ZHANG WENPENG

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A problem of D. H. Lehmer and its generalization (II)*

ZHANG WENPENG

Department of Mathematics, Northwest University, Xi'an, China

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Abstract. Let \( q > 2 \) be an odd number. For each integer \( x \) with \( 0 < x < q \) and \( (q, x) = 1 \), we define \( \hat{x} \) by \( \hat{x} \equiv 1 \pmod{q} \) and \( 0 < \hat{x} < q \). Let \( r(q) \) be the number of integers \( x \) with \( 0 < x < q \) for which \( x \) and \( \hat{x} \) are of opposite parity. The main purpose of this paper is to give a sharper asymptotic formula for \( r(q) \) for all odd numbers \( q \).

1. Introduction

Let \( q \) be an odd integer > 2. For each integer \( x \) with \( 0 < x < q \) and \( (q, x) = 1 \), we know that there exists one and only one \( \hat{x} \) with \( 0 < \hat{x} < q \) such that \( x\hat{x} \equiv 1 \pmod{q} \). Let \( r(q) \) be the number of cases in which \( x \) and \( \hat{x} \) are of opposite parity. For example, \( r(3) = 0 \), \( r(5) = 2 \), \( r(7) = 0 \), \( r(13) = 6 \). For \( q = p \) a prime, D. H. Lehmer \[1\] asks us to find \( r(p) \) or at least to say something nontrivial about it. It is known that \( r(p) \equiv 2 \) or 0 (mod 4) according to \( p \equiv \pm 1 \) (mod 4). About this problem, the author \[2\] obtained an asymptotic formula for \( r(p^2) \) and \( r(p_1p_2) \), where \( p, p_1 \) and \( p_2 \) are primes. In this paper, as an improvement of \[2\], we shall give an asymptotic formula for \( r(q) \) for all odd numbers \( q \). The constants implied by the \( O \)-symbols and the symbols \( <<, >> \) used in this paper do not depend on any parameter, unless otherwise indicated. By using estimates for character sum and Kloosterman sums, and the properties of Dirichlet L-functions, we prove the following two theorems:

THEOREM 1. For every prime \( p > 2 \) we have the asymptotic formula

\[
r(p) = \frac{1}{2} p + \frac{1}{\pi^2} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} \{ S(a, b; p) - S(a, -b; p) \\
+ 4S(\overline{a}, b; p) - 4S(\overline{a}, -b; p) - 4S(\overline{2a}, b; p) \\
+ 4S(\overline{2a}, -b; p) \} + O(\ln^3 p)
\]

where \( d\overline{d} \equiv 1 \pmod{p} \), \( S(m, n; p) = \sum_{d \pmod{p}} e \left( m \frac{d}{p} + n \frac{\overline{d}}{p} \right) \) is the Kloosterman sum, and \( e(y) = e^{2\pi iy} \).

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THEOREM 2. For every odd integer \( q > 2 \) we have

\[
r(q) = \frac{1}{2} \phi(q) + O(q^{1/2}d^2(q)\ln^2 q)
\]

where \( \phi(q) \) is the Euler function and \( d(q) \) is the divisor function.

From theorem 1 we can see that if we could get a nontrivial upper bound estimate for the mean value \( \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(\text{ma}, b; p) \), then we may obtain a more accurate asymptotic formula. From theorem 2 we can also deduce the following:

COROLLARY. For every prime \( p > 2 \) we have the asymptotic formula

\[
r(p) = \frac{1}{2} p + O(p^{1/2} \ln^2 p)
\]

2. Some lemmas

To complete the proofs of the theorems, we need some lemmas. First we have:

LEMMA 1. Let \( q > 2 \) be an odd number. Then we have

\[
r(q) = \frac{1}{2} \phi(q) - \frac{2}{\phi(q)} \sum_{\chi \equiv 1(q)} \chi(4) \left( \sum_{a=1}^{(q-1)/2} \chi(a) \right)^2
\]

where the summation is over all odd characters mod \( q \).

