrak{a}+\mathfrak{h}. \tag{2}$$

Let $\rho\in\mathfrak{a}^*$ and $\rho_m\in\mathfrak{a}_{mc}^*$ be given by half the trace of the adjoint actions on $\mathfrak{n}$, and on $\mathfrak{n}_1\cap\mathfrak{m}_c$, respectively.

Using the decomposition (2) a map $`\eta: U(\mathfrak{g}) \to U(\mathfrak{a})$ is defined by $u \equiv `\eta(u)$ modulo $(\mathfrak{n}_c + \mathfrak{m}_c)U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}_c$, and we obtain by restriction to $U(\mathfrak{g})^H$ a homomorphism, also denoted $`\eta$, from $\mathbb{D}(G/H) \simeq U(\mathfrak{g})^H/(U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}_c)$ into $S(\mathfrak{a})$. Let $\eta(D) \in S(\mathfrak{a})$ be defined by $\eta(D)(\lambda) = `\eta(D)(\lambda + \rho)$.

LEMMA 1. *We have*

$$\eta(D)(\lambda) = \gamma(D)(\lambda - \rho_m) \tag{3}$$

for all $D \in (G/H)$, $\lambda \in \mathfrak{a}_c^*$. *Moreover* $\eta(D) \in S(\mathfrak{a})^W$, *and* $\eta(D)$ *is independent of the choice of* Σ^+.

Proof. We first prove the following equation:

$$\rho_1 = \rho + \rho_m. \tag{4}$$

We have

$$\rho_1 = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+} (\dim \mathfrak{g}_c^\alpha)\alpha \quad \text{and} \quad \rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_\mathfrak{a} = 0} (\dim \mathfrak{g}_c^\alpha)\alpha.$$

Let

$$\bar{\rho} = \rho_1 - \rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_\mathfrak{a} \neq 0} (\dim \mathfrak{g}_c^\alpha)\alpha,$$

then it is clear that $\bar{\rho} = \rho$ on $\mathfrak{a}$. On the other hand, since the set of $\alpha \in \Sigma_1^+$ with $\alpha|_\mathfrak{a} \neq 0$ is $\sigma\theta$-invariant, we get that $\sigma\theta\bar{\rho} = \bar{\rho}$, and hence $\bar{\rho} = 0$ on $\mathfrak{a}_m$, so that in fact $\bar{\rho} = \rho$.

Since $\mathfrak{m}_c = \mathfrak{m}_c \cap \mathfrak{n}_1 + \mathfrak{a}_{mc} + \mathfrak{m}_c \cap \mathfrak{h}_c$ it follows from (1) and (2) that $`\eta(D)(\lambda) = `\gamma(D)(\lambda)$. From this and (4) we get (3).

The proof will be completed by using the following observation: Every element $w \in W$ can be represented by an element $\bar{w} \in N_{W_1}(\mathfrak{a})$; this element also normalizes $\mathfrak{a}_m$, and can be chosen so that $\bar{w}\rho_m = \rho_m$.

The W-invariance of $\eta(D)$ now follows from (3) and the W_1-invariance of $\gamma(D)$, in view of the above observation. By using this observation once more, it follows from (3) and the fact that γ is independent of the choice of the positive system Σ_1^+, that η is independent of the choice of Σ^+. □

Let $s: S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map, then the restriction of s to the set $S(\mathfrak{q})^H$ of H-invariants in $S(\mathfrak{q})$ gives rise to a linear bijection (also denoted by s) of $S(\mathfrak{q})^H$ with $\mathbb{D}(G/H)$ (see [15, p. 287, Thm. 4.9]). A differential operator $D \in \mathbb{D}(G/H)$ is called *homogeneous* if it is the image of a homogeneous element of $S(\mathfrak{q})^H$. For $P \in S(\mathfrak{q})^H$ let $r(P) \in S(\mathfrak{a})$ denote the restriction of P to $\mathfrak{a}$. Here P is identified with a polynomial on $\mathfrak{q}$ by means of the Killing form.

