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WILLIAM M. MCGOVERN

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A remark on differential operator algebras and an equivalence of categories

WILLIAM M. MCGOVERN*

Department of Mathematics, University of Washington, Seattle, WA, U.S.A. 98195

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Abstract. We prove a surjectivity result on rings of twisted differential operators on partial flag varieties stated by Vogan. In the process we derive an interesting parabolic analogue of the famous Bernstein-Gelfand equivalence of categories and exhibit a new class of modules for which Kostant's problem has a positive solution.

1. Introduction

Rings (and more generally sheaves) of twisted differential operators on partial flag varieties (called TDO's for short) were introduced by Beilinson and Bernstein in [1] and studied intensively by Borho and Brylinski [3, 5]. They have turned out to have very deep and extensive connections with the representation theory of complex reductive groups. Perhaps the most striking such connection is the category equivalence between modules of a fixed regular infinitesimal character over an enveloping algebra and quasi-coherent sheaves of modules over the corresponding sheaf of TDO's. A key step in the proof of this equivalence is a surjectivity result: given certain conditions on the twisting parameter, the corresponding ring of TDO's is actually a quotient of the enveloping algebra. (In general, all one can say is that a ring of TDO's is a primitive Dixmier algebra over a primitive quotient of the enveloping algebra.) Vogan has stated the same surjectivity result under rather different conditions on the twisting parameter. He does not give a proof, remarking only that the result is "fairly subtle" and "one of the keys to the Beilinson-Bernstein localization theory". Nor can one find a proof in [1] or [3]. The purpose of this paper is to prove Vogan's assertion, using ideas rather different from those of [3]. In fact, it will drop out of an interesting parabolic analogue of the Bernstein-Gelfand equivalence of categories between Harish-Chandra bimodules for complex groups and modules in category $\mathcal{O}$.

2. Notation and set-up

Let G be a complex connected reductive group with Lie algebra $\mathfrak{g}$, P a parabolic subgroup with Lie algebra $\mathfrak{p}$, and P_0 the commutator subgroup of

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P. Choose a Cartan subgroup H with Lie algebra $\mathfrak{h}$ contained in $\mathfrak{p}$ and a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ contained in the set of $\mathfrak{h}$-roots in $\mathfrak{p}$. Write

$$\begin{aligned}
P_0 &= M_0N, \text{ a Levi decomposition with } H \subset M_0,\\
\mathfrak{m}_0 &= \text{Lie } M_0,\\
T &= P/P_0, \text{ a torus,}\\
\mathfrak{t} &= \text{Lie } T.
\end{aligned}$$

$$\begin{aligned}
\Delta^+(\mathfrak{m}_0, \mathfrak{h}) &= \text{choice of positive roots of } \mathfrak{h} \text{ in } \mathfrak{m}_0 \text{ induced from } \Delta^+(\mathfrak{g}, \mathfrak{h}),\\
S &= \text{corresponding choice of simple roots of } \mathfrak{h} \text{ in } \mathfrak{m}_0,\\
\rho_0 &= \tfrac{1}{2}\Sigma_{\alpha\in\Delta^+(\mathfrak{m}_0, \mathfrak{h})}\, \alpha,\\
\rho &= \tfrac{1}{2}\Sigma_{\alpha\in\Delta^+(\mathfrak{g}, \mathfrak{h})}\, \alpha := \rho_0 + \rho_1,\\
Z &= G/P,\\
Y &= G/P_0,\\
\mathfrak{n} &= T_e^* Z,\\
\mathfrak{u} &= T_e^* Y.
\end{aligned}$$