Proof. From the definition of \( r(q) \) and the orthogonality of characters we get

\[
r(q) = \frac{1}{2} \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} \{1 - (-1)^{a+b}\}
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2} \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} (-1)^{a+b}
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \equiv 1(q)} \left( \sum_{a=1}^{(q-1)/2} (-1)^a \chi(a) \right)^2
\]

\[
= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \equiv \chi^*} \left( \sum_{a=1}^{(q-1)/2} (-1)^a \chi(a) \right)^2
\]

where \( \sum_{\chi \equiv \chi^*} \) denotes the summation over all nonprincipal characters mod \( q \).
Now if $\chi(-1) = 1$ and $\chi \neq \chi^0$, then we have

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 0 \quad (2)$$

while if $\chi(-1) = -1$, then

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 2 \sum_{a=1}^{(q-1)/2} \chi(2a) \quad (3)$$

Combining (1), (2) and (3) we may immediately deduce lemma 1. □

**Lemmas.**

**Lemma 2.** Let $q > 1$ be any odd number and let $\chi$ be any Dirichlet character modulo $q$, not necessarily primitive. Then

$$(1 - 2\chi(2)) \sum_{\gamma=1}^{q-1} \gamma \chi(\gamma) = \chi(2) \sum_{\gamma=1}^{(q-1)/2} \chi(\gamma)$$

*Proof.* (See reference [3]). □

**Lemma 3.** Let $m, n, q$ be integers with $q > 1$. Then

$$S(m, n; q) = \sum_{d=1}^{q-1} e \left( \frac{d}{q} \frac{m}{n} + \frac{d}{d(q)} \right) \ll (m, n, q)^{1/2} q^{1/2} d(q)$$

where $d(d) \equiv 1 \pmod{q}$, $d(q)$ is the divisor function, and $(m, n, q)$ denotes the greatest common factor of $m$, $n$ and $q$. $\sum_a$ denotes the summation over $a$ such that $(a, q) = 1$.

*Proof.* (See reference [4]). □

**Lemma 4.** Let $q$ be an odd integer > 2. Then for any integer $b$ we have the estimate

$$\sum_{\chi(-1) = -1} \tau(\chi(b)) L^2(1, \chi) \ll \phi(q) q^{1/2} d(q) \ln^2 q$$

where $L(s, \chi)$ is the Dirichlet L-function and $\tau(\chi)$ is the Gauss sum corresponding to $\chi$.

*Proof.* First for any integer $r$ with $(r, q) = 1$ we have

$$\sum_{\chi(-1) = -1} \chi(r) = \begin{cases} \frac{1}{2} \phi(q), & \text{if } r \equiv 1 \pmod{q}, \\ -\frac{1}{2} \phi(q), & \text{if } r \equiv -1 \pmod{q}, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$
Now for any \( N > 1 \), from the well-known Pólya-Vinogradov inequality (See Theorem 13.15 of [5]) we get for \( \chi \neq \chi^0 \)

\[
L(1, \overline{\chi}) = \sum_{n \leq N} \overline{\chi}(n) n + O \left( \frac{\sqrt{q \ln q}}{N} \right) \tag{5}
\]

If \( (b, q) > 1 \), then \( \chi(b) = 0 \) and in this case it is clear that lemma 4 holds. Now we suppose that \( (b, q) = 1 \). On noticing that \( |\tau(\chi)| \ll q^{1/2} \) and \( \sum_{n \leq N} \overline{\chi}(n) n \ll \ln N \), for \( N \geq q \), we obtain from (4) and (5)

\[
\sum_{\chi(-1) = -1} \tau^2(\chi) \chi(b)L^2(1, \overline{\chi}) = \sum_{\chi(-1) = -1} \tau^2(\chi) \chi(b) \left( \sum_{n \leq N} \overline{\chi}(n) n + O \left( \frac{\sqrt{q \ln q}}{N} \right) \right)^2
\]

