

# COMPOSITIO MATHEMATICA

W. DUKE

H. IWANIEC

## **Convolution $L$ -series**

*Compositio Mathematica*, tome 91, n° 2 (1994), p. 145-158

<[http://www.numdam.org/item?id=CM\\_1994\\_\\_91\\_2\\_145\\_0](http://www.numdam.org/item?id=CM_1994__91_2_145_0)>

© Foundation Compositio Mathematica, 1994, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Convolution $L$ -series

W. DUKE\*<sup>†</sup> and H. IWANIEC\*

*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, U.S.A.*

Received 18 November 1992; accepted in final form 18 March 1993

### 1. Introduction

In the series of papers [4]–[7] we have developed several techniques for estimating the coefficients of  $L$ -functions which satisfy standard functional equations. Inspired by these works we now begin to examine convolution series formed by multiplying the coefficients.

Suppose we have two Dirichlet series

$$\mathcal{A}(s) = \sum_1^{\infty} a_n n^{-s}$$

$$\mathcal{B}(s) = \sum_1^{\infty} b_n n^{-s}$$

which converge absolutely in the half-plane  $Re\ s > 1$ , which have analytic continuation to entire functions and which satisfy functional equations of the type

$$\mathcal{A}(1-s) = \Phi(s)\mathcal{A}(s)$$

$$\mathcal{B}(1-s) = \Psi(s)\mathcal{B}(s).$$

Here  $\Phi(s)$  and  $\Psi(s)$  are certain holomorphic functions in  $Re\ s > 0$  which have at most a polynomial growth on vertical lines. Furthermore we assume that  $\mathcal{A}(s)$ ,  $\mathcal{B}(s)$  have Euler products of degree  $k$ ,  $l$ , i.e.,

$$\mathcal{A}(s) = \prod_p \mathcal{A}_p(p^{-s})^{-1}$$

$$\mathcal{B}(s) = \prod_p \mathcal{B}_p(p^{-s})^{-1}$$

\*Research supported by NSF Grant No. DMS-9202022.

<sup>†</sup>Research supported by the Sloan Foundation.

If  $Re\ s > 1$ , where  $\mathcal{A}_p(T), \mathcal{B}_p(T)$  are polynomials in  $T$  of degree  $k, l$  respectively with the constant term  $\mathcal{A}_p(0) = \mathcal{B}_p(0) = 1$ . We shall study the convolution series

$$C(s) = \sum_1^\infty a_n b_n n^{-s}.$$

Clearly, this convolution series converges absolutely in  $Re\ s > 2$ . Our objective will be to prove the absolute convergence in  $Re\ s > 1$  and to establish the analytic continuation up to  $Re\ s > 1/2$  subject to some further natural conditions.

Motivating examples are the symmetric power  $L$ -functions attached to an automorphic form. These have been intensively studied along the lines of Langlands program (see the survey articles by F. Shahidi [8] and D. Bump [2]). In this context our approach to analytic continuation is more elementary but the results are not quite complete since we cannot reach the critical line  $Re\ s = 1/2$  and prove a functional equation for the convolution series  $C(s)$ . Nevertheless our applications illustrate what can be concluded directly from the existence of functional equations for Dirichlet series without appealing to automorphic theory.

As in [4]–[7] our approach requires suitable functional equations for the twisted series

$$\mathcal{A}(s, \chi) = \sum_1^\infty a_n \chi(n) n^{-s}$$

$$\mathcal{B}(s, \chi) = \sum_1^\infty b_n \chi(n) n^{-s}.$$

We assume that for any primitive character  $\chi \pmod{q}$  the twisted series are entire functions of finite order and that they satisfy the compatible functional equations

$$\mathcal{A}(1 - s, \chi) = \alpha_\chi q^{k(s-1/2)} \Phi_\nu(s) \mathcal{A}(s, \bar{\chi}) \tag{1}$$

$$\mathcal{B}(1 - s, \chi) = \beta_\chi q^{l(s-1/2)} \Psi_\nu(s) \mathcal{B}(s, \bar{\chi}) \tag{2}$$

with  $|\alpha_\chi| = |\beta_\chi| = 1$ . Here we allow the factors  $\Phi_\nu(s), \Psi_\nu(s)$  to depend on the parity index  $\nu = \chi(-1) = \pm 1$  but not on  $\chi$  itself. It is assumed that  $\Phi_\nu(s), \Psi_\nu(s)$  are holomorphic in  $Re\ s > 0$  where they have a polynomial growth on vertical lines, i.e.,

$$\Phi_\nu(s), \Psi_\nu(s) \ll |s|^B \quad \text{if } Re\ s = \sigma > 0, \tag{3}$$

where  $B > 0$  and the implied constant depend on  $\sigma$ . Usually the factors  $\Phi_v(s)$ ,  $\Psi_v(s)$  of functional equations are products of suitable gamma functions but we do not need to assume this property because the  $s$ -aspect plays no role in the method.

