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0. Introduction

Let \((x, y)\) denote a system of bihomogeneous coordinates on \(P^{n+1} \times P^{n+1}\) and \(\sigma\) the involution of \(P^{n+1} \times P^{n+1}\) defined by \(\sigma(x, y) = (y, x)\). In this paper we will check Grothendieck’s generalized Hodge conjecture for the general element of a family \(\pi: \mathcal{X} \to U\) of \(n\)-dimensional complete intersections \(\{X_t\}_{t \in U}\) in \(P^{n+1} \times P^{n+1}\) of \(\sigma\)-invariant divisors of bidegree \((1, 1)\) so that a general \(X_t\) admits a fixed point free involution \(\sigma\).

Our construction is motivated by the following classical 2-dimensional example (the Reye congruence [2] ex. VIII.19 p. 106).

Let \(P\) be a linear system of quadrics in \(P^3\) of projective dimension 3 satisfying

\[(1) \bigcap_{Q \in P} Q = \emptyset\]

\[(2)\text{ if } l\text{ is a line in } P^3\text{ which is the vertex of a quadric } Q \in P, \text{ then no other quadric of } P\text{ contains } l.\]

Let \(Y \subset \text{Gr}(P^1, P^3)\) denote the variety of lines which are contained in the intersection of all quadrics from a one dimensional linear subspace of \(P\). Then \(Y\) is an Enriques surface which is isomorphic to the quotient of the complete intersection in \(P^3 \times P^3\) of four \(\sigma\)-invariant divisors of bidegree \((1, 1)\).

We give a similar higher dimensional construction in section 1, together with a description of the topology and the cohomology of such varieties.

In section 2 we study the monodromy representation of the fundamental group \(\pi_1(U, t)\) on \(H^n(X_t, \mathbb{Q})\) for the family \(\pi: \mathcal{X} \to U\). If \(V^+\) and \(V^-\) denote the spaces of vanishing cycles respectively invariant and antiinvariant under \(\sigma\), the main result is that \(V^+\) and \(V^-\) are irreducible \(\pi_1(U, t)\)-subspaces of \(H^n(X_t, \mathbb{Q})\).

In section 3 we recall Grothendieck’s generalized Hodge conjecture (GGHC)
and introduce a suitable family

\[
T \rightarrow X_t \\
\downarrow \\
F
\]  

of algebraic 1-cycles on the general \(X_t\) in \(\mathcal{X} \rightarrow U\).

By using the infinitesimal cylinder map we show in section 4 that the morphism of Hodge structures deduced from (*)

\[
\Psi: H^n(X_t) \rightarrow H^{n-2}(F)
\]
is non-trivial and this, together with the irreducibility of \(V^\pm\) under the global monodromy representation, allows to conclude that the GGHC holds for \(t\) general in \(U\).

I would like to thank F. Bardelli for his helpful suggestions and encouragement.

1. Generalized Reye congruences

We start by giving the following generalization of the classical Reye congruence.

Let \(P\) be a linear system of quadrics in \(P^{n+1}\) of projective dimension \(n + 1\). We impose the following conditions on \(P\), which are satisfied if \(P\) is generic enough:

(i) \(\bigcap_{Q \in P} Q = \emptyset\),
(ii) if \(l\) is a line in \(P^{n+1}\) which is the vertex of a quadric \(Q \in P\), then there exists no \((n - 2)\)-dimensional linear system of quadrics in \(P\) containing \(l\).

Let \(Y\) be the variety of lines \(l \subset P^{n+1}\) contained in the quadrics of some \((n - 1)\)-dimensional linear system in \(P\) i.e. \(Y = \{l \subset P^{n+1}: \exists L \subset P \text{ proj dim } L = n - 1, L \supset l\}\).

If we let \((x, y)\) be a system of bihomogeneous coordinates on \(P^{n+1} \times P^{n+1}\) and \(\sigma\) be the involution defined by \((x, y) \rightarrow (y, x)\) we have the following:

**Proposition 1.1.** \(Y\) is isomorphic to \(X/\langle \sigma \rangle\) where \(X \subset P^{n+1} \times P^{n+1}\) is a smooth connected \(n\)-dimensional complete intersection of \(n + 2\) divisors of bidegree \((1, 1)\) invariant under \(\sigma\).

**Proof.** Let \(X\) be the subvariety of \(P^{n+1} \times P^{n+1}\) of pairs \((x, y)\) such that \(x\) and \(y\) are polar with respect to all the quadrics of \(P\). If we let \(Q_0, \ldots, Q_{n+1}\) be...
a basis of $P$, we can describe $X$ as the intersection of the divisors \{xQ_iy^T = 0\} 
\[i = 0, \ldots, n + 1.\] These divisors are invariant under $\sigma$. By the Jacobian criterion, $X$ is smooth and $n$-dimensional at a point $(x, y)$ if and only if the line $\langle x, y \rangle \subset P^{n+1}$ is not contained in the vertex of a quadric of $P$. This eventuality is excluded by condition (ii). By the Lefschetz hyperplane sections theorem (L.h.s.t.) we know that $h^0(X) = h^0(P^{n+1} \times P^{n+1}) = 1$, hence $X \subset P^{n+1} \times P^{n+1}$ is a smooth connected $n$-dimensional complete intersection. The fixed point set of the involution $\sigma$ of $P^{n+1} \times P^{n+1}$ is the diagonal. The induced action of $\sigma$ on $X$ is fixed point free because $xQ_iy^T = 0 \forall i$ contradicts (i). We can construct a map $p: X \to Y$ such that $p((x, y))$ is the line $\langle x, y \rangle$. In fact if $(x, y) \in X$, the quadrics of $P$ through $x$ and $y$ contain the line $\langle x, y \rangle$. The subspace 
\[L = \{Q \in P : xQx^T = 0 \} = yQy^T \]
has codimension two, whence $\langle x, y \rangle \in Y$. Conversely, let $l$ be a line of $Y$. The system $P$ induces on $l$ a pencil of 0-dimensional quadrics and there is exactly one pair of points $(x, y)$ polar with respect to all the quadrics of this pencil, whence $p$ induces an isomorphism between $X/\langle \sigma \rangle$ and $Y$.

In what follows we will study the varieties described before. From now on let $X$ and $Y$ be as in Prop. 1.1, $p: X \to Y$ the natural projection map, 
$p^*: H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ and $p_*: H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$. If $M$ is any module or vector space on which $\sigma$ acts, we denote by $M^+$ and $M^-$ the subspaces of invariant and antiinvariant elements of $M$ with respect to $\sigma$. Since $p: X \to Y$ is an unramified double cover and $Y$ is smooth, we have the following:

**PROPOSITION 1.2.** (1) $\chi_{\text{top}}(X) = 2\chi_{\text{top}}(Y)$ and $H^0(Y, \mathbb{Q}) = H^0(X, \mathbb{Q})^+$;
(2) the canonical bundle $K_X$ is trivial and $K_Y$ is trivial in case $n$ is odd;
(3) $H^n(X, \mathbb{Q})^+$ and $H^n(X, \mathbb{Q})^-$ are perpendicular with respect to the cup product;
(4) $H^{n,0}(X) \subseteq H^n(X, \mathbb{C})^+$ or $H^n(X, \mathbb{C})^-$ in case $n$ is respectively odd or even;
(5) the cup product over $H^n(X, \mathbb{Q})$ restricts to a non-degenerate alternating bilinear form on $H^n(X, \mathbb{Q})^+$ and on $H^n(X, \mathbb{Q})^-$.

**Proof.** (1) is obvious, being $p$ unramified;
(2) by definition of $X$ in $P^{n+1} \times P^{n+1}$ and the adjunction formula the triviality of $K_X$ follows $\forall n$. If $n$ is odd, the component of degree $n$ of the Todd class of $T_X$ is zero. The relations $c_i(T_Y) = c_i(p^*T_X) = p^*c_i(T_Y)$ and the injectivity of $p^*$ imply $(Td(T_Y))_n = 0$. Hence, by the Hirzebruch-Riemann-Roch formula, $\chi(O_Y) = 0$. By the L.h.s.t. we have $h^{i,0}(X) = 0$ and consequently $h^{i,0}(Y) = 0$ when $0 < i < n$. The relation $\chi(O_Y) = 0$ implies $h^{n,0}(Y) = 1$. Let $\alpha \in H^{n,0}(Y)$ be a generator, then $p^*(\alpha)$ is a generator of $H^{n,0}(X)$, hence div $p^*(\alpha) = 0$ and $\alpha$ cannot vanish at any point of $Y$, hence $K_Y = 0$;
(3) we can get the statement by computing the cup product between
elements \( a = (a + \sigma^*(a))/2 \) and \( b = (b - \sigma^*(b))/2 \) of \( H^n(X, \mathbb{Q})^+ \) and \( H^n(X, \mathbb{Q})^- \) respectively;

(4) since \( p^* \) maps \( H^n(Y, \mathbb{C}) \) isomorphically onto \( H^n(X, \mathbb{C})^+ \) and \( H^{n,0}(Y) \) onto \( H^{n,0}(X)^+ \), we have \( \mathbb{C} \simeq H^{n,0}(Y) \simeq H^{n,0}(X) \subset H^n(X, \mathbb{C})^+ \) if \( n \) is odd. If \( n \) is even \( 0 = H^{n,0}(Y) = H^{n,0}(X)^+ \) hence, by the non triviality of \( H^{n,0}(X) \), the statement follows;

(5) follows from (3) and the non-degeneracy of the cup product.