\[
= \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} \sum_{\chi(-1) = -1} \frac{1}{mn} \sum_{1 \leq m, n \leq N} \chi(a) \chi(b) \chi(c) \overline{\chi}(mn)e \left( \frac{a + c}{q} \right)
\]

\[
+ O \left( \frac{\phi(q)q^{3/2} \ln q \ln N}{N} \right)
\]

\[
= \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} \sum_{\chi(-1) = -1} \frac{1}{mn} \sum_{1 \leq m, n \leq N} \chi(abc \overline{c}n)e \left( \frac{a + c}{q} \right) + O \left( \frac{\phi(q)q^{3/2} \ln q \ln N}{N} \right)
\]

\[
= \frac{1}{2} \phi(q) \sum_{1 \leq m, n \leq N} \frac{1}{mn} \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} e \left( \frac{a + c}{q} \right)
\]

\[
- \frac{1}{2} \phi(q) \sum_{1 \leq m, n \leq N} \frac{1}{mn} \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} e \left( \frac{a + c}{q} \right) + O \left( \frac{\phi(q)q^{3/2} \ln q \ln N}{N} \right)
\]

\[
= \frac{1}{2} \phi(q) \sum_{1 \leq m, n \leq N} \frac{1}{mn} \{S(1, \overline{b}mn; q) - S(1, -\overline{b}mn; q)\} + O \left( \frac{\phi(q)q^{3/2} \ln q \ln N}{N} \right)
\]

Taking \( N = q \), applying lemma 3 we may immediately get

\[
\sum_{\chi(-1) = -1} \tau^2(\chi) \chi(b)L^2(1, \overline{\chi}) \ll \phi(q)q^{1/2}d(q)\ln^2 q.
\]

This completes the proof of the lemma 4.

\[\square\]

**Lemma 5.** Let \( p \) be a prime \( > 3 \). Then for any integer \( b \) with \( (b, p) = 1 \), we have
Proof. Let \( K(r, q) = \sum_{a=1}^{p-1} \frac{1}{ac} S(\tilde{b}, -ac; p) \), then we have the estimate

\[
\frac{2}{p-1} \sum_{x \equiv (-1) \pmod{p}} \tau^2(\chi) \chi(b)L^2(1, \chi) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ac} S(\tilde{b}, ac; p)
\]

Thus

This completes the proof of (7).

By applying (6) with \( q = p, N = p^2 + p \) we get

\[
K(r, q) \ll \frac{p^q}{\sin \left( \frac{\pi r}{p} \right)}
\]

Namely we have

\[
K(r, q) \left( 1 - e \left( \frac{\gamma}{p} \right) \right) = \sum_{a=1}^{p-2} (a + 1)^q - a^q) e \left( \frac{(a + 1)\gamma}{p} \right) + e \left( \frac{\gamma}{p} \right) - (p - 1)^q e \left( \frac{p\gamma}{p} \right)
\]

\[
\ll p^q + \sum_{a=1}^{p-2} ((a + 1)^q - a^q) \ll p^q.
\]

Thus

\[
|K(r, q)| \ll \frac{p^q}{\left| 1 - e \left( \frac{\gamma}{p} \right) \right|} \ll \frac{p^q}{\sin \left( \frac{\pi r}{p} \right)}.
\]

This completes the proof of (7). \( \square \)