LEMMA 2. *Let $D \in \mathbb{D}(G/H)$ be non-constant and let $D = s(P)$, $P \in S(\mathfrak{q})^H$. Then*

$$\deg(\eta(D) - r(P)) < \deg P = \text{order } D. \tag{5}$$

In particular, if D is homogeneous then $\deg \eta(D) = \text{order } D$ *if and only if* $r(P) \neq 0$.

Proof. That order $D = \deg P$ follows from the explicit expression for $s(P)$ in [15, p. 287, Thm. 4.9]. Let $r_1(P)$ denote the restriction of P to $\mathfrak{a}_1$, then it follows from [15, p. 305, Eq. (38)] that

$$\deg(\gamma(D) - r_1(P)) < \deg P. \tag{6}$$

It follows from (3) that $\eta(D) - r(P)$ and the restriction of $\gamma(D) - r_1(P)$ to $\mathfrak{a}$ have the same degree, and hence (5) follows from (6). If P is homogeneous, then either $\deg r(P) = \deg P$ or $r(P) = 0$, and the final statement follows from (5). □

Notice that $r_1(P)$ has the same degree as P (to see this, let P be homogeneous, then $\deg r_1(P) = \deg P$ unless $r_1(P) = 0$. But $r_1(P) = 0$ implies $P = 0$ by the H-invariance, because $\mathrm{Ad}(H)(\mathfrak{a}_1)$ contains an open subset of $\mathfrak{q}$). Hence it follows from (6) that also $\gamma(D)$ has this degree (which equals the order of D). Thus γ is a degree preserving isomorphism of $\mathbb{D}(G/H)$ onto $S(\mathfrak{a}_1)^{W_1}$.

However, a similar statement is not valid for $\eta(D)$; its degree can be strictly smaller than that of D. In fact η is not injective in general: Since $\mathbb{D}(G/H)$ and $S(\mathfrak{a})^W$ are polynomial algebras in $\dim \mathfrak{a}_1$ and $\dim \mathfrak{a}$ algebraically independent generators, respectively, η is not injective if $\mathfrak{a} \neq \mathfrak{a}_1$ (otherwise it would cause the existence of an injection of the quotient field of $\mathbb{D}(G/H)$ into the quotient field of $S(\mathfrak{a})^W$, which is impossible, since their transcendence degrees over $\mathbf{C}$ are $\dim \mathfrak{a}_1$ and $\dim \mathfrak{a}$, respectively (see [23, Ch. II, §12])). On the other hand, if $\mathfrak{a}_1 = \mathfrak{a}$, in which case the symmetric space G/H is called *split*, then η is injective since it equals γ. Examples of split symmetric spaces are the Riemannian symmetric spaces and the symmetric spaces of K_ε-type (see [18]). In the special case (the 'group case') of a semisimple Lie group G' considered as a symmetric space, where G is $G' \times G'$ and H is the diagonal, the notion of split for the space G/H coincides with the notion of split (also called a normal real form) for G'.

Notice also that η in general is not surjective. This can be seen already in the group case mentioned above, where $\mathbb{D}(G/H)$ is naturally isomorphic with $Z(\mathfrak{g}')$, the center of $U(\mathfrak{g}')$, and where η by transference under a suitable isomorphism can be identified with the natural homomorphism of $Z(\mathfrak{g}')$ into $\mathbb{D}(G'/K')$. It is known from [13, 16] that this homomorphism is surjective when G' is classical, but not surjective for certain exceptional groups G'.

For $v \in S(\mathfrak{a}_1)$ or $v \in S(\mathfrak{a})$ we define v^* *by* $v^*(\nu) = v(-\nu)$, where $\nu \in \mathfrak{a}_{1c}^*$ or $\nu \in \mathfrak{a}_c^*$.