Using standard identifications we may identify $\mathfrak{t}^*$ with $\mathfrak{u}/\mathfrak{n}$ and regard it as a subspace of $\mathfrak{h}^*$; more precisely, we have $\mathfrak{h}^* = \mathfrak{t}^* + (\mathfrak{h} \cap \mathfrak{m}_0)^*$. The "twisting parameters" of the introduction are elements ξ of $\mathfrak{t}^*$. Given such a ξ, we construct the Beilinson-Bernstein ring A_ξ of TDO's as follows. Start with the ring Diff G/P_0 of algebraic differential operators on G/P_0. There is a left G-action and a commuting right T-action on G/P_0. Differentiating these actions, we get a map from $U(\mathfrak{g}) \otimes U(\mathfrak{t})$ to Diff G/P_0, where $U(\cdot)$ as usual denotes the enveloping algebra of a Lie algebra. Set

$$A = \text{centralizer of } T \text{ in Diff } G/P_0; \tag{2.1}$$

then we get a map $\Phi: U(\mathfrak{g}) \to A$. Now bring ξ into the picture by letting I_ξ be the ideal of A generated by all $H + (\xi - \rho_1)(H)$ for all $H \in \mathfrak{t}$. These elements are central in A, since ρ_1 is trivial on Lie P_0. Finally, we set

$$A_\xi = A/I_\xi = \text{Diff}_\xi G/P, \tag{2.2}$$

the ring of (ξ-) TDO's on G/P. The map Φ induces a map

$$\Phi_\xi\colon U(\mathfrak{g}) \to A_\xi. \tag{2.3}$$

It is not difficult to see that A_ξ is finitely generated over $\Phi_\xi(U(\mathfrak{g}))$. In this paper we are primarily concerned with conditions under which Φ_ξ is surjective. Before we can state Vogan's sufficient condition, we need a definition.

DEFINITION 2.4. Assume that $\lambda \in \mathfrak{h}^*$ takes positive integral values on $\Delta^+(\mathfrak{m}_0, \mathfrak{h})^\vee$. We say that λ is dominant if it does not take a negative integral value on any positive coroot (relative to $\Delta^+(\mathfrak{g}, \mathfrak{h})$). We say that λ is $\mathfrak{n}$-antidominant if it does not take a positive integral value on any positive coroot not in $\Delta^+(\mathfrak{m}_0, \mathfrak{h})^\vee$.

The main result of this paper is

THEOREM 2.5 ([12, 3.9(c)]). *With notation as above, the map Φ_ξ is surjective whenever $\xi + \rho_0$ is dominant.*

The reason for the appearance of ρ_0 is that $\xi + \rho_0$, when regarded as an element of $\mathfrak{h}^*$ as above, is the infinitesimal character of A_ξ [3, 3.6]. What Borho and Brylinski show in [3, 3.8] is that Φ_ξ is surjective whenever $\xi + \rho_0$ is $\mathfrak{n}$-antidominant, using deep results of Conze-Berline and Duflo in [6]. Certainly Φ_ξ is not always surjective; a counterexample in type B_2 is given in [6, 6.5]. On the other hand, if $\mathfrak{p}$ happens to be a Borel subalgebra, then Φ_ξ is always surjective. Vogan gives an intersecting example in type C_4 where Φ_ξ fails to be surjective even though its image has full multiplicity in A_ξ. There should be many more examples of this last phenomenon; it is intimately related to the failure of certain nilpotent orbit closures to be normal.

3. Proof of Theorem 2.5 and an equivalence of categories

The first step of the proof of Theorem 2.5 is the same as that of the parallel result in [3].

LEMMA 3.1 ([3, 3.8]). *We may identify A_ξ with the ring $A'_\xi := \mathscr{L}(M_\mathfrak{p}(\xi + \rho_0), M_\mathfrak{p}(\xi + \rho_0))$ of G-finite maps from the generated Verma module*

$$M_\mathfrak{p}(\xi + \rho_0) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{\xi + \rho_0 - \rho}$$

to itself; here $\mathfrak{p}$ acts on the one-dimensional module $\mathbb{C}_{\xi+\rho_0-\rho}$ by making $\mathfrak{p}_0$ act trivially and $\mathfrak{t}$ act by the character $\xi + \rho_0 - \rho$.

Borho and Brylinski then complete their proof by invoking [6, 2.12, 4.7, 6.3] to show that the natural map $U(\mathfrak{g}) \to A'_\xi$ is surjective for $\mathfrak{n}$-antidominant $\xi + \rho$. These results do not apply if $\xi + \rho$ is dominant (the key difficulty being that $M_\mathfrak{p}(\xi + \rho_0)$ is not irreducible), so we must take another path.

