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An isoperimetric inequality for the area of plane regions defined by binary forms

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Abstract. In this paper, it is shown that if $F$ is a binary form with complex coefficients having degree $n \geq 3$ and discriminant $D_F \neq 0$, and if $A_F$ is the area of the region $|F(x, y)| \leq 1$ in the real affine plane, then $|D_F|^{1/(n-1)}A_F \leq 3B(1/3, 1/3)$, where $B(1/3, 1/3)$ denotes the Beta function with arguments of $1/3$. Consequently, if $F$ is a form with integer coefficients having non-zero discriminant and degree at least three, then $A_F \leq 3B(1, 1/3)$. The value $3B(1, 1/3)$, which numerically approximates to 15.8997, is attained in both inequalities for certain classes of cubic forms.

These inequalities are derived by demonstrating that the sequence $\{M_n\}$ defined by $M_n = \max |D_F|^{1/(n-1)}A_F$, where the maximum is taken over all forms of degree $n$ with $D_F \neq 0$, is decreasing, and then by showing that $M_3 = 3B(1, 1/3)$. It is conjectured that the limiting value of the sequence $\{M_n\}$ is $2\pi$.

1. Introduction

A binary form is a homogeneous polynomial in two variables, that is, a bivariate polynomial of the form

$$F(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ belong to some ring. If the coefficients are complex numbers, then the equation

$$|F(x, y)| = 1, \quad (x, y) \in \mathbb{R}^2$$

defines an algebraic curve which does not intersect itself. For, on converting to polar coordinates with the substitution

$$x = r \cos \theta, \quad y = r \sin \theta$$

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this equation becomes
\[ r = \frac{1}{|F(\cos \theta, \sin \theta)|^{1/n}}. \]

Hence, the region
\[ |F(x, y)| \leq 1, \quad (x, y) \in \mathbb{R}^2 \]

has a well-defined area which will be denoted by \( A_F \). The subject of this paper is the estimation of \( A_F \) over the class of forms with complex coefficients.

The problem of estimating the quantity \( A_F \) arises in the study of Thue equations. A *Thue equation* is a Diophantine equation of the form
\[ F(x, y) = h \]

where \( F \) is a binary form with rational integer coefficients which is irreducible and has degree \( n \geq 3 \), and \( h \) is a non-zero integer. In 1909, Thue [17] showed that the number of integer solutions of such an equation is finite.

In 1933, Mahler [11] gave an estimate for the number, \( Z_F(h) \), of solutions of the Thue inequality
\[ |F(x, y)| \leq h \]
in terms of the area, \( A_F(h) \), of the plane region \( |F(x, y)| \leq h, (x, y) \in \mathbb{R}^2 \). To be specific, he showed that if \( F \) is a binary form with rational integer coefficients which is irreducible and has degree \( n \geq 3 \), then
\[ |Z_F(h) - A_F(h)| \leq ch^{1/(n-1)} \]

where \( c \) is a number which depends only on \( F \). Notice, by the homogeneity of \( F \), that
\[ A_F(h) = A_F(1)h^{2/n} = A_Fh^{2/n}. \]

The number \( c \) and the quantity \( A_F \) were left unspecified by Mahler. However, he did show that \( A_F \) is finite when \( F \) is an irreducible binary form with integer coefficients and degree at least three.

More recently, Mueller and Schmidt [12] have given estimates for \( Z_F(h) \) which depend only on \( h \) and the number of non-zero terms occurring in \( F \). Estimates for \( A_F \) also appear in their work. In particular, they show that if \( F \) is an irreducible binary form with \( s + 1 \) non-zero coefficients, then
\[ A_F = O((ns^2)2s/n) \]

provided that \( n \geq 4s \). From this, they deduce that \( A_F \) is bounded when \( n \geq s \log s \). Notice that the conditions \( n \geq 4s \) and \( n \geq s \log s \) qualitatively mean that \( F \) has few coefficients.
Even more recently, Mueller and Schmidt [13] have shown that if $F$ is a binary form with integer coefficients, exactly $s + 1$ of which are nonvanishing, such that $a_0a_n \neq 0$ and $n > 2s$, then