LEMMA 3. *Let $D \in \mathbb{D}(G/H)$. Then $\gamma(D^*) = \gamma(D)^*$ and $\eta(D^*) = \eta(D)^*$.*

Proof. Choose $u \in U(\mathfrak{g})^H$ such that $D = R_u$, and let $v \mapsto \check{v}$ be the antiautomorphism of $U(\mathfrak{g})$ determined by $\check{v} = -v$ for $v \in \mathfrak{g}$. Using [15, Ch. I, Thm. 1.9 and Lemma 1.10] it is easily seen that $D^* = R_{\check{u}}$. The equality for γ will follow if we prove that $\gamma(\check{u}) = \gamma(u)^*$ for $u \in U(\mathfrak{g})^H$. Using [11, p. 16, Cor. 4] it is now seen that it suffices to consider the case of a Riemannian symmetric space, that is, we may assume that H is compact. In this special case, the statement is proved in [15, p. 307]. This proves that $\gamma(D^*) = \gamma(D)^*$.

From (3) we now get that

$$\eta(D^*)(\lambda) = \gamma(D^*)(\lambda - \rho_m) = \gamma(D)(-\lambda + \rho_m).$$

Using the fact that there exists an element w in the Weyl group of the root system of $\mathfrak{a}_m$ in $\mathfrak{m}$ such that $w\rho_m = -\rho_m$, and that this Weyl group is a subgroup of W_1, we get that

$$\gamma(D)(-\lambda + \rho_m) = \gamma(D)(-\lambda - \rho_m) = \eta(D)(-\lambda),$$

proving the equality for η. □

In the final section of this paper we relate $\eta(D)$ to the *radial part* of D with respect to the KAH decomposition. In particular we shall prove that the condition $\eta(D) = 0$ has the following strong consequence:

LEMMA 4. *Let $D \in \mathbb{D}(G/H)$ and assume that $\eta(D) = 0$. Then $Df = 0$ for all K-invariant smooth functions f on G/H.*

Convexity

We are now ready to state our main theorem:

THEOREM 2. *Let $D \in \mathbb{D}(G/H)$ be non-zero.*

(i) *If* $\deg \eta(D) = \text{order } D$ then

$$\operatorname{supp} f \subset X_S \Leftrightarrow \operatorname{supp} Df \subset X_S \Leftrightarrow \operatorname{supp} D^*f \subset X_S$$

for all $f \in C_c^\infty(X)$ and all convex, compact $W_{K \cap H}$-invariant sets $S \subset \mathfrak{a}$. In particular, X is D-convex, and D^* is injective on $C_c^\infty(X)$.

(ii) *If $\eta(D) = 0$ there exists for each closed ball $S \subset \mathfrak{a}$, centered at the origin, a function $f \in C_c^\infty(X)$ such that $D^*f = 0$ and* $\operatorname{supp} f = X_S$. *In particular, X is not D-convex, and D^* is not injective on $C_c^\infty(X)$.*

Proof. We first prove (i). The implication of $\operatorname{supp} Df \subset X_S$ from $\operatorname{supp} f \subset X_S$ is obvious. Assume $\operatorname{supp} Df \subset X_S$. Expanding f as a sum of K-finite functions, we have, since X_S is K-invariant, that f is supported in X_S if and only if all the summands are supported in X_S. Moreover, D can be applied termwise to the sum, and hence we see that we may assume f to be K-finite. Then the support of f is K-invariant, and it suffices to prove that $\operatorname{supp} f \cap AH \subset \exp(S)H$.

Let $m = \text{order } D$, then $m = \deg \eta(D)$ by the assumption on D. Let u_0 denote the homogeneous part of $\eta(D)$ of degree m, then $u_0 \neq 0$. Notice that u_0 is also the homogeneous part of $`\eta(D)$ of degree $m = \deg `\eta(D)$ for any choice of Σ^+.

Assume that $\operatorname{supp} f \cap AH \not\subset \exp(S)H$, and write

$$\operatorname{supp}_{\mathfrak{a}} f = \{Y \in \mathfrak{a} \mid \exp(Y)H \in \operatorname{supp} f\}.$$

Then $\operatorname{supp}_{\mathfrak{a}} f$ is compact and not contained in S. By the convexity of S there exists a non-empty open set of linear forms $\lambda \in \mathfrak{a}^*$ with the property that

$$0 < \max_{Y \in S} \lambda(Y) < \max_{Y \in \operatorname{supp}_{\mathfrak{a}} f} \lambda(Y). \tag{7}$$