We begin by noting that $M_\mathfrak{p}(\xi)$ is a projective object in the full subcategory $\mathcal{O}_\mathfrak{p}$ of category $\mathcal{O}$ consisting of the $\mathfrak{p}$-locally finite modules. Moreover, if ξ is regular, then by applying an appropriate translation functor to $M_{\mathfrak{p}}(\xi)$ one obtains a projective generator for the subcategory $\mathcal{O}_{\mathfrak{p},\xi}$ of modules in Ob $\mathcal{O}_{\mathfrak{p}}$ with infinitesimal character ξ [2]. The ring $A'_{\xi-\rho_0}$ should therefore have some equally pleasant properties in an appropriate category of Harish-Chandra bimodules. Our first step, which is of considerable interest in its own right, is to describe this category. To state the main result, we need some notation. Let $\lambda \in \mathfrak{h}^*$ take positive integral values on $\Delta^+(\mathfrak{m}_0, \mathfrak{h})^\vee$. Then one has the generalized Verma module $M_{\mathfrak{p}}(\lambda)$ relative to $\mathfrak{p}$ of highest weight $\lambda - \rho$; denote its unique irreducible quotient by $L(\lambda)$. Let W be the Weyl group of G and W_λ the integral Weyl subgroup relative to λ. As the terminology indicates, we may indeed regard W_λ as a subgroup of W. Denote by $W(\mathfrak{m}_0)$ the Weyl group of $\mathfrak{m}_0$; it too is a subgroup of W, and in fact a subgroup of W_λ. Write w_0, w_m, w_λ for the longest elements of W, $W(\mathfrak{m}_0)$, and W_λ, respectively. Then we have

THEOREM 3.2. *Fix a coset Λ of the weight lattice of $\mathfrak{g}$ in $\mathfrak{h}^*$ and a dominant regular representative λ of Λ taking integral values on $\Delta^+(\mathfrak{m}_0, \mathfrak{h})^\vee$. Then there is an equivalence of categories*

$$\mathcal{O}_{\mathfrak{p},\lambda} \cong \mathcal{HC}_{\mathfrak{p},\lambda}$$

between the subcategory $\mathcal{O}_{\mathfrak{p},\lambda} := \mathcal{O}_{\mathfrak{p},\Lambda}$ of $\mathcal{O}_{\mathfrak{p}}$ consisting of modules with all weights in Λ and the subcategory $\mathcal{HC}_{\mathfrak{p},\lambda}$ of all finitely generated Harish-Chandra bimodules over $U(\mathfrak{g})$ that are annihilated by Ann $L(w_\lambda w_m \lambda)$ *on the right. The equivalence is implemented by the maps*

$$M \mapsto \mathcal{L}(M_{\mathfrak{p}}(\lambda), M),\ X \mapsto X \otimes_{U(\mathfrak{g})} M_{\mathfrak{p}}(\lambda)$$

from $\mathcal{O}_{\mathfrak{p},\lambda}$ (resp. $\mathcal{HC}_{\mathfrak{p},\lambda}$) *to $\mathcal{HC}_{\mathfrak{p},\lambda}$* (resp. $\mathcal{O}_{\mathfrak{p},\lambda}$). *Moreover, we also have* $\mathcal{L}(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda)) \cong U(\mathfrak{g})/\text{Ann } L(w_\lambda w_m \lambda)$.

Proof. Of course, our starting point is the famous Bernstein-Gelfand equivalence of categories [2, 5.9(i)], which is precisely the first assertion above in the special case where $\mathfrak{p}$ is a Borel subalgebra $\mathfrak{b}$. (The second assertion, which is actually needed in the course of proving the first one, follows from a classical result of Kostant.) Using this equivalence, we will often allow ourselves to speak of the right annihilator RAnn M of an object M in $\mathcal{O}_\Lambda$ by identifying M with a Harish-Chandra bimodule. Then RAnn M depends on a choice of infinitesimal character, which will always be clear from context. In general, let $\mathfrak{b}$ be a Borel subalgebra contained in $\mathfrak{p}$. Then $\mathcal{O}_{\mathfrak{p},\lambda}$ is a full subcategory of $\mathcal{O}_{\mathfrak{b},\Lambda}$ and so is equivalent to a subcategory of $\mathcal{HC}_{\mathfrak{b},\lambda}$. We must identify this subcategory precisely. Recall that a typical $M \in \text{Ob } \mathcal{O}_{\mathfrak{b},\Lambda}$ lies in Ob $\mathcal{O}_{\mathfrak{p},\lambda}$ if and only if the shifted highest weights λ' of its irreducible subquotients $L(\lambda')$