From the above facts we get the following:

**CONCLUSION 1.3.** In the case \( n \) is odd (even) \( H^n(X, \mathbb{Q})^- (H^n(X, \mathbb{Q})^+) \) is a \( \mathbb{Q} \)-Hodge substructure of \( H^n(X, \mathbb{Q}) \), perpendicular to \( H^{n,0}(X) \) with respect to the cup product.

As regards the topology of the varieties \( X \) and \( Y \), we can state the following:

**PROPOSITION 1.4.**

\[
\begin{align*}
  b^i(X) &= \begin{cases} 
    k + 1 & i = 2k, \ 2n - 2k \ k = 0, \ldots, \lceil(n-1)/2\rceil \\
    0 & i = 2k + 1 \ k = 0, \ldots, N - 1 \ i \neq n
  \end{cases} \\
  b^i(Y) &= \begin{cases} 
    (k + 1)/2 & i = 2k, \ 2n - k \ k \ odd, \ 0 < k \leq \lceil(n-1)/2\rceil \\
    (k + 2)/2 & i = 2k, \ 2n - k \ k \ even, \ 0 \leq k \leq \lceil(n-1)/2\rceil \\
    0 & i = 2k + 1 \ k = 0, \ldots, n - 1 \ i \neq n
  \end{cases}
\end{align*}
\]

where \( \lceil(n-1)/2 \rceil \) means the greatest integer less or equal than \( (n-1)/2 \).

**Proof.** Since \( X \) is a smooth \( n \)-dimensional complete intersection of very ample hypersurfaces in \( P^{n+1} \times P^{n+1} \), we know, by the L.h.s.t., that \( H^i(X, \mathbb{Q}) \simeq H^i(P^{n+1} \times P^{n+1}, \mathbb{Q}) \) \( 0 \leq i < n \) and, by the hard Lefschetz theorem, that \( b_i(X) = b^{2n-i}(X) \). If \( i \neq n \) and \( \omega_1 \) and \( \omega_2 \) denote the Poincare' duals of the hyperplane sections respectively of the first and the second \( P^{n+1} \) restricted to \( X \), by the Kunneth formula, we have that \( H^i(X) \) \( \neq 0 \) only if \( i \) is even is spanned by \( \langle \omega_1^{i/2}, \omega_2^{i/2-1} \wedge \omega_2, \ldots, \omega_2^{i/2} \rangle \), hence the first part of the statement.

Changing basis, if we let \( \omega^+ = (\omega_1 + \omega_2)/2 \) and \( \omega^- = (\omega_1 - \omega_2)/2 \), we have: if \( i = 2k, \ k \ even \), \( H^i(X) = H^i(X)^+ \oplus H^i(X)^- = \langle (\omega^+)^{i/2}, (\omega^+)^{i/2-2} \wedge (\omega^-)^2, \ldots, (\omega^-)^{i/2}\rangle \oplus \langle (\omega^+)^{i/2-1} \wedge \omega^-, \ldots, (\omega^+)^{i/2-1} \omega^- \rangle \) hence \( b_i(Y) = \dim H^i(X)^+ = (k + 2)/2 \); if \( i = 2k, \ k \ odd \), \( H^i(X) = H^i(X)^+ \oplus H^i(X)^- = \langle (\omega^+)^{i/2}, \ldots, (\omega^+) \wedge (\omega^-)^{i/2-1} \rangle \oplus \langle (\omega^+)^{i/2-1} \wedge \omega^-, \ldots, (\omega^-)^{i/2} \rangle \) hence \( b_i(Y) = (k + 1)/2 \); if \( i \) is odd, \( b_i(Y) = b^i(X) = 0 \).

**REMARK 1.5.** \( b^*(X) \) can be computed, once we know \( c_n(X) \), from the relation \( \chi_{top}(X) = c_n(X) \) since all the \( b^i(X)'s \) for \( i \neq n \) are known. If we denote by \( T^{p+1}_*|_{p+1}X \) the tangent bundle to \( P^{n+1} \times P^{n+1} \) restricted to \( X \), by \( T_{X} \) the tangent bundle to \( X \) and by \( N_{X|p+1 \times p+1} \) the normal bundle to \( X \) in
\( P^{n+1} \times P^{n+1}, \) the value of \( c_n(X) \) comes from the following relation on the Chern polynomials

\[
C(T_{P^{n+1} \times P^{n+1}}) = C(T_X)C(N_{X|P^{n+1} \times P^{n+1}})
\]

By means of the values \( b_i(Y) \) for \( i \neq n \) previously computed and the relation \( \chi_{\text{top}}(X) = 2\chi_{\text{top}}(Y) \) we get

\[
b^*(Y) = \begin{cases} 
\frac{b^*(X)}{2} - \frac{n}{4} & n = 2k \text{ k even} \\
\frac{b^*(X)}{2} + \frac{n + 2}{4} & n = 2k \text{ k odd} \\
\frac{b^*(X)}{2} + \frac{n + 3}{4} & n = 2k + 1 \text{ k even} \\
\frac{b^*(X)}{2} + \frac{n + 1}{4} & n = 2k + 1 \text{ k odd}
\end{cases}
\]

Now we want to construct a family of complete intersections admitting a fixed point free involution \( \sigma \) like in Prop. 1.1; let \((x, y)\) be bihomogeneous coordinates in \( P^{n+1} \times P^{n+1} \) and \( \Delta = \{(x, y) \in P^{n+1} \times P^{n+1}: x = y\} \) the subspace of fixed points of \( \sigma \). Let \( R = H^0(P^{n+1} \times P^{n+1}, O(1, 1)) \) be the set of \((1, 1)\)-forms on \( P^{n+1} \times P^{n+1} \). We consider the following decomposition: \( R = S \oplus A \) where \( S = \{S_0, \ldots, S_{N^2 = n + 1}\} \) is the subspace of \( \sigma \)-invariant \((1, 1)\)-forms and \( A = \{A_{N+1}, \ldots, A_{(n+2)^2 - 1}\} \) is the subspace of \((1, 1)\)-forms of \( P^{n+1} \times P^{n+1} \) antiinvariant under \( \sigma \). We define the following maps: \( v_2: P^{n+1} \times P^{n+1} \to P(S^*) \) by

\[
v_2(x, y) = (\ldots, S_i(x, y), \ldots)_{i=0,\ldots,N}
\]

and the Segre embedding \( \eta: P^{n+1} \times P^{n+1} \to P(R^*) \) by

\[
\eta(x, y) = (S_0(x, y), \ldots, S_N(x, y), A_{N+1}(x, y), \ldots, A_{(n+2)^2 - 1}(x, y))
\]

We get the following commutative diagram

\[
\begin{array}{ccc}
P^{n+1} \times P^{n+1} & \xrightarrow{\eta} & \Sigma \\
\downarrow v_2 & & \downarrow \pi \\
Z & & \\
\end{array}
\]

where \( \Sigma \) is a smooth variety isomorphic to \( P^{n+1} \times P^{n+1} \), \( v_2 \) and the projection \( \pi \) are finite morphisms of degree 2 onto \( Z \) and \( Z \) is smooth off \( \text{Sing } Z = v_2(\Delta) = \pi(\Sigma \cap \text{Ann}(A_i)_{i=1,\ldots,(n+2)^2 - 1}) \).
The image by \( v_2 \) of a smooth complete intersection of \( n + 2 \) symmetric divisors of bidegree \((1, 1)\) \( X \subset P^{n+1} \times P^{n+1} \) is given by \( Y = Z \cap L \), where \( L \) is the \( n(n + 3)/2 \)-projective dimensional linear subspace of \( P(S^*) \) defined by \( L = \text{Ann}\langle Q_0, \ldots, Q_{n+1} \rangle \). We get therefore

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & Y = \Sigma \cap \pi^{-1}L \\
& v_2 \downarrow & \downarrow \pi \\
& & Y
\end{array}
\]

The morphisms \( v_2 : X \to Y \) and \( \pi : Y \to Y \) have degree 2 and fibres of cardinality one exactly over the points of \( L \cap \text{Sing } Z \). We will choose \( L \) generic in such a way that \( L \cap \text{Sing } Z = \emptyset \) and \( L \) and \( Z \) are transversal at each point of \( Z \cap L \). In particular \( Y \) will be smooth and the double coverings \( v_2 : X \to Y \) and \( \pi : Y \to Y \) will be unramified. We get therefore a parametrization of the varieties \( X \)'s by the points of a Zariski open subset \( U \) of \( \text{Gr}((n + 1)(n + 2)/2, S^*) \cong \text{Gr}(P^{n+1}, P(S)) \) with the generic one smooth. We denote \( n : X \to U \) the family of smooth complete intersections of \( n + 2 \) symmetric divisors of bidegree \((1, 1)\) \( P^{n+1} \times P^{n+1} \) admitting the involution \( \sigma \).