Since $u_0 \neq 0$ there exists a $\lambda \in \mathfrak{a}^*$ with $u_0(\lambda) \neq 0$, and satisfying (7). Let $Y_0 \in \operatorname{supp}_{\mathfrak{a}} f$ be a point where the value on the right side of (7) is attained. Then $Y_0 \notin S$ and we have that

$$\lambda(Y) \leqslant \lambda(Y_0), \quad (Y \in \operatorname{supp}_{\mathfrak{a}} f). \tag{8}$$

Let $a_0 = \exp Y_0$, then

$$a_0 H \notin \operatorname{supp} Df \tag{9}$$

by the assumption on $\operatorname{supp} Df$, and

$$a_0 H \in \operatorname{supp} f. \tag{10}$$

Choose a positive system Σ^+ such that λ is antidominant, and let $\mathfrak{n}$ and N be given correspondingly. Let Ω denote the open (see [21, Prop. 7.1.8]) subset $\Omega = NMAH$ of $X = G/H$, and define $g \in C^\infty(\Omega)$ by $g(nmaH) = \lambda(\log a)$ for $n \in N$, $m \in M$, $a \in A$. We claim that

$$f = 0 \quad \text{on } \{x \in \Omega \mid g(x) > g(a_0)\}. \tag{11}$$

To prove (11) let $x = nmaH \in \Omega \cap \operatorname{supp} f$. Then we must show that $g(x) \leqslant g(a_0)$, or equivalently, that $\lambda(\log a) \leqslant \lambda(Y_0)$. To see that this holds, write

$$nma = k \exp(Z)h, \quad (k \in K,\ Z \in \mathfrak{a},\ h \in H_e)$$

according to the $G = KAH_e$ decomposition; here H_e denotes the identity component of H. Then

$$\exp(Z)h \in KNMa = KMaN,$$

and by the convexity theorem of [1, Thm. 3.8] it follows that $\log a = U + V$, where U is contained in the convex hull of $W_{K \cap H}Z$, and V belongs to a certain subcone of the closed convex cone $\{V \in \mathfrak{a} \mid \langle V, Y \rangle \geqslant 0,\ Y \in \mathfrak{a}^+\}$, which is dual to the positive Weyl chamber $\mathfrak{a}^+$. In particular, $\lambda(V) \leqslant 0$ by the antidominance of λ, and hence

$$\lambda(\log a) \leqslant \lambda(U) \leqslant \max_{w \in W_{K \cap H}} \lambda(wZ).$$

Now $\exp(wZ)H = w \exp(Z)H = wk^{-1}xH$ for $w \in W_{K \cap H}$, and from $x \in \operatorname{supp} f$ and the K-invariance of the support we then see that $\exp(wZ)H \in \operatorname{supp} f$. Hence $wZ \in \operatorname{supp}_{\mathfrak{a}} f$, and we conclude by (8) that

$$\lambda(\log a) \leqslant \lambda(Y_0).$$

This implies (11).

Let $\sigma(D)$ be the principal symbol of D. We have

$$\sigma(D)(\mathrm{d}g(a_0)) = \frac{1}{m!} D((g - g(a_0))^m)(a_0). \tag{12}$$

It follows immediately from the definition of g that $R_u g = 0$ for $u \in U(\mathfrak{g})\mathfrak{h}_c$. Moreover, since g is left NM-invariant, and since $\mathfrak{n}$ and $\mathfrak{m}$ are normalized by A, we also have that $R_u g(a) = 0$ for $a \in A$, $u \in (\mathfrak{n} + \mathfrak{m})_c U(\mathfrak{g})$. Hence $Dg(a) = R_{'\eta(D)} g(a)$. Applying the same reasoning to the function $(g - g(a_0))^m$ we obtain that

$$D((g - g(a_0))^m)(a) = R_{'\eta(D)}(g - g(a_0))^m(a) = m!u_0(\lambda). \tag{13}$$

Combining (12) and (13) we obtain that $\sigma(D)(\mathrm{d}g(a_0)) = u_0(\lambda)$ and hence

$$\sigma(D)(\mathrm{d}g(a_0)) \neq 0 \tag{14}$$

by the assumption on λ.