all take positive integral values on $S^\vee$. Equivalently, thanks to [9, 5.2], M lies in $\mathrm{Ob}\,\mathcal{O}_{\mathfrak{p},\lambda}$ if and only if the right annihilator of any of its irreducible subquotients has τ-invariant not meeting S. But now the τ-invariant of a typical primitive ideal I of infinitesimal character λ fails to meet S if and only if I contains $\mathrm{Ann}\,L(w_\lambda w_m \lambda)$ [8, 5.20]. Hence an M in $\mathrm{Ob}\,\mathcal{O}_{\mathfrak{b},\Lambda}$ lies in $\mathrm{Ob}\,\mathcal{O}_{\mathfrak{p},\lambda}$ if and only if its right annihilator contains some power of $\mathrm{Ann}\,L(w_\lambda w_m \lambda)$. But $\mathrm{Ann}\,L(w_\lambda w_m \lambda)/\mathrm{Ann}\,L(w_\lambda \lambda)$ is idempotent ([7, 4.4], [9, 4.5]). Thus the categories $\mathcal{O}_{\mathfrak{p},\lambda}$ and $\mathscr{HC}_{\mathfrak{p},\lambda}$ are indeed equivalent via the Bernstein-Gelfand maps

$$M \mapsto \mathscr{L}(M(\lambda), M),\ X \mapsto X \otimes_{U(\mathfrak{g})} M(\lambda).$$

It remains to show that these maps coincide with those of the theorem. This is easy in the first case. Given $M \in \mathrm{Ob}\,\mathcal{O}_{\mathfrak{p},\lambda}$, the multiplicity of a typical finite-dimensional $\mathfrak{g}$-module E in the adjoint action on $\mathscr{L}(M(\lambda), M)$ is just the dimension of $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M \otimes E^*)$. Since $M \otimes E^*$ is $\mathfrak{m}_0$-locally finite, any $\mathfrak{g}$-map from $M(\lambda)$ to $M \otimes E^*$ must factor through the largest $\mathfrak{m}_0$-locally finite quotient $M_{\mathfrak{p}}(\lambda)$ of $M(\lambda)$. Hence the natural injection $\mathscr{L}(M_{\mathfrak{p}}(\lambda), M) \to \mathscr{L}(M(\lambda), M)$ is an isomorphism. In the second case, we must work harder; we begin by proving the second assertion of the theorem. Clearly $U := U(\mathfrak{g})/\mathrm{Ann}\,L(w_\lambda w_m \lambda)$ is an object in $\mathscr{HC}_{\mathfrak{p},\lambda}$, so it may be realized as $\mathscr{L}(M_{\mathfrak{p}}(\lambda), M)$ for some object M in $\mathcal{O}_{\mathfrak{p},\lambda}$. Since U contains a copy of the scalar field $\mathbb{C}$ as its subring of G-invariants, there is a nonzero homomorphism $\pi\colon M_{\mathfrak{p}}(\lambda) \to M$. Since the copy of $\mathbb{C}$ in U generates the latter as a $U(\mathfrak{g})$ bimodule, the image of π also generates M, whence π is surjective. Now it is easy to see that U is projective in $\mathscr{HC}_{\mathfrak{p},\lambda}$, whence M must be projective in $\mathcal{O}_{\mathfrak{p},\lambda}$. Since $M_{\mathfrak{p}}(\lambda)$ is indecomposable projective in $\mathrm{Ob}\,\mathcal{O}_{\mathfrak{p},\lambda}$, we get $M \cong M_{\mathfrak{p}}(\lambda)$, as desired.