$$A_F \leq \begin{cases} 60n^2s^2H^{-1/t} & \text{if } t \neq \frac{n}{2} \\ 60n^2s^2H^{-1/t}(1 + \frac{n}{2}\log H) & \text{if } t = \frac{n}{2} \end{cases}$$

where $H$ is the maximum of the absolute values of the coefficients of $F$ (often called the height of $F$) and $t = \max(q, n - q)$ with $q$ chosen such that $H = |a_q|$. The condition $n > 2s$ turns out to be essential. Their result shows, in particular, that $A_F$ is quite small for forms $F$ having few coefficients and height which is sufficiently large in terms of the degree.

Mueller and Schmidt considered the Newton polygon of the polynomial $F(x, 1)$ associated with the binary form $F$. One disadvantage of this approach is that it fails to capture the invariance of $A_F$ under linear transformations of the form. For example, the quantity $A_F$ is invariant under rotations of the region $|F(x, y)| \leq 1$ but the form $F$ and hence the polynomial $F(x, 1)$ are not. It is also worth noting that while the estimation of $A_F$ has been restricted to forms having integer coefficients, a more natural class of forms over which $A_F$ ought to be estimated is the class of forms with real coefficients.

In this paper, I will consider the slightly more general class of forms with complex coefficients and non-zero discriminant. It is a consequence of my results that if $F$ is such a form having integer coefficients and degree at least three, then $A_F \leq 3B(\frac{1}{3}, \frac{1}{3})$, where $B(\frac{1}{3}, \frac{1}{3})$ denotes the Beta function with arguments of $1/3$. It will soon become apparent that this bound is optimal.

2. Statement of Results

The general binary form $F(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ (with complex coefficients) has a factorization

$$F(x, y) = \prod_{i=1}^{n} (\alpha_i x - \beta_i y)$$

where each of the linear forms $\alpha_i x - \beta_i y$ has complex coefficients. Such a factorization need not be unique; however, the lines $\alpha_i x - \beta_i y = 0$ are uniquely determined by $F$. For a given factorization, the discriminant of $F$ is the quantity

$$D_F = \prod_{i < j} (\alpha_i \beta_j - \alpha_j \beta_i)^2.$$
ly, if \( a_0 \neq 0 \) and \( F \) has the factorization
\[
F(x, y) = a_0 \prod_{i=1}^{n} (x - \gamma_i y)
\]
then the \( \gamma_i \) are uniquely determined and
\[
D_F = a_0^{2n-2} \prod_{i < j} (\gamma_j - \gamma_i)^2.
\]

Let \( GL_2(\mathbb{R}) \) denote the group of \( 2 \times 2 \) real invertible matrices. For each \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \), let \( F_T \) denote the binary form given by
\[
F_T(x, y) = F(ax + by, cx + dy).
\]

Two forms \( F \) and \( G \) are said to be equivalent under \( GL_2(\mathbb{R}) \) if \( G = F_T \) for some \( T \in GL_2(\mathbb{R}) \). Similarly, let \( GL_2(\mathbb{Z}) \) denote the group of \( 2 \times 2 \) invertible matrices having integer coefficients, that is,
\[
GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}.
\]

Then, the forms \( F \) and \( G \) are equivalent under \( GL_2(\mathbb{Z}) \) if \( G = F_T \) for some \( T \in GL_2(\mathbb{Z}) \).

Let \( B(x, y) \) denote the Beta function of \( x \) and \( y \). The Beta function may be defined in terms of the Gamma function by the relation
\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}
\]
and has the integral representation
\[
B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt
\]
for \( x > 0 \) and \( y > 0 \) (see Abramowitz and Stegun [1]).

I will prove the following result.