However the signs  $\alpha_\chi, \beta_\chi$  of functional equations play the key part. Usually they are expressible in terms of the Gauss sum

$$\tau(\chi) = \sum_{x(\bmod q)} \chi(x)e_q(x)$$

where

$$e_q(x) = e\left(\frac{x}{q}\right) = e^{2\pi ix/q}$$

denotes the additive character. Let  $\varepsilon_\chi = \tau(\chi)q^{-1/2}$  be the sign of Gauss sum. It satisfies  $\bar{\varepsilon}_\chi = v\varepsilon_\chi = \varepsilon_\chi^{-1}$  for any primitive character.

Let us give two examples. If  $\mathcal{A}(s)$  is the  $k - 1$ th symmetric power  $L$ -function attached to a cusp form for the modular group then one expects (in accordance with the Langlands program) the functional equation for  $\mathcal{A}(s, \chi)$  to have the sign  $\alpha_\chi = \varepsilon_\chi^k$ . Another interesting example is the shifted Riemann zeta-function  $\mathcal{A}(s) = \zeta(ks - (k - 1)/2)$  which is useful for generating  $k$ th powers. In this case we have the functional equation for  $\mathcal{A}(s, \chi) = L(ks - (k - 1)/2, \chi^k)$  with the sign  $\alpha_\chi = \varepsilon_{\chi^k}$  provided  $\chi^k$  is primitive.

What truly matters in our argument is the Fourier transform

$$K_q(c) = \sum_{\chi(\bmod q)}^* \chi(c)\alpha_\chi\bar{\beta}_\chi \tag{4}$$

where the star restricts the summation to primitive characters. We can compute  $K_q(c)$  in two cases:

*Case 1.* Suppose  $\alpha_\chi = \beta_\chi$ . Then if  $(c, q) = 1$  we have

$$K_q(c) = \sum_{\chi(\bmod q)}^* \chi(c) = \sum_{w|(q,c-1)} \varphi(w)\mu(q/w). \tag{5}$$

For  $c = 1$  this yields the number of primitive characters to modulus  $q$ ,

$$K_q(1) = \sum_{sw=q} \mu(s)\varphi(w) = q \prod_{p|q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2.$$

If  $c \neq 1$  then  $K_q(c)$  is bounded on average in  $q$ . Moreover one senses a

“reciprocity law” for  $K_q(c)$  as  $w$  is switched into the complementary divisor of  $|c - 1|$  in (5).

Case 2. Suppose  $\alpha_x = v^{-h} \varepsilon_x^k$  and  $\beta_x = \varepsilon_x^l$  with  $h = l - k > 0$ . Then  $2k$  Gauss sums out of  $l + k$  annihilate themselves leaving  $\alpha_x \bar{\beta}_x = \varepsilon_x^h$ . Using (5) we infer that

$$\begin{aligned} q^{h/2} K_q(c) &= \sum_{\chi(\bmod q)}^* \bar{\chi}(c) \left( \sum_{x(\bmod q)} \chi(x) e_q(x) \right)^h \\ &= \sum_{sw=q} \mu(s) \varphi(w) \sum_{\substack{x_1, \dots, x_h(\bmod q) \\ x_1 \cdots x_h \equiv c(\bmod w)}}^* e_q(x_1 + \dots + x_h) \\ &= \sum_{\substack{sw=q \\ (s,w)=1}} \mu(s)^{h+1} \varphi(w) \sum_{\substack{x_1, \dots, x_h(\bmod w) \\ x_1 \cdots x_h \equiv c(\bmod w)}}^* e_w((x_1 + \dots + x_h)\bar{s}) \end{aligned}$$

where  $\bar{s}$  denotes the multiplicative inverse of  $s$  modulo  $w$ . Observe that the innermost sum is the generalized Kloosterman sum for which P. Deligne [3] has established the bound  $\tau_h(w)w^{(h-1)/2}$  (for prime modulus only but the extension to all moduli is straightforward). Employing Deligne’s bound we get

$$|K_q(c)| \leq \tau_h(q)q^{1/2}.$$

Of particular interest is the case of  $h = 1$  because the sum

$$q^{1/2} K_q(c) = \sum_{\substack{sw=q \\ (s,w)=1}} \mu^2(s) \varphi(w) e_w(c\bar{s}) \quad \text{if } (c, q) = 1 \tag{6}$$