From (9), (11) and (14) it follows by Holmgren's uniqueness theorem ([17, Thm. 5.3.1]) that $f = 0$ on a neighbourhood of a_0H, contradicting (10). This completes the proof of the first biimplication in (i). From Lemma 3 we get that D^* also satisfies the assumption of (i), and hence the remaining statements in (i) follow.

We now prove (ii). Let S be the ball of radius R centered at the origin, and let $\varphi \in C^\infty(\mathbb{R})$ be positive on $[0; R^2[$ and zero on $[R^2; \infty[$. Define $f(kaH) = \varphi(\|\log a\|^2)$ for $k \in K$, $a \in A$. Then $f \in C^\infty(X)$ by [10, Thm. 4.1], and we clearly have $\operatorname{supp} f = X_S$. Now (ii) follows from Lemma 4. □

COROLLARY 1

(i) *If $X = G/H$ is split, then X is D-convex and D is injective on $C_c^\infty(X)$ for all non-trivial invariant differential operators D.*
(ii) *If X is not split there exists a non-trivial invariant differential operator D, such that X is not D-convex and such that D is not injective on $C_c^\infty(X)$.*

REMARK 1. By regularization it follows that the statements of Theorem 2 and its corollary hold with $C_c^\infty(X)$ replaced by the space of compactly supported distributions on X.

REMARK 2. An explicit example of an operator D as in part (ii) of Theorem 2 and its corollary is given in [5] (see also [20]), where it is shown that the "imaginary part" C_I' of the Casimir operator on a complex semisimple Lie group G' is not solvable. Viewing G' as a symmetric space for $G' \times G'$ it is easily seen that $\eta(C_I') = 0$ (see [5, p. X.8]).

The radial part

Let $D \in \mathbb{D}(G/H)$. Choose a positive system Σ^+ and let $A^+ \subset A$ be the corresponding open chamber. Via the canonical map from G to G/H we identify A^+ with a submanifold of X. According to [15, p. 259] there exists a unique differential operator $\Pi(D)$ on A^+ such that $(Df)|_{A^+} = \Pi(D)(f|_{A^+})$ for all K-invariant smooth functions f on X. $\Pi(D)$ is called the *radial part* of D. The following result establishes a connection between $\Pi(D)$ and $\eta(D)$. It is a generalization of [12, p. 267, Lemma 26] (see also [15, p. 308, Prop. 5.23]).

Let $\mathfrak{R}^+$ denote the ring of analytic functions φ on A^+ which can be expanded in an absolutely convergent series on A^+ with zero constant term:

$$\varphi = \sum_{\nu \in \Lambda} c_\nu e^{-\nu}, \quad c_\nu \in \mathbb{C},\ c_0 = 0$$

where the sum is over the set $\Lambda = \mathbb{N}\Sigma^+$ and where $e^{-\nu}$ is defined by $e^{-\nu}(a) = e^{-\nu(\log a)}$.

PROPOSITION 1. *Let $D \in \mathbb{D}(G/H)$. There exist a finite number of elements $v_i \in S(\mathfrak{a})$ and functions $g_i \in \mathfrak{R}^+$ such that*

$$\Pi(D) = e^{-\rho} R_{\eta(D)} \circ e^{\rho} + \sum_i g_i R_{v_i} \tag{15}$$

on A^+. Moreover the order m of $\Pi(D)$ equals the degree of $\eta(D)$, and we can select the v_i such that

$$\deg v_i \leqslant m - 1 \tag{16}$$

for all i (where a negative degree of v_i means that $v_i = 0$). In particular, $\Pi(D) = 0$ if and only if $\eta(D) = 0$.

Proof. The existence of the v_i and g_i such that (15) holds follows from [2, Lemma 3.9]. It remains to prove (16) (from the lemma of loc. cit. we only get that $\deg v_i < \operatorname{order}(D)$, which is not sharp enough to conclude (16), because the order of $\Pi(D)$ in general may be smaller than that of D).