2. The monodromy action

The aim of this section is to study the monodromy action of \( \pi_i(U, t) \) on \( H_n(X_t, \mathbb{Q}) \) for the family \( \pi : \mathcal{X} \to U \) previously constructed. Following the notations introduced in section 1, we consider an \((n + 1)(n + 2)/2\)-dimensional linear space \( L' \subset P(S^*) \) satisfying the following conditions:

2.0.1. \( L' \) is transversal to \( Z \) at all points of \( L' \cap (Z \setminus \text{Sing } Z) \).

2.0.2. \( L' \cap \text{Sing } Z \) is a finite set of \( M = 2^{n+1} \) distinct points \( P_1, \ldots, P_M \) along which \( L' \) and \( \text{Sing } Z \) intersect transversely.

Define \( W_S = L' \cap Z \) and its counterimages \( W = v_2^{-1}(W_S) \) and \( W_R = \pi^{-1}(W_S) \).

For this choice of \( L' \) we get:

**PROPOSITION 2.1.** (1) \( W \) is a smooth complete intersection of \( n + 1 \) divisors belonging to \( P(S) \).

(2) \( W_R \) is a smooth complete intersection of \( n + 1 \) hyperplane sections of \( \Sigma \), symmetric with respect to the hyperplanes given by \{Ann \( A_i \)|\( i = N + 1, \ldots, (n + 2)^2 - 1 \), and intersecting \( \text{Ann}\langle A_{N+1}, \ldots, A_{(n+2)^2-1} \rangle \) in exactly \( 2^{n+1} \) points (the images of the points of \( W \) fixed by \( \sigma \)).

**Proof.** (1) Since the smoothness of \( W \setminus \Delta \) follows from the smoothness of \( W_S \setminus \text{Sing } Z \), we are interested in studying the points \((\bar{x}, \bar{x}) \in W \cap \Delta \). Here the
tangent space to $W \subset P^{n+1} \times P^{n+1}$ has dimension $n + 1$; the transversality of $W = \bigcap_{i=0}^{n} V(Q_i)$ and $\Delta$ follows from condition 2.0.2 and the isomorphism $v_2: \Delta \to \text{Sing } Z$, hence the statement.

(2) follows from (1) and the definition of $\eta$. \hfill \Box

We define the dual variety $D_S$ of $W_S$ in $L'$ by

$$D_S = \bar{W}_S \cup \left( \bigcup_{i=1}^{n+1} H_i \right)$$

where $\bar{W}_S$ is the closure of the set of hyperplanes in $L'$ which are tangent at some point of $W_S \setminus \text{Sing } W_S$ and $H_i$ is the set of hyperplanes of $L'$ passing through $P_i$; in the same way we define the dual variety $D_R$ of $W_R$ in $(\pi^{-1}L')^\vee$ by $D_R = \bar{W}_R = \{\text{hyperplanes in } \pi^{-1}L' \text{ which are tangent at some point of } W_S\}$.

We want to study the homology group $H_n(X_t, \mathbb{Q})$ for a general variety $X_t$ of the family previously constructed and to do this we choose a pencil $\mathcal{P}$ of hyperplanes of $W_S$ by choosing a line $\ell$ in $E$ such that $\ell$ and $D_S$ are transversal at each point of $\ell \cap D_S$. If we consider the counterimages by $v_2$ we get a pencil of hypersurfaces of $W$ with these properties:

(1) there are exactly $M = 2^{n+1}$ hypersurfaces $X_{R_i}, i = 1, \ldots, M$ with an ordinary double point at $P_i = v_2^{-1}(P_i)$ which is a fixed point of $\sigma$ and no other singular point;

(2) if $r$ is the number of points of $\ell \cap \bar{W}_S$, we have $r$ hypersurfaces $X_{T_i}$ with 2 ordinary double points $P^1_i$ and $P^2_i$ interchanged by $\sigma$ and no other singular point;

(3) all the other hypersurfaces $X_t$ of the pencil are smooth.

We now fix a base point $t \in \ell^* = \ell \setminus \{R_1, \ldots, R_M, T_1, \ldots, T_r\}$ and let $\delta_i$ be the vanishing cycle attached to the singularity $P_i \in X_{R_i}, i = 1, \ldots, M$ and $\delta^1_i, \delta^2_i$ the vanishing cycles attached to the singularities $P^1_i$ and $P^2_i$ of $X_{T_i}, i = 1, \ldots, r$.

We choose orientations on the $\delta^j_i$'s in such a way that

$$\sigma_{\#}(\delta^1_i) = \delta^2_i \quad i = 1, \ldots, r.$$

A local computation shows that

$$\sigma_{\#}(\delta_i) = (-1)^{n+1} \delta_i \quad i = 1, \ldots, M.$$

By the hard Lefschetz theorem, $H_n(X_t, \mathbb{Q}) = V \oplus I$ where $V$ is spanned by the vanishing cycles introduced above and $I$ (if $n$ is even) is the space spanned by the invariant cycles $[\omega^+]^{(n/2)-i}, [\omega^-]^i i = 0, \ldots, n/2$, Poincare' duals of the restrictions to $X_S$ of the cohomology classes $\omega^+$ and $\omega^-$ where the
multiplication stands for the intersection pairing in homology ([7] 4.1.8 p. 30). If we denote $\delta_i^\pm = (\delta_1^+ \pm \delta_2^+)/2$ we have the following decompositions:

$$H_n(X_t, \mathbb{Q})^+ = V^+ \oplus I^+$$

and

$$V^+ = \left\{ \langle \delta_i, \delta_k^+ \rangle \mid i = 1, \ldots, M \quad k = 1, \ldots, r \quad n \text{ odd} \right\}$$

$$\langle \delta_k^+ \rangle \quad k = 1, \ldots, r \quad n \text{ even}$$

and

$$I^+ = \begin{cases} [\omega^+]^{\left(\frac{n}{2}\right)-2i}. [\omega^-]^{2i} & i = 0, \ldots, \frac{n-2}{4}, \quad n = 2k \quad k \text{ odd} \\ [\omega^+]^{\left(\frac{n}{2}\right)-2i}. [\omega^-]^{2i} & i = 0, \ldots, \frac{n}{4}, \quad n = 2k \quad k \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$H_n(X_t, \mathbb{Q})^- = V^- \oplus I^-$$

and

$$V^- = \left\{ \langle \delta_i, \delta_k^- \rangle \mid i = 1, \ldots, M \quad k = 1, \ldots, r \quad n \text{ even} \right\}$$

$$\langle \delta_k^- \rangle \quad k = 1, \ldots, r \quad n \text{ odd}$$

and

$$I^- = \begin{cases} [\omega^+]^{\left(\frac{n}{2}\right)-(2i+1)}. [\omega^-]^{2i+1} & i = 0, \ldots, \frac{n-2}{4}, \quad n = 2k \quad k \text{ odd} \\ [\omega^+]^{\left(\frac{n}{2}\right)-(2i+1)}. [\omega^-]^{2i+1} & i = 0, \ldots, \frac{n}{4}-1, \quad n = 2k \quad k \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Now we can state the

**2.2. MONODROMY THEOREM.** $V^+$ and $V^-$ are simple submodules for the global monodromy representation on $H_n(X_t, \mathbb{Q})$.