By applying (6) with \( q = p, N = p^2 + p \) we get

\[
\frac{2}{p-1} \sum_{x \equiv (-1) \pmod{p}} \tau^2(\chi) \chi(b)L^2(1, \chi)
\]

\[
= \sum_{1 \leq m,n \leq p^2 + p} \frac{1}{mn} \{ S(1, \tilde{b}mn; p) - S(1, -\tilde{b}mn; p) \} + O \left( \frac{\ln^2 p}{\sqrt{p}} \right)
\]

\[
= \sum_{a=1}^{p-1} \sum_{n=0}^{p} \sum_{c=1}^{p-1} \sum_{m=0}^{p} \frac{1}{(np + a)(mp + a)} \{ S(1, \tilde{b}ac; p) - S(1, -\tilde{b}ac; p) \}
\]

\[
+ O \left( \frac{\ln^2 p}{\sqrt{p}} \right)
\]
It is clear that $M_1$ is the main term of lemma 5. Now we shall estimate the other three terms $M_2$, $M_3$ and $M_4$. Using the power series expansion of $(1 - x)^{-1}$ we get

\[
\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp + c)} S(1, \overline{bac}; p) = \sum_{a=1}^{p-1} \frac{1}{a} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{(m + 1)p - (p - c)} S(1, \overline{bac}; p)
\]

\[
= \sum_{a=1}^{p-1} \sum_{c=1}^{\infty} \frac{1}{a} \sum_{k=0}^{\infty} p^{k+1} \sum_{m=1}^{p} \frac{1}{(m + 1)^{k+1}} \sum_{c=1}^{p-1} (p - c)^k S(1, \overline{bac}; p)
\]

\[
= \sum_{a=1}^{p-1} \sum_{c=1}^{\infty} \frac{1}{a} \sum_{k=0}^{\infty} p^{k+1} \sum_{m=1}^{p} \frac{1}{(m + 1)^{k+1}} \sum_{c=1}^{p-1} c^k S(1, \overline{bac}; p)
\]

(9)

Applying (7) we get the estimate

\[
\sum_{c=1}^{p-1} c^k S(1, \overline{bac}; p) = \sum_{d=1}^{p-1} e\left(\frac{d}{p}\right) \sum_{c=1}^{p-1} c^k e\left(\frac{-\overline{bacd}}{p}\right)
\]

\[
\ll \sum_{d=1}^{p-1} \frac{p^n}{\sin\left(\frac{\pi \overline{bad}}{p}\right)} \ll p^{k+1} \ln p
\]

(10)

From (9) and (10) we immediately deduce the estimate

\[
\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp + c)} S(1, \overline{bac}; p)
\]
Similarly we can deduce that

\[ \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \sum_{c=1}^{p-1} \frac{1}{a(m^2 + c)} S(1, -bac; p) \ll \ln^3 p \]  

(12)

Combining (11) and (12) we get

\[ M_2 \ll \ln^3 p \]  

(13)

In the same way we get the estimates

\[ M_3 \ll \ln^3 p \]  

(14)

\[ M_4 \ll \ln^3 p \]  

(15)

Now lemma 5 follows at once from (8), (13), (14) and (15).

\[ \square \]

L E M M A 6. Suppose \( \chi \) is an odd character \( \mod q \), generated by the primitive character \( \chi_m \mod m \). Then we have

\[
\sum_{a=1}^{q} a\chi(a) = \frac{q}{m} \left( \prod_{p \mid q, p \not\mid m} \frac{1}{1 - \chi(p)} \right) \left( \sum_{a=1}^{m} a\chi_m(a) \right)
\]

Proof. Let \( l \) be the largest divisor of \( q \) that is coprime with \( m \). Then we have

\[
\sum_{a=1}^{q} a\chi(a) = \sum_{l=0}^{(q/ml) - 1} \sum_{j=1}^{ml} (ilm + j)\chi(ilm + j)
\]

\[
= \sum_{l=0}^{(q/ml) - 1} \sum_{j=1}^{ml} (ilm + j)\chi(j) = \frac{q}{ml} \sum_{j=1}^{lm} j\chi(j)
\]

\[
= \frac{q}{lm} \sum_{a=1}^{lm} \sum_{d \mid a \atop d \mid l} \mu(d) a\chi_m(a)
\]

\[
= \frac{q}{lm} \sum_{d \mid l} \mu(d) \sum_{a=1}^{lm} a\chi_m(a)
\]
LEMMA 7. Let $\chi$ be a primitive character mod $m$ with $\chi(-1) = -1$. Then we have

$$\frac{1}{m} \sum_{\gamma=1}^{m} \gamma\chi(\gamma) = -\frac{i}{\pi} \tau(\chi)L(1, \bar{\chi})$$

Proof. (See Theorem 12.11 and Theorem 12.20 of [5]).