can be transformed by means of the following ‘reciprocity’ formula

$$e_w(c\bar{s}) e_s(c\bar{w}) = e_q(c). \tag{7}$$

## 2. Statement of results

In this paper we explore Case 2 for  $k = 2$  and  $l = 3$ . It has been shown in [5] that both series formed by squaring the coefficients of  $\mathcal{A}(s)$  and  $\mathcal{B}(s)$  converge absolutely in  $Re\ s > 1$ . Hence by Cauchy’s inequality the convolution series  $\mathcal{C}(s)$  also converges absolutely in  $Re\ s > 1$ . Now we look deeper into the critical strip to prove analytic continuation in  $Re\ s > 1/2$ . For simplicity we shall assume more than the absolute convergence in  $Re\ s > 1$ , namely that the local polynomials  $\mathcal{A}_p(T)$  and  $\mathcal{B}_p(T)$  have bounded coefficients. This is indeed the case for  $L$ -functions attached to holomorphic cusp forms (the Ramanujan

conjecture proved by P. Deligne [3]). In general, one can probably avoid this condition by using the bounds on average.

**THEOREM 1.** *Suppose  $\mathcal{A}(s)$ ,  $\mathcal{B}(s)$  are Euler products of degree two and three with bounded coefficients such that  $\mathcal{A}(s, \chi)$ ,  $\mathcal{B}(s, \chi)$  are entire functions of finite order which satisfy the functional equations with signs  $\alpha_\chi = \varepsilon_\chi^2$  and  $\beta_\chi = \varepsilon_\chi^3$  respectively for all primitive characters. Then the convolution series  $\mathcal{C}(s)$  has analytic continuation without poles to the region  $Re s > 1/2$ .*

If we take for  $\mathcal{A}(s)$  the Hecke L-function attached to a cusp form for the modular group

$$L_1(s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

and for  $\mathcal{B}(s)$  we take the Shimura symmetric square L-function

$$L_2(s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}$$

then the convolution series  $\mathcal{C}(s)$  becomes  $L_1(s)L_3(s)P(s)$  where

$$L_3(s) = \prod_p (1 - \alpha_p^3 p^{-s})^{-1} (1 - \alpha_p^2 \beta_p p^{-s})^{-1} (1 - \alpha_p \beta_p^2 p^{-s})^{-1} (1 - \beta_p^3 p^{-s})^{-1}$$

is the symmetric cube L-function and  $P(s)$  is given by the product

$$P(s) = \prod_p (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})(1 - (\alpha_p + \beta_p)p^{-s})^{-1}$$

which converges absolutely in  $Re s > 1/2$ . By Theorem 1 we infer

**COROLLARY.** *The symmetric cube L-function  $L_3(s)$  attached to a Hecke eigencusp-form for the modular group has meromorphic continuation to the region  $Re s > 1/2$  whereas  $L_1(s)L_3(s)$  is holomorphic.*

**REMARKS.** The corollary is not new, it was proved in greater generality by F. Shahidi and others (see [8]) using quite different methods.

### 3. Applying the $\delta$ -symbol

Our approach to analytic continuation of  $\mathcal{C}(s)$  features estimates for finite sums of the type

$$\mathcal{D}(X) = \sum_n a_n b_n \eta^2 \left( \frac{n}{X} \right)$$

where  $\eta$  is any smooth function supported in the interval  $[1/2, 1]$ . We shall prove that

$$\mathcal{D}(X) \ll X^{1/2+\varepsilon} \tag{8}$$

and this shows through the Mellin transform that  $\mathcal{C}(s)$  is holomorphic in  $\operatorname{Re} s > 1/2$ .

We begin by writing

$$\mathcal{D}(X) = \sum_m \sum_n a_m b_n \eta\left(\frac{m}{X}\right) \eta\left(\frac{n}{X}\right) \delta_{mn}$$

where  $\delta_{mn}$  is the diagonal symbol of Kronecker. Then, as in [4], we use the formula

$$Y\delta_{mn} = \sum_{q|(m-n)} \left( \omega(q) - \omega\left(\frac{|m-n|}{q}\right) \right)$$

where  $\omega$  is any smooth function, compactly supported in  $\mathbb{R}^+$  and  $Y = \sum \omega(q)$ . We choose  $\omega(z) = \eta(z/\sqrt{X})$ , so  $Y \asymp \sqrt{X}$  and

$$Y\mathcal{D}(X) = \sum_q \sum_{m \equiv n \pmod{q}} \sum_{(mn, t) = 1} a_m b_n f\left(\frac{m}{X}, \frac{n}{X}, \frac{q}{\sqrt{X}}\right)$$

where

$$f(x, y, z) = \eta(x)\eta(y) \left( \eta(z) - \eta\left(\frac{|x-y|}{z}\right) \right).$$

Notice that  $f(x, y, z)$  is smooth and supported in the box  $[1/2, 1] \times [1/2, 1] \times [0, 1]$ .