Now one has a short exact sequence

$$0 \to K(\lambda) \to M(\lambda) \to M_{\mathfrak{p}}(\lambda) \to 0 \tag{3.3}$$

It is enough to show that the image of the natural map $X \otimes_{U(\mathfrak{g})} K(\lambda) \to X \otimes_{U(\mathfrak{g})} M(\lambda)$ is zero. Put $I := \mathrm{Ann}\,L(w_\lambda w_m \lambda)$. Then $IM(\lambda) \subset K(\lambda)$ and it suffices to show that equality holds (since $XI \otimes_{U(\mathfrak{g})} M(\lambda) = 0$). This follows since $I = \mathrm{Ann}\,M(\lambda)/K(\lambda)$ and $IM(\lambda)$ is the only submodule M' of $M(\lambda)$ with $I = \mathrm{Ann}\,M/M'$ [10, 4.3]. Hence both Bernstein-Gelfand maps coincide with their counterparts in the theorem. □

Theorem 2.5 follows at once for regular infinitesimal characters from Theorem 3.2. By combining [6, 5.8] with [13, 4.32], one obtains a formula for multiplicit is of K-types in $\mathscr{L}(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda))$ or $U(\mathfrak{g})/\mathrm{Ann}\,L(w_\lambda w_m \lambda)$. We will extend Theorems 2.5 and 3.2 to arbitrary infinitesimal characters in the next section; for now we conclude this section with

COROLLARY 3.4. *The socle Σ of $M_{\mathfrak{p}}(\lambda)$ is simple. If $\Sigma \cong L(w\lambda)$, then the left and right cells of w (regarded as an element of W_λ) coincide with those of $w_\lambda w_m$, $w_m w_\lambda$, respectively.*

Proof. It is well known and easy to check that $M_{\mathfrak{p}}(\lambda)$ may be obtained from a module induced from a one-dimensional $\mathfrak{p}$-module via a translation functor. It follows at once as in [11, §6] that Σ is simple and that its annihilator coincides with that of $M_{\mathfrak{p}}(\lambda)$. Now the result follows from Joseph's well-known formulas for the annihilators of simple Harish-Chandra bimodules [9, 5.2]. □

That socles Σ of generalized Verma modules with dominant highest weights are simple has long been known to the experts, but no one to my knowledge has given a formula for Σ. Corollary 3.4 suffices to pin down Σ exactly in type A, but not in the other types. Note that Theorem 3.2 shows in particular that Kostant's problem has a positive answer for $M_{\mathfrak{p}}(\lambda)$. One can similarly show that Kostant's problem has a positive solution for Σ.

4. Singular infinitesimal characters

We conclude the paper with two analogues of Theorem 3.2 for dominant weights λ that are not necessarily regular. In both cases one gets an equivalence of categories between a certain subcategory $\mathcal{O}_{\mathfrak{p},\lambda}$ of category $\mathcal{O}$ and the category $\mathcal{HC}_{\mathfrak{p},\Lambda}$ of Harish-Chandra bimodules defined in Theorem 3.2. The definition of $\mathcal{O}_{\mathfrak{p},\lambda}$ depends on whether or not λ is singular on S.

Again let Λ be a coset of the weight lattice in $\mathfrak{h}^*$ and λ an element of Λ^+, the set of dominant weights in Λ. Assume first that λ takes positive (integral) values on $S^\vee$. Let B_λ^0 be the set of roots corresponding to the simple reflections in W_λ fixing λ; then we have $B_\lambda^0 \cap S = \varnothing$. Denote by $\mathcal{O}_{\mathfrak{p},\lambda}$ the full subcategory of $\mathcal{O}_{\mathfrak{p},\Lambda}$ consisting of those modules that are $\mathcal{P}_\lambda$-presentable in the sense of Bernstein and Gelfand [2, §§1,5]. (Thus, in contrast to the last section, we do not have $\mathcal{O}_{\mathfrak{p},\lambda} = \mathcal{O}_{\mathfrak{p},\Lambda}$.) Then one has