Let

$$\Pi(D) = \sum_{\nu \in \Lambda} e^{-\nu} R_{v_\nu} \tag{17}$$

be the expansion of $\Pi(D)$ derived from (15), where $v_\nu \in S(\mathfrak{a})$ and where v_0 is given by $v_0(\lambda) = \eta(D)(\lambda + \rho)$. We claim that

$$\deg v_\nu \leqslant \deg v_0 - 1 \quad \text{for all } \nu \neq 0, \tag{18}$$

from which both the statement that order $\Pi(D) = \deg \eta(D)$ and (16) follow. We shall obtain (18) by means of a recursion formula for the v_ν, derived from the relation $L_X D = D L_X$, where L_X is the Laplace-Beltrami operator on X given in terms of the Casimir operator $\omega \in U(\mathfrak{g})^H$ by $L_X = R_\omega$.

The radial part of L_X is easily computed (see [10, Eq. (4.12)]):

$$\Pi(L_X) = J^{-1/2}(L_A \circ J^{1/2} - L_A(J^{1/2})) \tag{19}$$

where L_A is the Laplacian on A, and $J = \Pi_{\alpha \in \Sigma^+} (e^{\alpha} - e^{-\alpha})^{p_\alpha} (e^{\alpha} + e^{-\alpha})^{q_\alpha}$. Here p_α and q_α are certain integers given by root space dimensions, see [21, Thm. 8.1.1].

Put $\tilde{\Pi}(D) = J^{1/2} \Pi(D) \circ J^{-1/2}$, then it follows from the commutation relation $[L_X, D] = 0$ and (19) that $\tilde{\Pi}(D)$ commutes with $L_A - d$, where d is the function $J^{-1/2} L_A(J^{1/2})$. Expanding d in a power series $d(a) = \Sigma_{\gamma \in \Lambda} d_\gamma a^{-\gamma}$ on A^+ and expanding $\tilde{\Pi}(D)$ in analogy with (17) as

$$\tilde{\Pi}(D) = \sum_{\nu \in \Lambda} e^{-\nu} R_{\tilde{v}_\nu}$$

we obtain the following expression

$$\sum_{\nu,\gamma\in\Lambda} ([L_A, e^{-\nu}]R_{\tilde{v}_\nu} - d_\gamma e^{-\nu}[e^{-\gamma}, R_{\tilde{v}_\nu}]) = 0.$$

Comparing coefficients to $e^{-\nu}$ we get

$$[L_A, e^{-\nu}]R_{\tilde{v}_\nu} = \sum_{\gamma\in\Lambda,\nu-\gamma\in\Lambda} d_\gamma e^{-(\nu-\gamma)}[e^{-\gamma}, R_{\tilde{v}_{\nu-\gamma}}],$$

where the sum is finite. In this equation, if $\nu \neq 0$ and $\tilde{v}_\nu \neq 0$, the left side is a differential operator on A^+ of order $1+\deg\tilde{v}_\nu$, whereas the order of the operator on the other side is less than the maximum of the degrees of all $\tilde{v}_{\nu-\gamma}$, $\gamma\in\Lambda\backslash\{0\}$. In particular, it follows by an easy induction that $\deg\tilde{v}_\nu \leqslant \deg\tilde{v}_0 - 2$ for $\nu \neq 0$.

In the series

$$\Pi(D) = J^{-1/2}\tilde{\Pi}(D)\circ J^{1/2} = J^{-1/2}\sum_{\nu\in\Lambda} e^{-\nu}R_{\tilde{v}_\nu}\circ J^{1/2}$$

it is seen that the only contribution in degree $\deg\tilde{v}_0$ is obtained in the e^0 term. Hence v_0 and $\tilde{v}_0$ have the same degree (in fact it is easily seen that $\tilde{v}_0 = \eta(D)$), and v_ν has a lower degree for all other ν. From this the claimed property (18) of the v_ν follows.

The final statement of the proposition follows from the previous statements. □

PROOF OF LEMMA 4. Assume $\eta(D)=0$ and let f be smooth and K-invariant. It follows from the final statement of Proposition 1 that $Df = 0$ on A^+. Since Σ^+ was arbitrary we conclude that $Df = 0$ on an open dense subset of the submanifold AH of X. By $G = KAH$ and the K-invariance of f we conclude that $Df = 0$. □

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