Next we write by means of multiplicative characters

$$Y\mathcal{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sum_{\chi \pmod{q}}^* \sum_{\substack{(mn, t) = 1 \\ (mn, t) = 1}} \chi(m)\bar{\chi}(n) a_{rm} b_{rn} f\left(\frac{rm}{X}, \frac{rn}{X}, \frac{qrt}{\sqrt{X}}\right).$$

Further transformation of  $\mathcal{D}(X)$  requires factoring the coefficients  $a_{rm}, b_{rn}$  as well as relaxing the condition  $(mn, t) = 1$ . To this end we exploit the Euler products for  $\mathcal{A}(s)$  and  $\mathcal{B}(s)$ . There are numbers  $a_r(a) \ll r^\varepsilon$  defined for  $a|r$  such that for all  $m$  it holds that

$$a_{rm} = \sum_{am' = m} a_r(a) a_{m'}.$$

Also there are numbers  $b_r(b) \ll r^\epsilon$  defined for  $b|r^2$  such that for all  $n$  it holds that

$$b_{rn} = \sum_{bn'=n} b_r(b)b_{n'}.$$

The above factorizations yield

$$Y\mathcal{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a)b_r(b) \\ \times \sum_{\chi(\bmod q)}^* \sum_{(mn,t)=1} \chi(am)\bar{\chi}(bn)a_m b_n f\left(\frac{arm}{X}, \frac{brn}{X}, \frac{qrt}{\sqrt{X}}\right).$$

To relax the co-primality condition we again appeal to the Euler products. One can define numbers  $c_t(c) \ll t^\epsilon$  for  $c|t^2$  with the property that

$$a_m = \sum_{cm'=m} c_t(c)a_{m'}.$$

if  $(m, t) = 1$ , or else the sum vanishes. Also one can define numbers  $d_t(d) \ll t^\epsilon$  for  $d|t^3$  with the property that

$$b_n = \sum_{dn'=n} d_t(d)b_{n'}.$$

if  $(n, t) = 1$ , or else the sum vanishes. The above relations yield

$$Y\mathcal{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a)b_r(b) \sum_{(cd,q)=1} c_t(c)d_t(d) \\ \times \sum_{\chi(\bmod q)}^* \sum_m \sum_n \chi(acm)\bar{\chi}(bdn)a_m b_n f\left(\frac{acrm}{X}, \frac{bdrn}{X}, \frac{qrt}{\sqrt{X}}\right). \tag{9}$$

#### 4. Applying the functional equations

Now we are ready to execute the summation in  $m$  and  $n$ . Let us first consider a general character sum of the type

$$\Delta(\chi) = \sum_m \sum_n \chi(m)\bar{\chi}(n)a_m b_n g(m, n)$$

where  $g$  is a smooth function, compactly supported in  $\mathbb{R}^+ \times \mathbb{R}^+$ . Employing the functional equations for  $\mathcal{A}(s, \chi)$  and  $\mathcal{B}(s, \chi)$  by way of Mellin's transform



we infer

$$\Delta(\chi) = \alpha_\chi \bar{\beta}_\chi \sum_m \sum_n \bar{\chi}(m)\chi(n) a_m b_n g_\nu(mq^{-2}, nq^{-3}) q^{-5/2}$$

where  $g_\nu$  is an integral transform of  $g$  given by

$$g_\nu(x, y) = \frac{-1}{4\pi^2} \iint_{(\sigma_1, \sigma_2)} \check{g}(s_1, s_2) \Phi_\nu(s_1) \Psi_\nu(s_2) x^{-s_1} y^{-s_2} ds_1 ds_2$$

with  $\sigma_1, \sigma_2 > 0$  and

$$\check{g}(s_1, s_2) = \iint g(u, v) u^{-s_1} v^{-s_2} du dv.$$

Note that  $g_\nu$  depends only on the parity index  $\nu = \chi(-1) = \pm 1$  but not on the character itself. We put  $g^+ = g_1 + g_{-1}$  and  $g^- = g_1 - g_{-1}$  so  $2g_\nu = g^+ + \nu g^-$ . Summing over the primitive characters we evaluate the Fourier transform of  $\Delta(\chi)$  as follows