THEOREM 4.1. *With notation as above, Theorem 3.2 holds for $\mathcal{O}_{\mathfrak{p},\lambda}$ and $\mathcal{HC}_{\mathfrak{p},\lambda}$.*

Proof. One can repeat the proof of Theorem 3.2, starting from [2, 5.9(ii)] rather than [2, 5.9(i)]. Since B_λ^0 automatically lies in the τ-invariant of $w_\lambda w_m$, we can combine [4, 2.12] and [8, 5.19] to show that if M is an object in $\mathcal{O}_{\mathfrak{b},\lambda}$, then M lies in Ob $\mathcal{O}_{\mathfrak{p},\lambda}$ if and only if the radical of its right annihilator contains $I := \operatorname{Ann} L(w_\lambda w_m \lambda)$. Thus we get an equivalence of categories between $\mathcal{O}_{\mathfrak{p},\lambda}$ and the subcategory $\mathcal{HC}'_{\mathfrak{p},\lambda}$ of $\mathcal{HC}_{\mathfrak{b},\lambda}$ consisting of those bimodules annihilated by some power of I on the right. The equivalence is effected by the maps

$$M \mapsto \mathcal{L}(M_{\mathfrak{p}}(\lambda), M),\ X \mapsto X \otimes_{U(\mathfrak{g})} M(\lambda).$$

Now it follows as in the proof of Theorem 3.2 that $\mathscr{L}(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda)) \cong U(\mathfrak{g})/I$. Finally, we claim that $J := I^2 + \operatorname{Ann} L(w_\lambda \lambda) = I$ and $IM(\lambda) = K(\lambda)$ (notation (3.3)); given these facts, the rest of the proof of Theorem 3.2 carries over. To prove the claims, note that $U(\mathfrak{g})/J$ certainly lies in $\operatorname{Ob} \mathscr{HC}'_{\mathfrak{p},\lambda}$, whence $U(\mathfrak{g})/J \otimes_{U(\mathfrak{g})} M(\lambda) \cong M(\lambda)/I^2 M(\lambda)$ lies in $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$. Since $I^2 M(\lambda) \subset IM(\lambda) \subset K(\lambda)$ and $M_{\mathfrak{p}}(\lambda)$ is the largest $\mathfrak{m}_0$-locally finite quotient of $M(\lambda)$, we must have $JM(\lambda) = IM(\lambda) = K(\lambda)$, whence $J = I$ [10, 4.3], as desired. In particular, $\mathscr{HC}'_{\mathfrak{p},\lambda} = \mathscr{HC}_{\mathfrak{p},\lambda}$. $\square$

Theorem 4.1 implies that Theorem 2.5 holds in general. Finally, we consider what happens when λ is allowed to be singular on S. Fix Λ, Λ^+ as above and now assume only that $\lambda \in \Lambda^+$ takes (nonnegative) integral values on $S^\vee$. Define B_λ^0 as above and set $S_\lambda^0 = B_\lambda^0 \cap S$.

DEFINITION 4.2. Denote by $\mathcal{O}_{\mathfrak{p},\lambda}$ the full subcategory of $\mathcal{O}_\Lambda$ consisting of those modules M that are $\mathscr{P}_\lambda$-presentable and in addition satisfy the following condition: if N is a simple $U(\mathfrak{m}_0)$-subquotient of M, then $R \operatorname{Ann}_{U(\mathfrak{m}_0)} N$ has τ-invariant lying in S_λ^0. (This condition makes sense and is independent of the choice of regular integral infinitesimal character of $R \operatorname{Ann}_{U(\mathfrak{m}_0)} N$ because any such N is a simple highest weight $U(\mathfrak{m}_0)$-module of intergral infinitesimal character.) Also denote by $M_{\mathfrak{p}}(\lambda)$ the $U(\mathfrak{g})$-module induced from the simple $U(\mathfrak{p})$-module of highest weight $\lambda - \rho$ on which $\mathfrak{n}$ acts trivially.

Note that $\mathcal{O}_{\mathfrak{p},\lambda}$ is no longer subcategory of $\mathcal{O}_{\mathfrak{p},\Lambda}$ in this situation, as the modules in it are not in general $\mathfrak{m}_0$-locally finite.