In order to prove the theorem we recall:

**2.3. The Picard-Lefschetz formulas ([7] 6.3.3, p. 40)**

$$\rho_i(\gamma) = \gamma + \varepsilon(\gamma, \delta_i)\delta_i$$

and

$$\tau_i(\gamma) = \gamma + 2\varepsilon(\gamma, \delta_i^+)\delta_i^+ + 2\varepsilon(\gamma, \delta_i^-)\delta_i^-$$
where

\[ \varepsilon = (-1)^{(\alpha + 1)(\alpha + 2)/2}, \gamma \in H_\alpha(X_\nu, \mathbb{Q}), \rho_i : H_\alpha(X_\nu, \mathbb{Q}) \to H_\alpha(X_\nu, \mathbb{Q}) \]

is the Picard-Lefschetz monodromy transformation associated to an elementary loop \( r_i \) in \( l^* \) based at \( s \) and encircling \( R_i \) but no other \( R_j \) for \( i \neq j \) or \( T_k \). Now \( \tau_i : H_\alpha(X_\nu, \mathbb{Q}) \to H_\alpha(X_\nu, \mathbb{Q}) \) is the monodromy transformation associated to an elementary loop \( t_i \) in \( l^* \) based at \( s \) and encircling \( T_i \) but no other \( T_j \) for \( i \neq j \) or \( R_k \);

2.4. The following propositions:

If we denote with the same symbols \( t_i \) and \( r_i \) the homotopy classes of \( t_i \) and \( r_i \) in \( \pi_1(l^*, s) \) and in \( \pi_1(L \setminus D, s) \), the following relations hold in \( \pi_1(L \setminus D, s) \):

1. \( \forall i, j = 1, \ldots, M, r_ir_j = r_jr_i \) ([1] Prop. 3.6, p. 179);
2. \( t_i, t_j \) are mutually conjugate \( \forall i, j = 1, \ldots, r \) and there exists \( u \in \pi_1(L \setminus D, s) \) such that \( u(\delta_i^+) = \pm \delta_j^+ \) ([7] 7.3.5, p. 46).

We also need

**Lemma 2.5.** (1) \( \forall i, j = 1, \ldots, M \)

\[
(\delta_i, \delta_j) = \begin{cases} 
0 & n \text{ odd} \\
0 & n \text{ even} \delta_i, \delta_j \text{ linearly independent} \\
\pm (-1)^n/2 & n \text{ even} \delta_i, \delta_j \text{ linearly dependent.}
\end{cases}
\]

2. \( \forall \delta_h, h = 1, \ldots, M \) there exists \( \delta_i^+ \) such that \( (\delta_h, \delta_i^+) \neq 0 \) (\( \delta_i^+ \) if \( n \) is odd, \( \delta_i^- \) if \( n \) is even).

**Proof.** (1) By 2.4(1) we know that \( \forall i, j, \rho_i \rho_j = \rho_j \rho_i \), hence for each \( \gamma \in H_\alpha(X_\nu, \mathbb{Q}) \) we have:

\[
\rho_i \rho_j(\gamma) = \gamma + \varepsilon(\gamma, \delta_i) \delta_j + \varepsilon(\gamma, \delta_j) \delta_i + \varepsilon^2(\gamma, \delta_i)(\delta_j, \delta_i) \delta_i,
\]

\[
\rho_j \rho_i(\gamma) = \gamma + \varepsilon(\gamma, \delta_j) \delta_i + \varepsilon(\gamma, \delta_i) \delta_j + \varepsilon^2(\gamma, \delta_i)(\delta_i, \delta_j) \delta_j,
\]

which gives

\[
(\gamma, \delta_i)(\delta_j, \delta_i) \delta_i = (\gamma, \delta_j)(\delta_i, \delta_j) \delta_j.
\]

By definition of vanishing cycle, the self-intersection number

\[
(\delta_i, \delta_j) = \begin{cases} 
0 & n \text{ odd} \\
(-1)^n/2 & n \text{ even.}
\end{cases}
\]
In case $n$ is odd, if $\delta_i, \delta_j$ are linearly dependent, the assertion is obvious; if they are independent, then there exists $\gamma_i \in H_n(X_r, \mathbb{Q})$ such that $(\gamma_i, \delta_i) = 0$ and $(\gamma_i, \delta_j) \neq 0$, hence, by $(*)$, the assertion follows.

If $n$ is even and $\delta_i, \delta_j$ are non zero and linearly dependent in $H_n(X_r, \mathbb{Q})$, we note that $\delta_i = \pm \delta_j$, in fact, there exist two rational numbers $a \neq 0$ and $b \neq 0$ such that $a\delta_j + b\delta_i = 0$, we have that

$$(\delta_i, \delta_j) = -\frac{b}{a} (\delta_i, \delta_i) = -\frac{b}{a} (-1)^{n/2} = -\frac{a}{b} (\delta_j, \delta_j) = -\frac{a}{b} (-1)^{n/2},$$

which gives $a = \pm b$ i.e. $\delta_i = \pm \delta_j$ and $(\delta_i, \delta_j) = \pm (-1)^{n/2}$.

If $\delta_i, \delta_j$ are independent, $(*)$ implies that for each $\gamma \in H_n(X, \mathbb{Q})$

$$(\gamma, \delta_j)(\delta_j, \delta_j) = (\gamma, \delta_i)(\delta_i, \delta_j) = 0,$$

but if we choose, for example $\gamma = \delta_i$, we get $(\delta_i, \delta_j)^2 = \pm 2(\delta_i, \delta_j) = 0$ and the assertion follows.

(2) Let $n$ be odd. In (1) it is proved that, given a vanishing cycle $\delta_h$, $(\delta_h, \delta_j) = 0$ for $j$. If $(\delta_h, \delta_i^+) = 0 \forall i = 1, \ldots , r$ the intersection pairing in $H_n(X_r, \mathbb{Q})^+$ would be degenerate, but this is a contradiction.

If $n$ is even, we note first the following facts.

Let $l$ be the line in $L^V$ introduced at the beginning of this section, corresponding to the $P_i$ of hypersurfaces $\{X\}_{i \in \mathbb{N}}$ of $W$ admitting the involution $\sigma$.

(i) If $\alpha$ is a hyperplane of the pencil $P_i$ passing through the image by $\nu_2$ of a fixed point $P_2$ of $W_R$ by the symmetry of $W_R$ and the meaning of $\pi^{-1}$, we have that $\pi^{-1}(\alpha)$ is tangent to $W_R$ at $\eta(P_2)$ i.e. $\cap H_i$ belongs to $\tilde{W}_R \cap \{A_i = 0\}_{i = N+1, \ldots , (n+2)^2-1}$.

(ii) If $\beta \in P_i$ is tangent to $W_S$ at a point $P_2 \notin \text{Sing} W_S$, its counterimage $\pi^{-1}(\beta)$ is tangent to $W_R$ at the two counterimages $P^1_\beta$ and $P^2_\beta$ of $P_\beta$. This implies that its corresponding point $B \in \tilde{W}_R$ is double for $\tilde{W}_R$, otherwise there would exist only one tangency point between $\pi^{-1}(\beta)$ and $W_R$.

To prove the assertion, we construct a general Lefschetz pencil of hyperplane sections of $W_R$ whose fibres no longer admit the involution $\sigma$. In particular, we may consider $L^V$ as a subspace of $(\pi^{-1}L)^V$ and consequently the line $l$ as a line in $L^V \subset (\pi^{-1}L)^V$; we choose a line $l'$ in $(\pi^{-1}L)^V$ 'close enough to $l'$ with the following properties:

- $l'$ is contained in $(\pi^{-1}L)^V$ but not in $L^V$;
- $l \cap l' = t$ where $t \in l \setminus \{R_1, \ldots , R_M, T_1, \ldots , T_r\}$;
- $l'$ and $D_R$ are transversal at each point of $l \cap D_R$. 

Let $R_i, T_{i1}$ and $T_{i2}$ be the points of $l \cap D_R$ ‘close’ to $R_i$ and $T_i \in l \cap D_R$ and $\delta_i, \delta_{i1}, \delta_{i2}$ the corresponding vanishing cycles. By choosing a suitable path in $(\pi^{-1}L)^\vee$, we can construct a $(n + 1)$-chain $\Gamma$ whose boundary is given by $\delta_i - \delta_i$. This implies that $\delta_i$ and $\delta_i$ are homologous in $X_t$. The same argument shows that $\delta_{i1}$ and $\delta_{i2}$ are homologically equivalent to $\delta_1$ and $\delta_2$ respectively and, as a consequence, $(\delta_i, \delta_i^1) = (\delta_i, \delta_{i1})$; $(\delta_i, \delta_i^2) = (\delta_i, \delta_{i2})$; $(\delta_i, \delta_i^1) = (\delta_{i1}, \delta_{i2})$.

To show that for each given $\delta_h$ there exists a $\delta_i^-$ such that $(\delta_h, \delta_i^-) \neq 0$, we show that $(\delta_h, \delta_i^1) \neq 0$, since

$$(\delta_h, \delta_i^-) = (\delta_h, \delta_i^1) - (\delta_h, \delta_i) = (\delta_h, \delta_i^1) - (\sigma_*\delta_h, \sigma_*\delta_i)$$

$$= (\delta_h, \delta_i^1) - (-\delta_h, \delta_i^1) = 2(\delta_h, \delta_i^1).$$

Suppose there exist no such $\delta_i^1$'s and, correspondingly, no $\delta_{i1}$'s such that $(\delta_h, \delta_{i1}) \neq 0$. By our choice of $l$, corresponding to a Lefschetz pencil $\mathcal{P}_i$, we know by the classical Lefschetz theory that, if we denote by $l^* = l \setminus \{R_i, T_{j1}, T_{j2}\}_{i=1,...,M; j=1,...,r}$, there exists an element $u \in \pi_1(l^*, t)$ such that $u(\delta_h) = \delta_{i1}$, but, by the Picard Lefschetz formulas, this implies that there exists at least a $\delta_k \neq \delta_h$ $k \neq h$ such that $(\delta_h, \delta_k) \neq 0$ and this $\delta_k$ must be one of the $\delta_i$ by 2.5 (1). This gives a contradiction.