$$\sum_{\chi(\text{mod } q)}^* \chi(e) \Delta(\chi) = \frac{1}{2} \sum_{(mn, q)=1} a_m b_n K_q(\pm e \bar{m} n) g^\pm(mq^{-2}, nq^{-3}) q^{-5/2}$$

for any  $(e, q) = 1$ . In particular this together with (9) gives

$$2Y \mathcal{D}(X) = X^2 \sum_{rt < \sqrt{X}} \sum_{\substack{a|r \\ (ab, t)=1}} \sum_{b|r^2} \sum_{\substack{c|t^2 \\ d|t^3}} a_r(b) b_r(b) c_t(c) d_t(d) (abcd)^{-1} \mathcal{E} \tag{10}$$

where we have put

$$\mathcal{E} = \sum_{\substack{m \\ (abcdmn, q)=1}} \sum_n \sum_q q^{-5/2} \varphi(qt)^{-1} a_m b_n K_q(\pm acn \overline{bdm}) F^\pm \left( \frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}} \right)$$

and  $F^\pm = F_1 \pm F_{-1}$ , where  $F(u, v, z)$  are the integral transforms of  $f(x, y, z)$  given by

$$F_\nu(u, v, z) = \frac{-1}{4\pi^2} \iint_{(\sigma_1, \sigma_2)} \check{f}(s_1, s_2, z) \Phi_\nu(s_1) \Psi_\nu(s_2) u^{-s_1} v^{-s_2} ds_1 ds_2 \tag{11}$$

with

$$\check{f}(s_1, s_2, z) = \iint f(x, y, z) x^{-s_1} y^{-s_2} dx dy$$

on the lines  $Re s_1 = \sigma_1 > 0$  and  $Re s_2 = \sigma_2 > 0$ .

**5. Applying the reciprocity transformation**

Let  $(uv, q) = 1$ . By (6) and (7) we get

$$q^{1/2}K_q(u\bar{v}) = e\left(\frac{u}{vq}\right) \sum_{\substack{sw=q \\ (s,w)=1}} \mu^2(s)\varphi(w)e\left(\frac{-u\bar{w}}{vs}\right).$$

Since  $\varphi(qt)^{-1} = \varphi(st)^{-1}\varphi(w)^{-1}\sigma((t, w))$  with  $\sigma(h) = \varphi(h)h^{-1}$  we obtain

$$\mathcal{E} = \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw,abcdmn)=1}} \mu^2(s) \frac{\sigma((t, w))}{\varphi(st)} e\left(\pm \frac{acn\bar{w}}{bdms}\right) G(m, n, sw)$$

where for notational simplicity we have put

$$G(m, n, q) = q^{-3}e\left(\frac{\pm acn}{bdmq}\right) F^\pm\left(\frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}}\right). \tag{12}$$

In the sequel we shall denote  $u = \pm acn\delta^{-1}$ ,  $v = bdm\delta^{-1}$  where  $\delta = (acn, bdm)$ . Furthermore we split  $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$ , where

$$\begin{aligned} \mathcal{E}_0 &= \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw,\delta uv)=1}} \frac{\mu(vs)\sigma((t, w))}{\varphi(st)\varphi(sv)} G(m, n, sw), \\ \mathcal{E}_1 &= \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw,\delta uv)=1}} \mu^2(s) \frac{\sigma((t, w))}{\varphi(st)} \left( e\left(\frac{u\bar{w}}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)} \right) G(m, n, sw). \end{aligned}$$

**6. Estimating  $G(m, n, q)$**

First let us estimate the transform (11). The partial derivatives of  $f(x, y, z)$  are bounded by

$$f^{(ijk)}(x, y, z) \ll z^{-i-j-k}$$

with the implied constant depending on  $i, j, k$  only. Hence, by a repeated partial integration the Mellin transform satisfies

$$\frac{z^k \partial^k}{\partial z^k} \check{f}(s_1, s_2, z) \ll (1 + |s_1|z)^{-A}(1 + |s_2|z)^{-A}$$

for  $s_1, s_2$  on the vertical lines  $Re s_1 = \sigma_1 > 0, Re s_2 = \sigma_2 > 0$ , where  $A$  is an arbitrary positive number, the implied constant depending on  $\sigma_1, \sigma_2, A$  and  $k$  only. Since  $\Phi_v(s_1)$  and  $\Psi_v(s_2)$  have at most a polynomial growth we obtain