LEMMA 4.3. *Fix $\lambda_1, \lambda_2 \in \Lambda^+$ and write W_1, W_2 for the parabolic subgroup of W_λ corresponding to $B_{\lambda_1}^0, B_{\lambda_2}^0$. Fix w_1, w_2 of maximal length in their left cosets $w_1 W_1, w_2 W_2$, respectively. Choose any regular representative λ' of Λ^+. Then there is a finite-dimensional module E such that $L(w_2\lambda_2)$ is a subquotient of $L(w_1\lambda_1) \otimes E$ if and only if* $\operatorname{Ann} L(w_2^{-1}\lambda') \supset \operatorname{Ann} L(w_1^{-1}\lambda')$ *(this last condition is well known to be independent of the choice of λ').*

Proof. This is well known if both λ_1 and λ_2 are regular [8, 7.13]. In general, suppose that $L(w_2\lambda_2)$ appears in (i.e., is a subquotient of) $L(w_1\lambda_1) \otimes E$. Translating $L(w_2\lambda_2)$ off the $B_{\lambda_2}^0$ walls, we deduce that $L(w_2\lambda')$ appears in $L(w_1\lambda_1) \otimes E$ (with a different E). Translating $L(w_1\lambda')$ onto the $B_{\lambda_1}^0$ walls, we see that $L(w_2\lambda')$ also appears in $L(w_1\lambda') \otimes E$ (again with a different E) [8, 4.12, 4.13]. Hence $\operatorname{Ann} L(w_2^{-1}\lambda') \supset \operatorname{Ann} L(w_1^{-1}\lambda')$, since λ' is regular. A similar argument shows that $L(w_2\lambda_2)$ appears in $L(w_1\lambda_1) \otimes E$ for some E whenever $\operatorname{Ann} L(w_2^{-1}\lambda') \supset \operatorname{Ann} L(w_1^{-1}\lambda')$ and completes the proof. $\square$

COROLLARY 4.4. *$M_{\mathfrak{p}}(\lambda)$ is the unique largest quotient of $M(\lambda)$ belonging to* $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$. *In general, a typical $\mathscr{P}_\lambda$-presentable module M lying in* $\operatorname{Ob} \mathcal{O}_\Lambda$ *belongs to* $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$ *if and only if its simple subquotients $L(w\mu)$ satisfy the following* *condition: if $\mu \in \Lambda^+$ and $w \in W_\lambda$ is chosen to have maximal length in its coset wW_μ^0, then $\tau_\Lambda(w^{-1}) \cap S \subset S_\lambda^0$.*

Proof. The $U(\mathfrak{m}_0)$-submodule N of $M_\mathfrak{p}(\lambda)$ generated by its highest weight vector is simple of highest weight λ, whence one checks directly that it satisfies Definition 4.2. Any other simple $U(\mathfrak{m}_0)$-subquotient N' of $M_\mathfrak{p}(\lambda)$ appears in $N \otimes E$ for some E, whence by [8, 5.19] and Lemma 4.3 it too satisfies Definition 4.2. Hence $M_\mathfrak{p}(\lambda) \in \operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$. The same argument shows that the criterion of the corollary for lying in $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$ is sufficient; to see its necessity, look at the $U(\mathfrak{m}_0)$-submodule of any $U(\mathfrak{g})$-subquotient $L(w\mu)$ of M generated by the highest weight vector of $L(w\mu)$. It remains to show that any quotient M' of $M(\lambda)$ lying in $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$ is actually a quotient of $M_\mathfrak{p}(\lambda)$; for this it suffices to show that the $\mathrm{U}(\mathfrak{m}_0)$-submodule M'' generated by the highest weight vector of M' is simple. Definition 4.2 guarantees that $\operatorname{Ann}_{U(\mathfrak{m}_0)} M''$ is maximal of infinitesimal character λ. We are therefore reduced to proving the following claim: if $\nu \in \Lambda^+$, so that $I_\nu := \operatorname{Ann} L(\nu)$ is a maximal ideal, then $M(\nu)/I_\nu M(\nu)) \cong L(\nu)$ (so that the only quotient of $M(\nu)$ with annihilator I_ν is $L(\nu)$; we apply this fact to Verma modules over $U(\mathfrak{m}_0)$). To prove the claim, note that the equivalence of categories yields $\mathscr{L}(M(\nu), M(\nu)) \otimes_{U(\mathfrak{g})} M(\nu) \cong L(\nu)$. But $U(\mathfrak{g})/I_\nu$ embeds into $\mathscr{L}(L(\nu), L(\nu))$, which embeds into $\mathscr{L}(M(\nu), L(\nu))$, so that $U(\mathfrak{g})/I_\nu \otimes_{U(\mathfrak{g})} M(\nu) \cong M(\nu)/I_\nu M(\nu)$ embeds into $L(\nu)$. The claim follows at once. (We also recover the known result that $\mathscr{L}(L(\nu), L(\nu)) \cong U(\mathfrak{g})/I_\nu$.) $\square$