**Proof of the monodromy theorem.** Suppose $n$ is odd. We know that $V^-$ is spanned by $\langle \delta_i^- \rangle$. If $F \subset V^-$ is a nontrivial $\pi_1$-invariant subspace, by the non-degeneracy of the intersection pairing on $V^-$, there exists $x \in F$ and some $\delta_i^-$ such that $(x, \delta_i^-) \neq 0$, but then, by the Picard Lefschetz formulas and the $\pi_1$-invariance of $F$, it follows that $\delta_i^-$, $\delta_i^+$ and 2.4 (2) implies that $F = H_n(X_t, \mathbb{Q})$. If $F \subset V^+$ is a nontrivial $\pi_1$-invariant subspace and $x \in F$, always by the nondegeneracy of the intersection pairing, there exists a vanishing cycle $\delta$ such that $(x, \delta) \neq 0$. By the same arguments as before, $\delta \in F$. If $\delta = \delta_h$ then by 2.5 (2) there exists a $\delta_i^+$ such that $(\delta_h, \delta_i^+) \neq 0$ and $\delta_i^+ \in F$. By 2.4 (2) and the $\pi_1$-invariance of $F$ all the $\delta_i^+$'s belongs to $F$. To finish the proof we note that $\forall \delta_k$ there exists a cycle $\delta_k^+$ such that $(\delta_k, \delta_k^+) \neq 0$ and by applying the transformation $\rho_k$ to $\delta_k^+$, we conclude that $\delta_k \in F$ for $k = 1, \ldots, M$ and we are done.

The same proof holds if $n$ is even by changing plus into minus.

3. A family of algebraic one cycles

In this section we want to construct a family of algebraic one cycles on the general variety $X_t$ of the family $\mathcal{X} \to U$ introduced in section 1. Let $X = X_t$ denote the variety we have fixed. If $Q_0, \ldots, Q_{n+1}$ denotes a fixed basis for the
linear system \( P \) satisfying the imposed generality conditions and \( W \) the smooth \((n + 1)\)-dimensional complete intersection given by \( W = \{(x, y) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} : xQ_0 y^T = 0, \ i = 1, \ldots, n + 1\} \), we can think of \( X \) as the hypersurface of \( W \) given by \( \{xQ_0 y^T = 0\} \) or, equivalently, as the hypersurface given by \( \{xQ_0 y^T = 0\} \) where \( Q_0 \) represents a rank \( n \) quadric of \( P \) which, together with \( Q_1, \ldots, Q_{n+1} \), spans the linear system.

Let \( F_n(X) \) be the variety of such quadrics i.e. the variety of quadrics in \( \mathbb{P}^{n+1} \) of rank \( n \) containing \( X \).

**Proposition 3.1.** \( F_n(X) \) is a \((n - 2)\)-dimensional variety whose singular locus, given by \( \{Q \in F_n(X) : \text{rank } Q < n\} \) has dimension \((n - 5)\).

**Proof.** It is well known that the dimension of the affine variety of quadrics in \( \mathbb{P}^{n+1} \) of rank \( n \) is \([(n + 2)(n + 3)/2] - 3 \) and that its singular locus is given by the quadrics of rank strictly smaller than \( n \). For a general choice of the \((n + 1)\)-dimensional linear system \( P \), the assertion follows. \( \square \)

Let \( Q_0' \in F_n(X) \); after a projective automorphism we can always arrange \( xQ_0' y^T = \Sigma_{i=0}^{n-1} x_i y_i \) so that it is immediate to see that \( xQ_0' y^T = 0 \) contains a \( P^1 \times P^{n+1} \) given by \( x_0 = \cdots = x_{n-1} = 0 \) and the corresponding \( P^{n+1} \times P^1 \) under the involution \( \sigma \).

Let us denote \( C_1 = X \cap (P^1 \times P^{n+1}) \) and \( C_2 = X \cap (P^{n+1} \times P^1) \).

**Proposition 3.2.** \( C_1 \) and \( C_2 \) are smooth rational curves on \( X \), complete intersections in \( W \) of the hyperplanes \( x_0 = \cdots = x_{n-1} = 0 \) and \( y_0 = \cdots = y_{n-1} = 0 \) respectively.

**Proof.** We will prove the assertion for \( C = C_1 \); the same proof holds for \( C_2 = \sigma(C_1) \) interchanging \( x \) with \( y \). Let \( \Gamma = P^1 \times P^{n+1} \) and \( W \) as above. By the exact sequence

\[
0 \to H^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{I}_\Gamma(1, 1)) \to H^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{O}(1, 1)) \\
\to H^0(\Gamma, \mathcal{O}_\Gamma(1, 1)) \to 0
\]

knowing that \( h^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{O}(1, 1)) = (n + 2)^2 \) and \( h^0(\Gamma, \mathcal{O}_\Gamma(1, 1)) = 2(n + 2) \), we have \( h^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{I}_\Gamma(1, 1)) = n(n + 2) \) hence, in the space of all divisors of bidegree \((1, 1)\) in \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \), the space of symmetric divisors of \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \) containing \( \Gamma \) has projective dimension \( \geq n(n + 2) + [(n + 2)(n + 3)/2] - (n + 2)^2 - 1 = [(n + 2)(n - 1)/2] - 1 \geq n + 1 = \dim P \) \( \forall n \geq 3 \) hence we can choose a quadric \( Q_0' \) in \( P(\mathcal{S}) \) containing \( \Gamma \) which is not linearly dependent on the quadrics defining \( W \) and such that \( C_1 = \Gamma \cap W \) is a smooth complete intersection. As regards the rationality, let \( \omega_1 \) and \( \omega_2 \) denote the first Chern classes of the hyperplane bundles of the first and the second \( \mathbb{P}^{n+1} \) respectively. It is immediate to see that

(1) \( C \) is algebraically equivalent to \( \omega_1^n + \omega_2^n \) in \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \);
(2) \( \Omega^{2+2}_{P^{n+1}} \simeq -(n+2)\omega_1 - (n+2)\omega_2; \)
(3) \( \det N_{W|P^{n+1}} \simeq (n+1)\omega_1 + (n+1)\omega_2 \) where \( N_{W|P^{n+1}} = \bigoplus^{n+1} \mathcal{C}(1, 1); \)
(4) \( \det N_{C|W} = n\omega_1 \) where \( N_{C|W} = \bigoplus^n \mathcal{C}(1, 0). \)

By the adjunction formula we have \( \Omega^+_{P^{n+1}} = -\omega_1 - \omega_2 \) and the class of \( \Omega_C \) in \( \text{Pic}(C) \) is represented by the divisor \( [(n-1)\omega_1 - \omega_2]_C \). Hence, by restriction to \( C \), we see that \( \deg \Omega_C = \deg([[(n-1)\omega_1 - \omega_2][\omega_1^n(\omega_1 + \omega_2)^{n+1}]) = -2 \) and the assertion follows.

By taking a desingularization \( \tilde{F} \) of \( F_n(X) \) as a parameter space for such quadrics \( Q_0 \), we get a family of curves

\[
\begin{array}{ccc}
T & \longrightarrow & X_t \\
\downarrow f & & \downarrow \pi \\
\tilde{F} & \longrightarrow & F \\
\end{array}
\]

By the presence of the involution \( \sigma \) which, given a point \( Q_0 \in F_n(X_t) \), interchanges the \( P^1 \times P^{n+1} \subset Q_0 \) into \( P^{n+1} \times P^1 \), we can consider a Stein factorization of the map \( f \)

\[
\begin{array}{ccc}
T & \stackrel{p}{\longrightarrow} & F = F_1 \cup F_2 \\
\downarrow f & & \downarrow h \\
\tilde{F} & \longrightarrow & F \\
\end{array}
\]

where \( h: F \to \tilde{F} \) is an unramified double cover, \( F \) is smooth and has two irreducible components \( F_1 \) and \( F_2 \). In fact, there is no closed path \( \gamma: [0, 1] \to \tilde{F} \) which, lifted to a path \( \tilde{\gamma} \) on \( F \), admits \( \tilde{\gamma}(0) \) and \( \tilde{\gamma}(1) \) lying on distinct sheets of the covering \( h: F \to \tilde{F} \).

