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Dimension of families of space curves

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In this paper we consider families of smooth curves immersed or embedded in a fixed projective space and parametrized by a complete variety. For immersed curves we will give a sharp dimension bound (Theorem 1) which slightly improves the main result of [CR]. For embedded rational and elliptic curves in $\mathbb{P}^3$, we will prove that there do not exist any nontrivial families as above (Theorem 3). In addition, we will prove the ampleness of a certain adjoint line bundle associated to a family of immersed curves (Theorem 2).

To set things up precisely, by a *family of immersed* (resp. *embedded*) *curves in* $\mathbb{P}^n$ we shall mean a diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{p} & \mathbb{P}^n \\ {\scriptstyle\pi}\downarrow & & \\ \Lambda & & \end{array} \tag{1}$$

in which Λ is an irreducible variety, π is a smooth proper morphism of relative dimension 1 with connected fibres $Y_\lambda = \pi^{-1}(\lambda)$, for all $\lambda \in \Lambda$, p is an immersion (resp. embedding) when restricted on Y_λ. Such a family is said to be *closed* if Λ is a complete variety and *nondegenerate* (resp. *effectively parametrized*) if the natural map

$$\Lambda \to \mathrm{Hilb}_{\mathbb{P}^n}$$
$$\lambda \mapsto Y_\lambda$$

is generically finite. (resp. finite)

THEOREM 1. *For any closed, nondegenerate family* (1) *of immersed curves of genus* $g \geqslant 1$ *in* $\mathbb{P}^n$, *the following sharp estimate holds*:

$$\dim \Lambda \leqslant n - 2. \tag{2}$$

REMARK. In [CR], the slightly weaker estimate $\Lambda \leqslant n - 1$ was obtained

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under the slightly stronger hypothesis that the Y_λ are *embedded* in $\mathbb{P}^n$ without, however, the hypothesis of genus $g \geqslant 1$; the sharp bound in the case of genus 0 remains unknown to us.

PROOF OF THEOREM. We begin by establishing sharpness, using a modification of an idea first suggested to us by R. Lazarsteld and already used in [CR]. Given n, start with some closed nondegenerate $(n-2)$-dimensional family $\{Y_\lambda\}$ in $\mathbb{P}^N$ for $N \gg 0$; as the tangent variety $T(Y_\lambda) \subset \mathbb{P}^N$ of each Y_λ is 2-dimensional, we have $\dim \bigcup_\lambda T(Y_\lambda) \leqslant n$, hence a generic $(N-n-1)$-plane $L \subset \mathbb{P}^N$ will be disjoint from $\bigcup_\lambda T(Y_\lambda)$. Projecting from L, we obtain a closed nondegenerate $(n-2)$-dimensional family of immersed curves in $\mathbb{P}^n$, thus establishing sharpness.

Turning now to the estimate (2), our proof will be based on combining part of the method of [CR] with an idea of R. Braun (cf. [S]), which in turn is similar to ideas already used earlier by J. Harris [H] and D. Mumford [M].

Let

$$P = P^1_{\mathscr{Y}/\Lambda}(\mathscr{O}(1))$$

denote the *relative principal parts sheaf* of $\mathscr{O}_{\mathscr{Y}}(1)$, i.e. the fibre of P at $y \in Y_\lambda \subset \mathscr{Y}$ is $(p^*\mathscr{O}(1))_y \otimes \mathscr{O}_y/(m_y^2 + \pi^* m_\lambda)$, where m denotes maximal ideal. Thus we have an exact sequence

$$0 \to \Omega^1_{\mathscr{Y}/\Lambda}(1) \to P \to \mathscr{O}_{\mathscr{Y}}(1) \to 0 \tag{3}$$

By the construction of principal parts sheaves, the natural map

$$(n+1)\mathscr{O}_{\mathscr{Y}} \to \mathscr{O}_{\mathscr{Y}}(1)$$

lifts to a map

$$\varphi\colon (n+1)\mathscr{O}_{\mathscr{Y}} \to P.$$

Now thanks to our assumption that p is an imersion on each Y_λ, we conclude that φ is *surjective*. As $\operatorname{Ker}\varphi$ is a rank-$(n-1)$ vector bundle, its n-th Chern class vanishes, hence in the Chow ring of y we obtain the relation

$$\left[\frac{1}{c(P)}\right]_n = 0. \tag{4}$$

Now put $H = c_1(\mathcal{O}_{\mathscr{Y}}(1)), K = c_1(\Omega^1_{\mathscr{Y}/L}) \in A^*(\mathscr{Y})$. Then in view of (3), (4) reads

$$(-1)^n(H^n + H^{n-1}(K + H) + \cdots + (K + H)^n) = 0. \tag{5}$$

A basic result of Arakelov (cf. [M]) states that the class K is *numerically effective* on $\mathscr{Y}$ and in particular (5) yields

$$(K + H)^n = 0. \tag{6}$$

Assuming $\dim \Lambda = n - 1$ (i.e. $\dim \mathscr{Y} = n$), we will now show that (6) leads to a contradiction. Note that H need not be ample on $\mathscr{Y}$, hence ditto for $K + H$, so that it is not a priori obvious that $(K + H)^n > 0$. However, we may argue as follows. Put