THEOREM 1. Let \( F \) be a binary form with complex coefficients having degree \( n \geq 3 \) and discriminant \( D_F \neq 0 \). Then
\[
|D_F|^{1/(n-1)} A_F \leq 3B \left( \frac{1}{3}, \frac{1}{3} \right).
\]
This bound is attained precisely when $F$ is a cubic form which, up to multiplication by a complex number, is equivalent under $GL_2(\mathbb{R})$ to the form $xy(x - y)$.

Since the discriminant of a form with integer coefficients is an integer, Theorem 1 immediately provides the following estimate for $A_F$.

**COROLLARY 1.** If $F$ is a binary form with integer coefficients having non-zero discriminant and degree at least three, then

$$A_F \leq 3B \left( \frac{1}{3}, \frac{1}{3} \right).$$

This bound is attained for forms with integer coefficients which are equivalent under $GL_2(\mathbb{Z})$ to $xy(x - y)$.

The approximate numerical value of $3B(\frac{1}{3}, \frac{1}{3})$ is 15.8997. Notice that Theorem 1 cannot be extended to quadratic forms since $|D_F|^{1/2}A_F$ is infinite for the form $F(x, y) = x^2 - y^2$. In fact, if $F$ is a quadratic form with real coefficients, then $|D_F|^{1/2}A_F$ is infinite when $D_F > 0$ but equals $2\pi$ when $D_F < 0$. Notice further, that the condition $D_F \neq 0$ in Theorem 1 is required to exclude pathological examples where $D_F = 0$ and $A_F$ is infinite.

The quantity $|D_F|^{1/(n-1)}A_F$ is natural to consider since it is absolutely invariant with respect to $GL_2(\mathbb{R})$ (while the quantity $A_F$ is not). Indeed, if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, then

$$A_F = \iint_{|F(x, y)| \leq 1} dx \, dy$$

$$= \iint_{|F(ax + by, cu + dv)| \leq 1} |\det T| \, du \, dv$$

$$= |\det T| A_{F_T}$$

and

$$D_{F_T} = (\det T)^{n(n-1)}D_F;$$

hence

$$|D_{F_T}|^{1/(n(n-1))}A_{F_T} = |D_F|^{1/(n(n-1))}A_F$$

for all $T \in GL_2(\mathbb{R})$. The quantity $|D_F|^{1/(n(n-1))}A_F$ is also invariant with respect to replacing $F$ by $kF$ for any complex number $k$ since

$$|D_{kF}| = |k|^{2(n-1)}|D_F| \quad \text{and} \quad A_{kF} = \frac{1}{|k|^{2/n}}A_F.$$
On the other hand, $|D_F|^{1/n(n-1)}A_F$ is not invariant with respect to $GL_2(\mathbb{C})$ since, for example, the forms $x^2 + y^2$ and $x^2 - y^2$ are equivalent under $GL_2(\mathbb{C})$ but the area of the region $|x^2 + y^2| \leq 1$ is finite while the area of the region $|x^2 - y^2| \leq 1$ is infinite.

The proof of Theorem 1 relies on reducing the estimation of $|D_F|^{1/n(n-1)}A_F$ for a general form to the estimation of $|D_F|^{1/6}A_F$ over cubic forms and on demonstrating that the quantity $|D_F|^{1/6}A_F$ is maximized over the cubic forms by a form $F$ for which the polynomial $F(x, 1)$ has three distinct real roots. It is straightforward to show that, up to multiplication by a complex number, any such form is equivalent under $GL_2(\mathbb{R})$ to $xy(x - y)$. The inequality

$$|D_F|^{1/n(n-1)}A_F \leq 3B \left( \frac{1}{3}, \frac{1}{3} \right)$$

then follows from a routine area calculation.

The principal ideas can be generalized to give the proof a more inductive flavour and to provide more insight into the nature of $|D_F|^{1/n(n-1)}A_F$. This is the content of Theorem 2 and Theorem 3 below.