3. Proof of the theorems

In this section, we shall complete the proofs of the theorems. First we prove theorem 2. From lemmas 2, 4, 6 and lemma 7 we know that for every odd integer $q > 2$,

$$\sum_{\chi(-1) = -1} \chi(4) \left( \sum_{a=1}^{(q-1)/2} \chi(a) \right)^2$$

$$= \frac{1}{q^2} \sum_{\chi(-1) = -1} (1 - 2\chi(2))^2 \left( \sum_{\gamma=1}^{q-1} \gamma\chi(\gamma) \right)^2$$

$$= \sum_{m|q} \sum_{\chi \mod m \chi(-1) = -1} \frac{1}{m^2} (1 - 2\chi(2))^2 \left( \prod_{p|m} (1 - \chi_m(p)) \right)^2 \left( \sum_{a=1}^{m} a\chi_m(a) \right)^2$$

$$= -\frac{1}{\pi^2} \sum_{m|q} \sum_{\chi \mod m \chi(-1) = -1} (1 - 2\chi_m(2))^2 \left( \prod_{p|m} (1 - \chi_m(p)) \right)^2 \tau^2(\chi_m)L^2(1, \bar{\chi}_m)$$

$$= -\frac{1}{\pi^2} \sum_{m|q} \sum_{d|m} \mu(d) \sum_{\chi \mod m \chi(-1) = -1} (1 - 2\chi(2))^2 \left( \prod_{p|m} (1 - \chi(p)) \right)^2 \tau^2(\chi)L^2(1, \bar{\chi})$$
\[\ll \sum_{m \mid q} \sum_{k \mid m} 2^{\omega(q) - \omega(m)} k^{1/2} d(k) \phi(k) \ln^2 k\]

\[\ll \phi(q) q^{1/2} d^2(q) \ln^2 q\]  (16)

where \(\omega(n)\) denotes the number of all distinct prime divisors of \(n\). Now lemme 1 and (16) imply that

\[r(q) = \frac{1}{2} \phi(q) + O(q^{1/2} d^2(q) \ln^2 q)\]

This completes the proof of theorem 2. \[\square\]

Now we prove theorem 1. Let \(p\) be a prime \(>2\). Note that every odd character \(\chi \mod p\) is a primitive character \(\mod p\). Thus, from lemma 2 and lemma 7 we derive

\[
\sum_{\chi \equiv (-1)} \chi(4) \left( \sum_{a=1}^{(p-1)/2} \chi(a) \right)^2
\]

\[\quad = -\frac{1}{\pi^2} \sum_{\chi \equiv (-1)} (1 - 2\chi(2))^2 \tau^2(\chi) L^2(1, \overline{\chi})\]

\[\quad = -\frac{1}{\pi^2} \sum_{\chi \equiv (-1)} (1 - 4\chi(2) + 4\chi(4)) \tau^2(\chi) L^2(1, \overline{\chi})\]  (17)

From (17), lemma 1 and lemma 5 we deduce that

\[r(p) = \frac{1}{2} p + \frac{1}{\pi^2} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} \{S(1, ab; p) - S(1, -ab; p)\}
\]

\[+ 4S(\overline{4}, ab; p) - 4S(\overline{4}, -ab; p) - 4S(\overline{2}, ab; p)
\]

\[+ 4S(\overline{2}, -ab; p) + O(\ln^3 p)\]  (18)

Notice that

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(m, ab; p) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(ma, b; p).
\]

Together with (18) this implies theorem 1.
References