We will think, from now on, of the family \( T \to F \) as the union of two families of curves on \( X_t \), \( \{(C_1)_r\}_{r \in F_1} \), and \( \{(C_2)_s\}_{s \in F_2} \), interchanged by the action of \( \sigma \).

If \( i \) denotes the map from \( F \) to the component of the Hilbert scheme parametrizing such curves on \( X_t \), it is not difficult to see that \( i \) is generically injective.

**Proposition 3.3.** If we let \( C \) be a rational curve of one of the above families (for example \( C = (C_1)_r \)) then

1. \( \det N_{C|X} = \mathcal{O}(-2); \)
2. \( N_{C|W} = \bigoplus_{r} \mathcal{O}(1); \)
3. \( N_{X|W|C} = \mathcal{O}(n + 2); \)
4. \( N_{C|X} = \bigoplus_{r} \mathcal{O}(-2) \).

**Proof.** (1) This follows from the adjunction formula and the triviality of \( \Omega_C^x \).
(2) $C$ is a complete intersection in $W$ of $n$ divisors of bidegree $(1, 0)$ hence $N_{C|W} = \bigoplus_{i=1}^n \mathcal{O}(1, 0)$. If $\omega_1$ and $\omega_2$ are the Chern classes introduced in the proof of Prop. 3.2, the restriction to $C$ of $\omega_1$ is equivalent to $\omega_1^{n+1}(\omega_1 + \omega_2)^{n+1} = \omega_2^{n+1} \omega_1^{n+1}$. Therefore it has degree 1 and $\mathcal{O}_C(1, 0) = \mathcal{O}_C(1)$;

(3) in the same way, being $N_{X|W|C} = \mathcal{O}_C(1, 1)$, the restriction to $C$ of $\omega_1 + \omega_2$ has degree $n + 2$;

(4) as seen in Prop. 3.1, we know that the parameter space $F$ for our family of curves has dimension $n - 2$ hence, in the generic point, the tangent space to the component of the Hilbert scheme parametrizing such curves must have dimension greater or equal than $n - 2$, thus $h^0(N_{C|X}) \geq n - 2$. If $N_{C|X} = \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i)$ denotes the decomposition of the $(n - 1)$-bundle $N_{C|X} \rightarrow C$, by the Riemann-Roch formula we get

$$
\chi(N_{C|X}) = \sum_{i=1}^{n-1} h^0(\mathcal{O}(a_i)) - \sum_{i=1}^{n-1} h^0(\mathcal{O}(-2 - a_i)) = n - 3
$$

hence $\sum_{i=1}^{n-1} h^0(\mathcal{O}(-2 - a_i)) = \sum_{i=1}^{n-1} h^0(\mathcal{O}(a_i)) + 3 - n \geq n - 2 + 3 - n = 1$. This implies there exists at least an index $j$, $1 \leq j \leq n - 1$, such that $a_j \leq -2$. Let us consider on $C$ the normal bundle sequence

$$
0 \rightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \rightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}(1) \rightarrow \mathcal{O}(n + 2) \rightarrow 0;
$$

by tensoring with $\mathcal{O}(-1)$ we get the corresponding cohomology exact sequence

$$
0 \rightarrow \bigoplus_{i=1}^{n-1} H^0(\mathcal{O}(a_i - 1)) \rightarrow \bigoplus_{i=1}^{n-1} H^0(\mathcal{O}) \xrightarrow{f} H^0(\mathcal{O}(n + 1))
\rightarrow H^1(\mathcal{O}(a_i - 1)) \rightarrow 0.
$$

We want to prove the injectivity of $f$. If this is the case, $\bigoplus_{i=1}^{n-1} H^0(\mathcal{O}(a_i - 1)) = 0$, hence $\forall i$ we get $a_i < 1$. By this and the previous relations $\sum_{i=1}^{n-1} a_i = -2$ and $\exists j$ such that $a_j \leq -2$ the assertion follows.

If $\sum_{i=0}^{n-1} x_i y_i = 0$ and $x_0 = \cdots = x_{n-1} = 0$ are respectively the equations of $X$ and $C$ in $W$, we see that the map

$$
f : \bigoplus_{i=1}^{n} H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(n + 1))
$$

is computed by:

$$(\alpha_1, \ldots, \alpha_n) \rightarrow \sum_{i=1}^{n} \alpha_i y_i$$
where \( y_i \) denote the restrictions to \( C \) of the \( n \) sections \( y_1, \ldots, y_n \in H^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{O}(0, 1)) \). If the sections \( y_i \) were linearly independent, \( f \) would be injective. Let us tensor by \( \mathcal{O}(0, 1) \) the exact sequence

\[
0 \to \mathcal{F}_C \to \mathcal{O}_{\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}} \to \mathcal{O}_C \to 0;
\]

in cohomology we get

\[
0 \to H^0(\mathcal{F}_C \otimes \mathcal{O}(0, 1)) \to H^0(\mathcal{O}_{\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}}(0, 1)) \overset{r}{\to} H^0(\mathcal{O}_C(0, 1)) \to 0
\]

where \( H^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{O}(0, 1)) \) is spanned by the \( n + 2 \) independent sections \( y_0, \ldots, y_{n+1} \) and, using the same argument as in (1) and (2), \( \mathcal{O}_C(0, 1) \simeq \mathcal{O}_C(n + 1) \) whence \( H^0(\mathcal{C}(0, 1)) \cong H^0(\mathcal{C}(n + 1)) \). Being the curve \( C \) defined as the complete intersection in \( \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \) of \( n + 1 \) symmetric forms of type \((1, 1)\) and \( n \) forms of type \((1, 0)\), \( H^0(\mathcal{C}(0, 1)) = 0 \) hence the restriction of the \( n \) independent sections \( y_1, \ldots, y_n \in H^0(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}, \mathcal{O}(0, 1)) \) to \( C \) remain independent and we are done.

Given our family

\[
\begin{array}{ccc}
T & \overset{q}{\longrightarrow} & X \\
p & \downarrow & \\
F & & \\
\end{array}
\]

let us consider the induced morphism of Hodge structure of type \((-1, -1)\)

\[
p_*q^*: H^n(X, \mathbb{Q}) \to H^{n-2}(F, \mathbb{Q})
\]

(the so called ‘integration over the fibre’). In the next section we prove the nontriviality of

\[
\Phi = p_*q^*_{(n-1,1)}: H^{n-1,1}(X) \to H^{n-2,0}(F).
\]

which implies the nontriviality of \( p_*q^* \).

4. The infinitesimal cylinder map

In order to study the relations between these families of curves and the cohomology of \( X \), let us consider the cylinder map

\[
q_\ast b^*: H_{n-2}(F) \to H_n(X),
\]

where $\gamma$ maps to $\bigcup_{i \in \mathbb{C}_i} C_i$. 

\[
\begin{array}{ccc}
\gamma & \overset{q_\ast b^*}{\longrightarrow} & H_n(X) \\
\end{array}
\]
the restriction to $H^{n-1,1}(X)$ of the corresponding map in cohomology

$$\Phi = p_* q_{(n-1,1)}^*: H^{n-1,1}(X) \to H^{n-2,0}(F).$$

and the composition map

$$\tau^* = r\Phi: H^1(\Omega_X^{n-1}) \to H^0(\Omega_F^{n-2}) \to \Omega_F^{n-2}.$$ 

where $r$ denotes the restriction to $0 \in F$.

To give a formula for $\tau^*$, if $C = C_0$, let

(1) $\psi^*: H^1(X, \Omega_X^{n-1}) \to H^1(C, /\wedge^{n-2} N_{C|X}^e \otimes \Omega^1_C) \cong H^0(\wedge^{n-2} N_{C|X}^e)*$

be the composition of the restriction map together with the map induced by the exact sequence

$$0 \to /\wedge^{n-1} N_{C|X}^e \to \Omega_X^{n-1} \to /\wedge^{n-2} N_{C|X}^e \otimes \Omega^1_C \to 0;$$

(2) $\eta^*: H^0(\wedge^{n-2} N_{C|X}^e)* \to /\wedge^{n-2} H^0(N_{C|X}^e)*$ be the dual of the natural map $\eta: /\wedge^{n-2} H^0(N_{C|X}^e) \to H^0(\wedge^{n-2} N_{C|X}^e)$;

(3) $\rho^*: /\wedge^{n-2} H^0(N_{C|X}^e)* \to \Omega_F^{n-2}$ be the dual of the map induced by the Kodaira Spencer map ([6] Def. 4, p. 150).

By the same arguments as in ([4] Thm. 2.25, p. 827) we have the following:

PROPOSITION 4.1. $\tau^* = \rho^* \eta^* \psi^*$.