**THEOREM 2.** Suppose that

$$|D_F|^{1/(n-1)(n-2)}A_F \leq C$$

for all forms $F$ of degree $n - 1$ with $D_F \neq 0$. Then

$$|D_F|^{1/n(n-1)}A_F < C$$

for all forms $F$ of degree $n$ with $D_F \neq 0$. Hence, if

$$M_n = \max |D_F|^{1/n(n-1)}A_F$$

where the maximum is taken over all forms $F$ of degree $n$ with $D_F \neq 0$, then

$$M_3 > M_4 > M_5 > \cdots$$

**THEOREM 3.** The quantity $|D_F|^{1/n(n-1)}A_F$ is maximized over the class of forms of degree $n$ with complex coefficients and non-zero discriminant by a form $F$ with real coefficients for which the polynomial $F(x, 1)$ has $n$ distinct real roots. In fact, if $F$ is a form of degree $n$ for which the polynomial $F(x, 1)$ has at least one non-real root, then

$$|D_F|^{1/n(n-1)}A_F < M_n$$
where $M_n$ is as defined in the statement of Theorem 2.

It is convenient to adopt the convention that if $F$ is a form which has $y$ as a factor, then the polynomial $F(x, 1)$ has a root at infinity (denoted $\infty$); similarly, if $F$ has $x$ as a factor, then $F(1, y)$ has a root at infinity. Throughout this paper, a root at infinity will be considered a real root. With this convention, the slopes of the asymptotes of the curve $|F(x, y)| = 1$ are the real roots of the polynomial $F(1, y)$.

I originally established the monotonicity of the sequence $\{M_n\}$ by applying the generalized form of Hölder’s inequality to a certain integral representation of $A_F$. Enrico Bombieri has since suggested to me that this result may be established in a slightly simpler way by appealing to the inequality between arithmetic and geometric means. The details of both approaches will be given in Section 4.

Theorem 3 will be established in Section 5 by considering the quantity $|D_F|^{1/(n-1)}A_F$ as a function of $n$ complex variables and then appealing to an appropriate maximum principle. I am very grateful to Professor Bombieri for suggesting this approach.

In light of Theorem 3, it is natural to wonder whether the assumption in Theorem 1 that $F$ have complex coefficients is an unnecessary complication. However, it will soon become apparent that this assumption is required for the induction in Theorem 2 to work.

In a subsequent paper, I will examine more closely the nature of the sequence $\{M_n\}$. Based on that work, I believe that the following is true.

CONJECTURE 1. The sequence $\{M_n\}$ defined by

$$M_n = \max |D_F|^{1/(n-1)}A_F$$

where the maximum is taken over all forms of degree $n$ with complex coefficients and discriminant $D_F \neq 0$, decreases monotonically to the value $2\pi$.

Coincidentally, the conjectured limiting value of this sequence is equal to the value of $|D_F|^{1/2}A_F$ when $F$ is a quadratic form with real coefficients and negative discriminant.

3. An integral representation for $A_F$

As mentioned in the Introduction, the curve $|F(x, y)| = 1$ may be expressed in polar form as

$$r = \frac{1}{|F(\cos \theta, \sin \theta)|^{1/n}}.$$
Hence, from Calculus,

\[ A_F = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta \]

\[ = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{|F(\cos \theta, \sin \theta)|^{2/n}}. \]

Now the curve \(|F(x, y)| = 1\) is symmetric about the origin, and so using an appropriate substitution we have

\[ A_F = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{|F(\cos \theta, \sin \theta)|^{2/n}} \]

\[ = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\left(\cos^2 \theta + \sin^2 \theta \right)^{2/n}} \]

\[ = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{|F(1, \tan \theta)|^{2/n}} \]

\[ = \int_{-\infty}^{\infty} \frac{dv}{|F(1, v)|^{2/n}}. \]

Similarly,

\[ A_F = \int_{-\infty}^{\infty} \frac{du}{|F(u, 1)|^{2/n}}. \]