$$s = \dim(p(\mathscr{Y})).$$

Then using again the nefness of K, we have

$$(K + H)^n \geqslant \tbinom{n}{s} H^s \cdot K^{n-s},$$

and suffice to show that

$$H^s \cdot K^{n-s} > 0. \tag{7}$$

To this end, let

$$Z = p^{-1}(Q) \subset \mathscr{Y}$$

be a generic fibre of p. Then Z is $(n - s)$-dimensional and may be identified with the set of $\lambda \in \Lambda$ such that $p(Y_\lambda) \ni Q$. Moreover

$$H^s \cdot K^{n-s} = \deg(p(Y_\lambda)) \cdot (K|_Z)^{n-s},$$

hence to prove (7) it will suffice to prove $(K|_Z)^{n-s} > 0$. For this, consider the natural ("Gauss") map

$$\begin{aligned} \gamma: Z &\to \mathbb{P}(T_Q \mathbb{P}^n) \\ \lambda &\mapsto (T_Q Y_\lambda \subset T_Q \mathbb{P}^n) \end{aligned}$$

Then as Z contains a generic point of Λ it follows from ([CR]), Proposition

1.5) that γ is *generically finite*. Moreover, clearly $K|_Z = \gamma^* \mathcal{O}(1)$, so that $(K|_Z)^{n-s} > 0$. □

THEOREM 2. *In the situation of Theorem* 1, *assume moreover that* $\mathcal{Y}/\Lambda$ *is effectively parametrized. Then* $\Omega_{\mathcal{Y}/\Lambda}(1)$ *is ample on* $\mathcal{Y}$.

Proof. Using the Nakai-Moisezon criterion, it suffices to prove that for any irreducible r-dimensional subvariety $Z \subset \mathcal{Y}$ we have, with notation as above,

$$(K + H)^r|_Z > 0.$$

This can be proved as above, using the fact that if $F = p^{-1}(Q) \cap Z \subset \Lambda \times Q$ is a general fibre of $p|_Z$, then the map

$$\gamma: F \to \mathbb{P}(T_Q \mathbb{P}^n)$$

is *finite*, by effective parametrization of $\mathcal{Y}/\Lambda$. □

THEOREM 3. *There is no nondegenerate closed family of nondegenerate embedded rational or elliptic curves in* $\mathbb{P}^3$.

Proof. If not, then there is such a family (1) with Λ a smooth complete curve. We will use a relative version of the *double-point formula* [F], which we now recall. Consider the fibred product

$$\begin{array}{ccccc} & & \mathcal{Y} \underset{\Lambda}{\times} \mathcal{Y} & & \supset \Delta = \mathcal{Y} \\ & {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} & \\ \mathcal{Y} & & & & \mathcal{Y} \end{array}$$

put $L = p^* \mathcal{O}(1)$, $Q = p^* T_{\mathbb{P}_3}(-1)$, $b = L^2 = \deg(p_* \mathcal{Y})$, $d = \deg(p(Y_\lambda))$. Then the virtual number of double points in the family $\{p(Y_\lambda)\}$ is given by

$$\delta = \frac{1}{2} c_3(p_1^* L \otimes p_2^* Q \otimes \mathcal{O}(-\Delta)).$$

In particular, in our case we must have $\delta = 0$. Now we compute that

$$\begin{aligned} 2\delta &= c_3(p_1^* L \otimes p_2^* Q) - \Delta \cdot c_2(p_1^* L \otimes p_2^* Q) + \Delta^2 \cdot c_1(p_1^* L \otimes p_2^* Q) - \Delta^3 \\ &= p_1^* c(L) p_2^* c_2(Q) + p_1^* c_1(L)^2 p_2^* c_1(Q) - c_2(L \otimes Q) - K \cdot c_1(L \otimes Q) - K^2 \\ &= 2bd - 6b - 4KH - K^2 \qquad (8) \end{aligned}$$

Now in the elliptic case we have $K = \mathcal{O}$, so (8) reads $2\delta = 2b(d - 3)$ which is

>0. Consider next the rational case, where $\mathscr{Y}/\Lambda$ is a ruled surface.
Write

$$\mathscr{Y} = \mathbb{P}(E)$$

where E is a rank-2 vector bundle on Λ with $c_1(E) = 0$ or -1. We will assume $c_1(E) = -1$ as the other case is similar but simpler. Write

$$D = c_1(\mathscr{O}_{\mathbb{P}(E)}(1))$$

and let F be the class of a fibre on $\mathscr{Y}$. Then we have

$$K = -2D + F,$$
$$H = dD + fF, \ f \in \mathbb{Z}$$

As $b = H^2 \geqslant 2$ and $D^2 = -c_1(E) = 1$, we have $b = d^2 + 2fd \geqslant 2$, hence

$$HK = -d - 2f = -\frac{b}{d} \leqslant -\frac{2}{d} < 0.$$

Since $K^2 = 0$, we see from (8) that

$$2\delta \geqslant 2bd - 6b + 4 = 2b(d-3) + 4 > 0. \qquad \square$$

In the case of elliptic curves, the foregoing argument generalizes readily to $\mathbb{P}^n$, $n \geqslant 4$, yielding.

THEOREM 4. *There is no nondegenerate* $(n-2)$*-dimensional family of embedded elliptic curves of degree* $\geqslant n+1$ *in* $\mathbb{P}^n$, $n \geqslant 3$.

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