$$u^i v^j z^k F_v^{(ijk)}(u, v, z) \ll u^{-\sigma_1} v^{-\sigma_2} z^{-2B}$$

for any  $\sigma_1, \sigma_2 > 0$  and some constant  $B > 0$ . This yields

$$u^i v^j z^k F_v^{(ijk)}(u, v, z) \ll (1 + u)^{-A} (1 + v)^{-A} (uz^2)^{-B} (uv)^{-\varepsilon}$$

for any  $\varepsilon, A > 0$  and some constant  $B > 0$ . Finally, by a change of variables we conclude from the above and (12) that

$$m^i n^j q^k G^{(ijk)}(m, n, q) \ll q^{-3} \left(1 + \frac{mX}{acrq^2}\right)^{-A} \left(1 + \frac{nX}{bdrq^3}\right)^{-A} X^\varepsilon \tag{13}$$

for  $m, n, q \geq 1$ , where  $\varepsilon, A > 0$  are arbitrary and the implied constant depends on  $\varepsilon, A, i, j, k$  only.

Recall that  $G(m, n, q)$  vanishes if  $qrt > \sqrt{X}$ . Therefore, by (13) all terms in  $\mathcal{E}_0$  and  $\mathcal{E}_1$  are very small except for

$$\begin{aligned} 1 \leq m \leq M \quad &\text{with } M = acr^{-1}t^{-2}X^\varepsilon \\ 1 \leq n \leq N \quad &\text{with } N = bdr^{-2}t^{-3}X^{1/2+\varepsilon} \\ QX^{-\varepsilon} < q < Q \quad &\text{with } Q = (rt)^{-1}X^{1/2}. \end{aligned} \tag{14}$$

Notice that the above conditions imply  $M \leq X^\varepsilon, N \leq X^{1/2+\varepsilon}$ , and  $Q < X^{1/2}$ .

### 7. Estimating $\mathcal{E}_0$

First we execute the summation over  $n$  in  $\mathcal{E}_0$  by an appeal to the following general result:

LEMMA 1. *Let  $G(n)$  be a smooth function on  $\mathbb{R}^+$  whose derivatives satisfy*

$$n^j G^{(j)}(n) \ll \left(1 + \frac{n}{N}\right)^{-A}$$

for some  $N \geq 1$  and any  $A > 0$ . Then we have

$$\sum_{(n,t)=1} b_{rn} G(n) \ll (rtN)^\varepsilon. \tag{15}$$

*Proof.* Using the functional equation for  $\mathcal{B}(s)$  by contour integration we infer that

$$\sum_n b_n G(n) \ll N^\epsilon.$$

Hence

$$\sum_{(n,t)=1} b_{rn} G(n) = \sum_{\substack{b|r^2 \\ (b,t)=1}} b_r(b) \sum_{d|t^3} d_t(d) \sum_n b_n G(bdn) \ll (rtN)^\epsilon.$$

The sum over  $n$  in  $\mathcal{E}_0$  is of the type (15). More precisely we have

$$\mathcal{E}_0 = \sum_m a_m \sum_{\delta|bdm} \sum_{\substack{(s,w)=1 \\ (sw,abcdmm)=1}} \sum_{(acn,bdm)=\delta} \frac{\mu(vs)\sigma((t,w))}{\varphi(st)\varphi(sv)} \sum_{\substack{(n,sw)=1 \\ (acn,bdm)=\delta}} b_n G(m, n, sw).$$

Therefore, by Lemma 1 we obtain

$$\begin{aligned} \mathcal{E}_0 &\ll \sum_m |a_m| \sum_{q>Q} t^{-1} q^{-3} \left(1 + \frac{mX}{acrq^2}\right)^{-A} X^\epsilon \\ &\ll t^{-1} M Q^{-2} X^\epsilon \ll acrt^{-1} X^{\epsilon-1}. \end{aligned} \tag{16}$$

### 8. Estimating $\mathcal{E}_1$

We replace  $\sigma((t,w))$  by

$$\sigma((t,w)) = \sum_{\tau|(t,w)} \tau^{-1} \mu(\tau)$$

and relax the condition  $(w, \delta u) = 1$  using Möbius inversion to get

$$\begin{aligned} \mathcal{E}_1 &= \sum_m \sum_n a_m b_n \sum_{\substack{\tau|t \quad v|\delta u \\ (\tau,\delta uv)=(v,v)=1}} \tau^{-1} \mu(\tau) \mu(v) \sum_{(s,\tau\delta uv)=1} \mu^2(s) \varphi(st)^{-1} \\ &\quad \times \sum_{(w,sv)=1} \left( e\left(\frac{u\tau v w}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)} \right) G(m, n, \tau vsw). \end{aligned}$$