This representation for \(A_F\) reveals several of the difficulties to be overcome when estimating the quantity \(|D_F|^{1/n(n-1)}A_F\). To see this, suppose that

\[ F(x, y) = (\alpha_1 x - y) \cdots (\alpha_n x - y) \]

where \(\alpha_1, \ldots, \alpha_n\) are distinct real numbers, and consider

\[ A_F = \int_{-\infty}^{\infty} \frac{dv}{|(v - \alpha_1) \cdots (v - \alpha_n)|^{2/n}}. \]

Notice that this integral has singularities at \(\alpha_1, \ldots, \alpha_n\) corresponding to the asymptotes

\[ y - \alpha_1 x = 0, \ldots, y - \alpha_n x = 0 \]

of the curve \(|F(x, y)| = 1\). The behaviour of \(A_F\) depends on the relative separation of the roots \(\alpha_1, \ldots, \alpha_n\). For example, if all the \(\alpha\)'s were close to zero, then the resulting integral would be close to
and so $A_F$ would become arbitrarily large. In fact, if at least half the roots cluster to a point, the resulting integral has an accumulated singularity with exponent at least one, and hence is divergent. Notice, however, that when the $\alpha$'s are close together, the quantity

$$|D_F|^{1/n(n-1)} = \prod_{i < j} (\alpha_j - \alpha_i)^{2/|D_F|^{1/n(n-1)}}$$

is close to zero (as must be the case if $|D_F|^{1/n(n-1)}A_F$ is to remain bounded). On the other hand, the squares of the differences $(\alpha_j - \alpha_i)^2$ could be quite large resulting in an arbitrarily large discriminant (and an arbitrarily small $A_F$, although this is not immediately obvious).

4. Proof of Theorem 2

In view of the integral representation for $A_F$ given in the previous section and the invariance of $|D_F|^{1/n(n-1)}A_F$ with respect to $GL_2(\mathbb{R})$ and with respect to replacing $F$ by $kF$ for any complex number $k$, it is apparent that the analysis of $|D_F|^{1/n(n-1)}A_F$ over the class of forms of degree $n$ with complex coefficients and non-zero discriminant is equivalent to the analysis of the quantity

$$\prod_{i > j} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|v - \alpha_1| \cdots |v - \alpha_n|^{2/n}}$$

over all $n$-tuples $(\alpha_1, \ldots, \alpha_n)$ of distinct complex numbers. In this section, I will demonstrate that the sequence $\{M_n\}$ defined by

$$M_n = \max |D_F|^{1/n(n-1)}A_F,$$

where the maximum is taken over all forms of degree $n$ with $D_F \neq 0$, is decreasing, by applying Hölder's inequality to the integral

$$\int_{-\infty}^{\infty} \frac{dv}{|v - \alpha_1| \cdots (v - \alpha_n)^{2/n}}.$$
Put
\[ f(z) = (z - \alpha_1) \cdots (z - \alpha_n), \]
\[ f_i(z) = \frac{f(z)}{z - \alpha_i}, \]
and let \( D_f \) and \( D_{f_i} \) denote the discriminants of \( f \) and \( f_i \) respectively. Notice that
\[ \prod_{i=1}^{n} f_i(z) = f(z)^{n-1} \]
and
\[ \prod_{i=1}^{n} |D_{f_i}| = |D_f|^{n-2}. \]

Hence
\[ \int_{-\infty}^{\infty} \frac{dv}{|f(v)|^{2/n}} = \int_{-\infty}^{\infty} \frac{dv}{\prod_{i=1}^{n} |f_i(v)|^{2/n(n-1)}}. \]

Applying the generalized form of Hölder’s inequality to the latter integral, with each exponent equal to \( n \), we have
\[ \int_{-\infty}^{\infty} \frac{dz}{\prod_{i=1}^{n} |f_i(z)|^{2/n(n-1)}} < \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} \frac{dz}{|f_i(z)|^{2/(n-1)}} \right)^{1/n}. \]

This inequality is strict since for \( i \neq j \), there is no constant \( k \) for which \( |f_i(v)| = k|f_j(v)| \) for almost all \( v \).