At last we can state

THEOREM 4.5. *With notation as above, Theorem* 3.2 *holds for $\mathcal{O}_{\mathfrak{p},\lambda}$ and $\mathscr{HC}_{\mathfrak{p},\lambda}$.*

Proof. We imitate the proof of Theorem 4.1. Using [8, 5.19] and [4, 2.12] one may rephrase the necessary and sufficient condition of Corollary 4.4 for a $\mathscr{P}_\lambda$-presentable module M to lie in $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$ as follows: whenever $L(w\mu)$ is a simple subquotient of $M, \mu \in \Lambda^+$, and $w \in W_\lambda$ has maximal length in its W_μ^0 coset, then $\operatorname{RAnn} L(w\mu) \supset \operatorname{Ann} L(w_\lambda w_m w_0 \lambda')$, where $\operatorname{RAnn} L(w\mu)$ is taken to have infinitesimal character λ' (notation (4.3)) and w_0 is the longest element in the parabolic subgroup of W_λ corresponding to S_λ^0. Equivalently, M is in $\mathcal{O}_{\mathfrak{p},\lambda}$ if and only if its simple subquotients $L(w\mu)$ satisfy $\operatorname{RAnn} L(w\mu) \supset \operatorname{Ann} L(w_\lambda w_m \lambda)$, where $\operatorname{RAnn} L(w\mu)$ is taken to have infinitesimal character λ. By Corollary 4.4, the maps $M \mapsto \mathscr{L}(M_\mathfrak{p}(\lambda), M)$, $X \mapsto X \otimes_{U(\mathfrak{g})} M(\lambda)$ define equivalences of categories between $\mathcal{O}_{\mathfrak{p},\lambda}$ and the subcategory $\mathscr{HC}'_{\mathfrak{p},\lambda}$ defined as in the proof of Theorem 4.1. Arguing again as in the proof of Theorem 3.2 and using the projective indecomposability of $M_\mathfrak{p}(\lambda)$ in $\operatorname{Ob} \mathcal{O}_{\mathfrak{p},\lambda}$, we see that $\mathscr{L}(M_\mathfrak{p}(\lambda), M_\mathfrak{p}(\lambda)) \cong U(\mathfrak{g})/I$ (notation (4.1)). Arguing as in the proof of Theorem 4.1 and using Corollary 4.4 again, we show that $I^2 + \operatorname{Ann} L(w_\lambda \lambda) = I$ (so that $\mathscr{HC}'_{\mathfrak{p},\lambda} = \mathscr{HC}_{\mathfrak{p},\lambda}$) and that the map from $\mathscr{HC}'_{\mathfrak{p},\lambda}$ to $\mathcal{O}_{\mathfrak{p},\lambda}$ may be replaced by the one in Theorem 3.2. $\square$

That $I/\mathrm{Ann}\,L(w_\lambda\lambda)$ is idempotent in the settings of Theorems 4.1 and 4.5 seems to be new. If λ is regular, then $\mathrm{Ann}\,L(w_\lambda w\lambda)/\mathrm{Ann}\,L(w_\lambda\lambda)$ is idempotent whenever w is the longest element of any parabolic subgroup of W_λ (not just of W). I do not know whether this is still true if λ is singular.

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