First, we shall show that small  $s$  contribute very little. To this end we establish the following general result:

LEMMA 2. Let  $G(w)$  be a smooth function supported in  $[0, W]$  whose derivatives

satisfy  $G^{(j)} \ll W^{-j}$ . Let  $g < W^{1-\varepsilon}$ . Then

$$\sum_{w \equiv a \pmod{g}} G(w) = \frac{1}{g} \sum_w G(w) + O(W^{-A}).$$

*Proof.* The Poisson summation gives

$$\frac{1}{g} \sum_h e\left(\frac{-ah}{g}\right) \hat{G}\left(\frac{h}{g}\right)$$

where  $\hat{G}$  is the Fourier transform of  $G$ . Integrating by parts one proves that

$$\hat{G}(y) \ll (1 + |y|W)^{-A}.$$

Hence our sum is equal to  $g^{-1}\hat{G}(0) + O(W^{-A})$ . This shows that the sum does not depend on  $a \pmod{g}$  up to the error term  $O(W^{-A})$ , giving the result.

**COROLLARY.** *If  $(a, g) = 1$  and  $g < W^{1-\varepsilon}$  then*

$$\sum_{(w,g)=1} \left( e\left(\frac{a\bar{w}}{g}\right) - \frac{\mu(g)}{\varphi(g)} \right) G(w) \ll W^{-A}. \tag{17}$$

The hypothesis of the above corollary to Lemma 2 is satisfied for  $G(w) = G(m, n, \tau vsw)$  in the range  $g = vs \leq (\tau vs)^{-1} QX^{-2\varepsilon}$  by virtue of (13), where  $Q = (rt)^{-1} X^{1/2}$ . We put

$$S = (\tau v v)^{-1/2} Q^{1/2} X^{-\varepsilon},$$

so (17) can be used for all  $s \leq S$ . Therefore, the contribution of this range to  $\mathcal{E}_1$  is

$$\mathcal{E}_{11} \ll X^{-A}. \tag{18}$$

In the remaining range of  $s > S$  we estimate the contribution of the part  $-\mu(vs)\varphi(vs)^{-1}$  trivially as follows:

$$\begin{aligned} \mathcal{E}_{12} &\ll \sum_m \sum_n |a_m b_n| \sum_{\tau|t} \tau^{-1} \sum_{v|\delta u} \varphi(t)^{-1} \varphi(v)^{-1} S^{-2} \sum_q |G(m, n, \tau vq)| \tau(q) \\ &\ll t^{-1} M N Q^{-3} X^\varepsilon \ll abcdt^{-3} X^{\varepsilon-1}. \end{aligned} \tag{19}$$

Now we are left with

$$\begin{aligned} \mathcal{E}_{13} &= \sum_m \sum_n a_m b_n \sum_{\substack{\tau|t \\ (\tau, \delta uv) = (v, v) = 1}} \tau^{-1} \mu(\tau) \mu(v) \\ &\quad \times \sum_{\substack{(s, \tau \delta uv) = 1 \\ s > S}} \mu^2(s) \varphi(st)^{-1} \sum_{(w, sv) = 1} e\left(\frac{u\tau vw}{vS}\right) G(m, n, \tau vsw). \end{aligned}$$

After changing the order of summation we estimate as follows:

$$\mathcal{E}_{13} \ll \varphi(t)^{-1} \sum_m |a_m| \sum_{\tau|t} \tau^{-1} \sum_{(v, v) = 1} \sum_{\delta|bdm} \mathcal{H} \tag{20}$$

where  $\mathcal{H}$  is a sum in  $s, w, n$  given by

$$\mathcal{H} = \sum_{\substack{(s, \tau v) = 1 \\ s > S}} \varphi(s)^{-1} \sum_{\substack{(w, sv) = 1 \\ \tau vsw < Q}} \left| \sum_{\substack{(n, \tau s) = 1, v|acn \\ (acn, bdm) = \delta}} b_n e\left(\frac{u\tau vw}{vS}\right) G(m, n, \tau vsw) \right|.$$

By virtue of (13) we can restrict the summation to the range (14) up to a small error term  $O(X^{-A})$ . Moreover we can separate  $n$  from the other variables of  $G(m, n, q)$  by any standard technique at no cost. In fact the integral representation (11) yields the desired separation without effort. We obtain

$$\mathcal{H} \ll Q^{-3} X^\varepsilon \sum_{\substack{(s, \tau v) = 1 \\ s > S}} s^{-1} \sum_{\substack{(w, sv) = 1 \\ \tau vsw < Q}} \left| \sum_{\substack{n > N, v|acn \\ (n, s) = 1}} \beta_n e\left(\frac{u\tau vw}{vS}\right) \right| + X^{-A}$$

for some  $\beta_n \ll b_n$ . For estimating this sum we shall use the large sieve inequality (see [1]).