Now suppose that
\[ \int_{-\infty}^{\infty} \frac{dz}{|f_i(z)|^{2/(n-1)}} \leq C |D_{f_i}|^{1/(n-1)(n-2)} \]
for \( i = 1, 2, \ldots, n \). Then
\[ \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} \frac{dz}{|f(z)|^{2/(n-1)}} \right\}^{1/n} \leq \prod_{i=1}^{n} \left\{ \frac{C}{|D_{f_i}|^{1/(n-1)(n-2)}} \right\}^{1/n} \]

\[ = \frac{C}{\prod_{i=1}^{n} |D_{f_i}|^{1/\left(1/(n-1)(n-2)\right)}} \]

\[ = \frac{C}{|D_f|^{1/n(n-1)}} \]

and so

\[ \int_{-\infty}^{\infty} \frac{dv}{|f(v)|^{2/n}} < \frac{C}{|D_f|^{1/n(n-1)}}. \]

Consequently,

\[ M_3 > M_4 > M_5 \cdots \]

as required.

The monotonicity of the sequence \( \{M_n\} \) can also be established by appealing to the inequality between arithmetic and geometric means, in the form

\[ x_1 x_2 \cdots x_n \leq \frac{1}{n} \left( x_1^n + x_2^n + \cdots + x_n^n \right). \]

Indeed, let

\[ F(x, y) = (\alpha_1 x - y) \cdots (\alpha_n x - y) \]

and put

\[ F_i(x, y) = \frac{F(x, y)}{\alpha_i x - y}. \]

Then

\[ \prod_{i=1}^{n} F_i(x, y) = F(x, y)^{n-1}, \]

\[ \prod_{i=1}^{n} |D_{F_i}| = |D_F|^{n-2} \]
and so

\[ |D_F|^{1/n(n-1)} |F(1, v)|^{-2/n} = \prod_{i=1}^{n} |D_{F_i}|^{1/n(n-1)(n-2)} F_i(1, v)^{-2/n(n-1)} \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} |D_{F_i}|^{1/(n-1)(n-2)} F_i(1, v)^{-2/(n-1)}. \]

The latter inequality is strict for all but finitely many \( v \) since for \( i \neq j \),

\[ |D_{F_i}|^{1/(n-2)} F_i(1, v)|^{-2} = |D_{F_j}|^{1/(n-2)} F_j(1, v)|^{-2} \]

for at most two values of \( v \). Hence,

\[ |D_F|^{1/n(n-1)} A_F < \frac{1}{n} \sum_{i=1}^{n} |D_{F_i}|^{1/(n-1)(n-2)} A_{F_i} \]

and the inequality \( M_n < M_{n-1} \) follows as before.

5. Proof of Theorem 3

As noted in the previous section, the analysis of the quantity \( |D_F|^{1/n(n-1)} A_F \) is equivalent to the analysis of the quantity

\[ Q(\alpha_1, \ldots, \alpha_n) = \prod_{i < j} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|v - \alpha_1| \cdots |v - \alpha_n|^{2/n}} \]

where \( \alpha_1, \ldots, \alpha_n \) are distinct complex numbers. In this section, I will show that if at least one of the \( \alpha_i \) is non-real, then

\[ Q(\alpha_1, \ldots, \alpha_n) < M_n. \]

It will then follow that \( Q \) is maximized at a point \( (\alpha_1, \ldots, \alpha_n) \) for which each \( \alpha_i \)

is real (or \( \infty \)).

Throughout this section, I will adopt the convention that if one of the \( \alpha \)'s is infinite, say \( \alpha_n \), then

\[ Q = \prod_{1 \leq i < j \leq n-1} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|v - \alpha_1| \cdots |v - \alpha_{n-1}|^{2/n}}. \]

Before discussing the details of the proof of Theorem 3, let us recall the following terminology from the theory of functions.
A continuous real-valued function $u$ of a single complex variable $z = x + iy$ is \emph{harmonic} if it has continuous partial derivatives of the second order and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ 

A continuous real-valued function $v$ of a single complex variable is said to be \emph{subharmonic} if, in any region of the complex plane, $v$ is less than or equal to the harmonic function $u$ which coincides with $v$ on the boundary of the region. A subharmonic function need not be continuous; however, this assumption allows one to simplify the definition to some extent. There are several equivalent definitions of subharmonicity; the one given here highlights the property of convexity.