Proof. Let $\Delta \in F$ be a polycylinder with coordinates $t_1, \ldots, t_{n-2}$, $t = 0$ its origin and let us choose local coordinates $z, w_1, \ldots, w_{n-1}$ on $X$ such that $C = C_0$ is given by $w_1 = \cdots = w_{n-1} = 0$. Locally, $C_t$ will be given by $w_i = f_i(z, t)$, where $f_i(z, t)$ is holomorphic and, by the condition $f_i(z, 0) = 0$, we can write

$$f_i(z, t) = \sum_j \frac{\partial f_i(z, t)}{\partial t_j} \bigg|_{t=0} t_j + [2]$$

where $[2]$ are terms of order $\geq 2$ in $t$.

If $\xi \in H^{n-1,1}(X)$, locally, we can write

$$\xi = \sum_{i=1}^{n-1} (-1)^i \xi_i(z, w)dz \wedge d\bar{z} \wedge dw_1 \wedge \cdots \wedge \widehat{dw_i} \wedge \cdots \wedge dw_{n-1} + [n-1]$$

where $[n-1]$ are terms which either do not involve $dz$ or do not involve $d\bar{z}$.

By definition of $\Phi = p_* q_{(n-1,1)}^*: H^{n-1,1}(X) \to H^{n-2,0}(F)$, we get $\Phi(\xi) =$
The composition of $\Phi$ with the restriction map gives therefore
\[
\tau^*(\xi) = (\int_C \det A(z, 0)dz \wedge d\bar{z})dt_1 \wedge \cdots \wedge dt_{n-2}.
\]

On the other hand $\psi^*(\xi) \in H^0(\bigwedge^{n-2}N_{C|X})^*$ is the element which, by Kodaira-Serre duality, corresponds to
\[
\sum_{i=1}^{n-1} (-1)^{i+1} \xi_i(z)dz \wedge d\bar{z} \otimes dw_1 \wedge \cdots \wedge dw_{i-1} \wedge \cdots \wedge dw_{n-1} \in H^1(C, \Omega^1_C \otimes \bigwedge^{n-2}N_{C|X})
\]
i.e.
\[
\psi^*(\xi) = \left( x \rightarrow \int_C x \otimes \sum_{i=1}^{n-1} (-1)^{i+1} \xi_i(z)dz \wedge d\bar{z} \otimes dw_1 \wedge \cdots \wedge dw_i \wedge \cdots \wedge dw_{n-1} \right)
\]
\[\forall x \in H^0(\bigwedge^{n-2}N_{C|X}).\]

Furthermore $\rho : \bigwedge^{n-2}T_{F,0} \to \bigwedge^{n-2}H^0(N_{C|X})$ acts as follows:
\[
\rho \left( \sum_{i=1}^{n-1} \frac{\partial}{\partial t_i} \wedge \cdots \wedge \frac{\partial}{\partial t_{n-2}} \right) = \sum_{i=1}^{n-1} \frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_i} \wedge \cdots \wedge \frac{\partial}{\partial w_{n-1}}
\]
where $(-1)^{i+1} \xi_i$ is the cofactor of the element $\xi_i$ in the matrix $A$. Therefore we have
\[
\rho^* \eta^* \psi^*(\xi) : \bigwedge^{n-2}T_{F,0} \overset{\rho}{\to} \bigwedge^{n-2}H^0(N_{C|X}) \overset{\eta}{\to} H^0(\bigwedge^{n-2}N_{C|X}) \to C
\]
\[
\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_{n-2}} \to \int_C \det A(z, 0)dz \wedge d\bar{z}
\]
which, thought as an element of $\Omega^{n-2}_{F,0}$ is exactly $\Phi^*(\xi)$. 

What we want to do now, is to describe $\eta^*$ and $\psi^*$ in our situation. We will follow the notations previously introduced supposing, for example, $0 \in F_1$
and consequently \( C = (C_1)_{0} \).

To describe \( \eta^{*} \) let us consider the dual map \( \eta: \wedge^{n-2}H^{0}(N_{C|X}) \to H^{0}(\wedge^{n-2}N_{C|X}) \).

By Prop. 3.3(4) the map \( \eta \) may be written

\[
\wedge^{n-2}H^{0}(\bigoplus_{i=1}^{n-2} \mathcal{O}e_{i} \oplus \mathcal{O}(-2)) \to H^{0}(\wedge^{n-2}(\bigoplus_{i=1}^{n-2} \mathcal{O}e_{i} \oplus \mathcal{O}(-2)))
\]

\[
\simeq H^{0}(\mathcal{O}(e_{1} \wedge \cdots \wedge e_{n-2})).
\]

Since \( e_{1} \wedge \cdots \wedge e_{n-2} \) gets mapped to \( e_{1} \wedge \cdots \wedge e_{n-2} \), \( \eta \) is an isomorphism of one dimensional vector spaces.

To study \( \psi^{*} \) let us consider

\[ 0 \to N_{C|w}^{*} \to T_{w}^{*} \otimes \mathcal{O}_{C} \to T_{X}^{*} \to 0 \tag{a} \]

and the induced sequence

\[ 0 \to \wedge^{n}N_{C|w}^{*} \to \Omega_{W}^{n} \otimes \mathcal{O}_{C} \to \wedge^{n-1}N_{C|w}^{*} \otimes \Omega_{C}^{1} \to 0 \tag{\dot{a}} \]

which tensored with \( N_{X|w} \) gives

\[ 0 \to \wedge^{n}N_{C|w}^{*} \otimes N_{X|w} \to \Omega_{W}^{n} \otimes N_{X|w} \otimes \mathcal{O}_{C} \to \wedge^{n-1}N_{C|w}^{*} \otimes \Omega_{C}^{1} \otimes N_{X|w} \to 0. \tag{\ddot{a}} \]

Furthermore let us consider the sequence

\[ 0 \to N_{X|w}^{*} \to T_{w}^{*} \otimes \mathcal{O}_{X} \to T_{X}^{*} \to 0 \tag{b} \]

which taking exterior \( n \)-powers and tensoring with \( N_{X|w} \) induces

\[ 0 \to \Omega_{X}^{n-1} \to \Omega_{w}^{n} \otimes N_{X|w} \otimes \mathcal{O}_{X} \to \Omega_{X}^{n} \otimes N_{X|w} \to 0. \tag{\dddot{b}} \]

Let us put \((\dot{a})\) and \((\dddot{b})\) \( \otimes \mathcal{O}_{C} \) into the following diagram:

\[
\begin{array}{ccc}
(\ast) & (\dot{a}) \\
0 & 0 \\
\downarrow & \downarrow \\
\wedge^{n-1}N_{C|X}^{*} & \wedge^{n}N_{C|w}^{*} \otimes N_{X|w} \\
\downarrow & \downarrow \\
0 & \Omega_{X}^{n-1} \otimes \mathcal{O}_{C} & \Omega_{w}^{n} \otimes N_{X|w} \otimes \mathcal{O}_{C} & \Omega_{X}^{n} \otimes N_{X|w} \otimes \mathcal{O}_{C} & 0 \tag{\dddot{b}} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Omega_{C} \otimes \wedge^{n-2}N_{C|X}^{*} & \wedge^{n-1}N_{C|w}^{*} \otimes \Omega_{C}^{1} \otimes N_{X|w} & \Omega_{X}^{n} \otimes N_{X|w} \otimes \mathcal{O}_{C} \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]
The isomorphism \( \alpha \), which is equivalent to \( \det N_{C|W} \cong \det N_{C|X} \otimes \det N_{X|W} \otimes \mathcal{O}_C \), comes from the exact sequence

\[
0 \to N_{C|X} \to N_{C|W} \to N_{X|W} \otimes \mathcal{O}_C \to 0.
\]

A simple diagram chase now shows that the map \( \beta \) is injective.