LEMMA 3. For any complex numbers  $\beta_n$  it holds that

$$\sum_{\substack{s < T \\ (s, v) = 1}} \sum_{\substack{x \pmod{vs} \\ x \pmod{vs} \neq 0}}^* \left| \sum_{\substack{n < N \\ (n, s) = 1}} \beta_n e\left(\frac{nx}{vs}\right) \right|^2 \ll (vT^2 + N) \sum_n |\beta_n|^2.$$

From Lemma 3 by Cauchy’s inequality we deduce the following:

COROLLARY. If  $(ab, v) = 1$  it holds that

$$\sum_{\substack{sw < Q \\ (vs, abw) = 1 \\ s > S}} s^{-1} \left| \sum_{\substack{n < N \\ (n, s) = 1}} \beta_n e\left(an \frac{bw}{vs}\right) \right| \ll (vQ)^{1/2} \left(1 + \frac{Q}{vS}\right)^{1/2} \left(1 + \frac{N}{vS^2}\right)^{1/2} \left(\sum |\beta_n|^2\right)^{1/2}.$$

The corollary provides an estimate for a sum of the type we have in  $\mathcal{H}$ . It gives

$$\begin{aligned} \mathcal{H} &\ll Q^{-3} X^\varepsilon \left(\frac{vQ}{\tau v}\right)^{1/2} \left(1 + \frac{Q}{\tau v S^2}\right)^{1/2} \left(1 + \frac{(v, ac)N}{v S^2}\right)^{1/2} \left(\frac{(v, ac)}{v} N\right)^{1/2} \\ &\ll v^{-1}(v, ac)v^{1/2}Q^{-5/2}(1 + NQ^{-1})^{1/2}N^{1/2}X^\varepsilon \\ &\ll v^{-1}(v, ac)bdr^{3/2}t(1 + bdr^{-1}t^{-2})^{1/2}X^{\varepsilon-1} \ll v^{-1}(v, ac)bdr^2t^{3/2}X^{\varepsilon-1} \end{aligned}$$

because  $b|r^2, d|t^3$ , so  $bd \leq r^2t^3$ . Hence by (20) we derive

$$\mathcal{E}_{13} \ll bdr^2t^{1/2}MX^{\varepsilon-1} \ll abcdrt^{-3/2}X^{\varepsilon-1}. \tag{21}$$

Gathering together (18), (19), and (21) we conclude that

$$\mathcal{E}_1 \ll abcdrt^{-3/2}X^{\varepsilon-1}. \tag{22}$$

From (16) and (22) we obtain

$$\mathcal{E} \ll abcdrt^{-1}X^{\varepsilon-1}. \tag{23}$$

Finally by (10) and (23) we conclude (8). This completes the proof of Theorem 1.

**References**

1. E. Bombieri: Le grand crible dans la théorie analytique des nombres, *Astérisque* 18 (1973).
2. D. Bump: The Rankin-Selberg method. In: *Number Theory, Trace Formulas and Discrete Groups*. Oslo, Norway (1987), Academic Press (1989).
3. P. Deligne: Le conjecture de Weil I, *Publ. Math. I.H.E.S.* 43 (1974), 273–307.
4. W. Duke and H. Iwaniec: Estimates for coefficients of  $L$ -functions. I. In: *Automorphic Forms and Analytic Number Theory*. CRM Publications, Montreal (1990) pp. 43–47.
5. W. Duke and H. Iwaniec: Estimates for coefficients of  $L$ -functions. II. In: *Analytic Number Theory*. Proceedings of the Amalfi Conference (1989), Università di Salerno (1992) pp. 71–82.
6. W. Duke and H. Iwaniec, Estimates for coefficients of  $L$ -functions. III. In: *The Séminaire de Théorie des Nombres*. Paris (1989–90) pp. 113–120.
7. W. Duke and H. Iwaniec: Estimates for coefficients of  $L$ -functions. IV. *Amer. J. Math.* 116 (1993) 1–11.
8. F. Shahidi: Automorphic  $L$ -functions. In: *Automorphic Forms, Shimura Varieties, and  $L$ -Functions*. Proceedings of the conference in Ann Arbor (1988), Academic Press, Boston (1990) pp. 415–437.