By means of adjunction, the isomorphism \( \alpha \) and the duality on bundles, we obtain isomorphisms

(i) \( \Omega^1_C \otimes \wedge^{n-2} N^*_{C|X} \cong \Omega^n_X \otimes \wedge^{n-1} N_{C|X} \wedge^{n-2} N^*_{C|X} \cong \Omega^X_X \otimes N_{C|X} \);

(ii) \( \wedge^{n-1} N^*_{C|X} \otimes \Omega^1_C \otimes N_{X|W} \cong N_{C|W} \otimes \wedge^{n-1} N^*_{C|X} \otimes \Omega^1_C \otimes N_{X|W} \cong N_{C|W} \otimes \Omega^X_X \)

which allow us to replace the last line of the preceding diagram with

\[
0 \to \Omega^X_X \otimes N_{C|X} \to \Omega^X_X \otimes N_{C|W} \to \Omega^X_X \otimes N_{X|W} \otimes \mathcal{O}_C \to 0.
\]

To describe \( \psi^* \) let us consider

\[
0 \to \Omega^{X-1}_X \to \Omega_W \otimes N_{X|W} \to \Omega_X \otimes N_{X|W} \to 0
\]

and the induced diagram

\[
\begin{array}{ccccccc}
H^0(\Omega^n_W \otimes N_{X|W}) & \to & H^0(\Omega^n_X \otimes N_{X|W}) & \to & H^1(\Omega^n_X \otimes N_{X|W}) & \to & H^1(\Omega^n_W \otimes N_{X|W}) \\
\downarrow & & \downarrow & & \downarrow \psi^* & & \downarrow \\
H^0(\Omega^n_C \otimes N_{C|W}) & \to & H^0(\Omega^n_X \otimes N_{X|W} \otimes \mathcal{O}_C) & \to & H^1(\Omega^n_X \otimes N_{C|X}) & \to & H^1(\Omega^n_C \otimes N_{C|W}) \\
\end{array}
\]

\[
H^0(\wedge^{n-2} N_{C|X})^*
\]

**PROPOSITION 4.2.** In our situation \( \psi^* \) is a non trivial surjective map.

**Proof.** The above diagram becomes

\[
\begin{array}{ccccccc}
H^0(\Omega_w^n \otimes \mathcal{C}(1, 1)) & \to & H^0(\mathcal{C}(1, 1)) & \to & H^1(\Omega_{X}^{n-1}) & \to & H^1(\Omega_w^n \otimes \mathcal{C}(1, 1)) \\
\downarrow & & \downarrow b & & \downarrow \psi^* & & \downarrow \\
H^0(\mathcal{C}(n+2)) & \to & H^0(\mathcal{C}(n+2)) & \to & H^1(\mathcal{C}(n+2)) & \to & 0 \\
a & & a & & \end{array}
\]

where \( a \) and \( b \) are surjective maps. The surjectivity of \( a \) is obvious; let us
consider the following commutative diagram

\[
\begin{array}{ccc}
H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(1, 1)) & \xrightarrow{c} & H^0(\mathcal{O}(1, 1)) \\
\downarrow r & & \downarrow b \\
H^0(\mathcal{O}_X(1, 1)) & \rightarrow & H^0(\mathcal{O}_c(n+2))
\end{array}
\]

We know \(h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(1, 1)) = (n + 2)^2\) while an easy computation shows that \(\text{dim Ker } c = n(n + 2) + n + 1 = n^2 + 3n + 1\). Then \(\text{dim Im c} = (n + 2)^2 - (n^2 + 3n + 1) = n + 3 = h^0(\mathcal{O}_c(1, 1))\) i.e. \(c\) and hence \(b\) are surjective. By the non triviality of \(H^1(N_C|X)\) and the surjectivity of \(a\) and \(b\) the assertion follows.

By Proposition 4.1 and 4.2 we get the following:

CONCLUSION. \(\Phi = p_*q^*|(n-1,1): H^{n-1,1}(X) \rightarrow H^{n-2,0}(F)\) is not trivial.

5. The GGHC and the conclusion

Let us recall the GGHC for a smooth connected complex projective variety \(X\) ([5], p. 300). We denote \(F^*H'(X, C)\) the Hodge filtration defined by

\[F^pH^i(X, C) = \bigoplus_{p' + q = 1} H^{p',q}(X)\]

and \(F''H'(X, Q)\) the arithmetic filtration defined by

\[F''H^i(X, Q) = \left\{ \eta \in H^i(X, Q) : \exists a \text{ Zariski closed set } Z \subseteq X \text{ with codim } Z \geq p \text{ and } \eta|_{X \setminus Z} = 0 \right\}\]

We note that \(F^*H^i\) is defined over \(C\) and generally it is not induced by a corresponding filtration defined over \(Q\) while \(F''H^i\) is defined over \(Q\) and it can be shown, by using standard exact sequences and [3] thm. 8.2.7–8.2.8 p. 40, that \(F^pH^i\) is the space spanned by the images of the Gysin morphisms \(H^{i-2q}(Y, Q) \rightarrow H^i(X, Q)\) for any desingularization \(Y\) of closed subschemes \(Z \subseteq X\) of pure codimension \(q \geq p\). As a consequence, we have that \(F''H^i(X, Q)\) spans a \(Q\)-Hodge substructure of \(H^i(X, C)\) contained in \(F^pH^i(X, C) \cap H^i(X, Q)\). In particular, if we denote \(F''M^i\) the maximal \(Q\)-Hodge substructure of \(F''H^i(X, C)\), we have that \(F''H^i(X, Q) \subseteq F''M^i\).

The GGHC states that this is an equality, i.e. \(F''M^i \subseteq F''H^i(X, Q)\).

For \(i = 2p\), this is nothing else that the usual Hodge conjecture for rational cohomology classes of type \((p, p)\). In fact in this case \(F''M^{2p}\) coincides with
$H^{p,p}(X, C) \cap H^{2p}(X, Q)$ and $F'_{p}H^{2p}(X, Q)$ is the space of $2p$-cohomology classes supported by subvarieties of $X$ of codimension $\geq p$, therefore exactly $p$ (or, equivalently, which are Poincaré duals of such subvarieties).

If $i \neq n$, we saw in Prop. 1.5 that $H^i(X)$ is spanned by the Poincaré duals of intersections of hyperplane sections of $P^{n+1} \times P^{n+1}$ restricted to $X$, hence, in this case, the GGHC is easily checked.

If $i = n$, let us consider the family

$$T \xrightarrow{q} X$$

$$p |$$

$$F$$

of algebraic one cycles on $X$ introduced in section 3 and the induced sequence

$$H_{n-2}(F, Q) \to H_n(T, Q) \to H_n(q(T), Q) \to H_n(X, Q).$$

It is not difficult to see that $\dim q(T) = n - 1$.

**PROPOSITION 5.1.** (1) The maximal $Q$-Hodge substructure $F^1M^n$ contained in $F^1H^n(X, C) \cap H^n(X, Q)$ is $H^n(X, Q)^-$ if $n$ is odd and $V^+ \oplus I^+ \oplus I^-$ if $n$ is even.

(2) If $n$ is even, the maximal $Q$-Hodge substructure $F^{n/2}M^n = H^{n/2,n/2}(X, C) \cap H^n(X, Q)$ is $I^+ \oplus I^-$. The proof is a straightforward consequence of the $\pi_1$-invariance of the maximal $Q$-Hodge substructures $F^1M^n$ and Thm. 2.2.

**PROPOSITION 5.2.** The image of the Gysin morphism

$$\lambda: H^{n-2}(q(T), Q) \to H_n(q(T), Q) \to H_n(q(T), Q) \to H_n(X, Q) \to H^n(X, Q),$$

where $q(T)$ denotes a desingularization of $q(T)$, coincides with $H^n(X, Q)^-$ if $n$ is odd. If $n$ is even, $V^+ \subseteq \text{Im} \lambda \subseteq V^+ \oplus I^+ \oplus I^-.$

**Proof.** We know the following facts:

— $\lambda$ is nontrivial by section 4;
— $\text{Im} \lambda$ is contained in $F^1H^n(X, Q)$ since $\text{codim} q(T) = 1$;
— $\text{Im} \lambda$ generates a $Q$-Hodge substructure of $H^n(X)$ invariant under monodromy and contained in $F^1H^n(X, C) \cap H^n(X, Q)$.

If $n$ is odd, by 5.1 (1), $\text{Im} \lambda \subseteq H^n(X, Q)^- = F^1M^n$. By the above facts and the irreducibility of $H^n(X, Q)^-$ under monodromy they must coincide.
If $n$ is even, by 5.1 (1), $\text{Im} \lambda \subseteq V^+ \oplus I^+ \oplus I^-$. We know by the previous results that $\text{Im} \lambda \cap V^+ \neq 0$ hence, by the same irreducibility argument as before, $V^+ \subseteq \text{Im} \lambda$.

CONCLUSION 5.3. (1) In case $n$ is even, the classical Hodge conjecture holds;

(2) the GGHC holds for $F^1M^n$.

Proof. (1) By the meaning of $I^+ \oplus I^-$, the classical Hodge conjecture

$$H^{n/2, n/2}(X, \mathbb{C}) \cap H^n(X, \mathbb{Q}) = F^{n/2}M^n = I^+ \oplus I^- \subseteq F^{n/2}H^n(X, \mathbb{Q})$$

is exactly the assertion 5.1 (2).

(2) If $n$ is odd, by 5.1 (1) and 5.2, we get $F^1M^n = H^n(X, \mathbb{Q})^- = \text{Im} \lambda \subseteq F^1H^n(X, \mathbb{Q})$. If $n$ is even, by 5.2 we know that $V^+ \subseteq \text{Im} \lambda \subseteq F^1H^n(X, \mathbb{Q})$. On the other hand, $I^+ \oplus I^- = F^{n/2}H^n(X, \mathbb{Q})) = F^1H^n(X, \mathbb{Q})$ hence,

$$F^1M^n = V^+ \oplus I^+ \oplus I^- \subseteq F^1H^n(X, \mathbb{Q}).$$

References