An important property of subharmonic functions is that they satisfy the maximum principle. The \emph{maximum principle for subharmonic functions} states that a non-constant subharmonic function has no maximum in its region of definition. Consequently, the maximum of a subharmonic function on a closed bounded set is attained on the boundary of the set.

The generalizations of these concepts to functions of several complex variables are respectively the notions of pluriharmonicity and plurisubharmonicity. A continuous real-valued function of several complex variables is said to be \emph{plurisubharmonic} if its restriction to any complex line is subharmonic on that line. The function is \emph{pluriharmonic} if its restriction to any complex line is harmonic on that line. A \emph{complex line} in $\mathbb{C}^n$ is a set of the form

$$\{a + b\zeta; \zeta \in \mathbb{C}\}$$

where $a, b \in \mathbb{C}^n$. Notice that any positive linear combination of plurisubharmonic functions is plurisubharmonic. Further, the composition of a plurisubharmonic function and a monotonically increasing convex function is plurisubharmonic (see Gunning [8] or Hormander [10]).

Now consider the quantity

$$Q(\alpha_1, \ldots, \alpha_n) = \prod_{i<j} |\alpha_j - \alpha_i|^{2/(n-1)} \int_{-\infty}^{\infty} \frac{dv}{(v - \alpha_1) \cdots (v - \alpha_n)^{2/n}}$$

as a function of the complex variables $\alpha_1, \ldots, \alpha_n$. This function is plurisubharmonic on the region

$$\mathcal{R} = \mathbb{C}^n \setminus \bigcup_{i=1}^{n} \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{C}^n: \alpha_i \in \mathbb{R}\}.$$
To see this, it suffices, by linearity, to show that each of the functions \( q_v \) given by

\[
q_v(x_1, \ldots, x_n) = \frac{\prod_{l<j} |x_j - x_l|^{2/n(n-1)}}{|(v - x_1) \cdots (v - x_n)|^{2/n}}
\]

with \( v \) a real number, is plurisubharmonic on \( \mathbb{R} \). Since the exponential function is convex and monotonically increasing, it suffices to demonstrate this for

\[
\log q_v(x_1, \ldots, x_n) = \frac{2}{n(n-1)} \sum_{i<j} \log |x_j - x_i| - \frac{2}{n} \sum_{i=1}^n \log |v - x_i|.
\]

Now, for \( i \neq j \), the function \((x_1, \ldots, x_n) \mapsto \log |x_j - x_i|\) is plurisubharmonic on \( \mathbb{C}^n \); in fact, it is pluriharmonic on \( \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \neq x_j \} \). Further, the function \((x_1, \ldots, x_n) \mapsto -\log |v - x_i|\) with \( v \) a real number, is pluriharmonic except on the hyperplane \( \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_i = v \} \). Hence, \( Q \) is plurisubharmonic on \( \mathbb{R} \) as claimed.

In particular, for fixed values of \( x_1, \ldots, x_{n-1} \), the quantity \( Q(x_1, \ldots, x_n) \), when viewed as a function of \( x_n \), is subharmonic in the upper and lower half planes. Moreover, it is continuous and non-constant on \( \mathbb{C} \). Hence, by the maximum principle for subharmonic functions,

\[
Q(x_1, \ldots, x_n) \leq Q(x_1, \ldots, x_{n-1}, r_n)
\]

for some real number \( r_n \) (possibly \( \infty \)). Moreover, this inequality is strict for non-real values of \( x_n \).

Now the quantity \( Q(x_1, \ldots, x_{n-1}, r_n) \) when viewed as a function of the complex variables \( x_1, \ldots, x_{n-1} \) is plurisubharmonic on

\[
\mathbb{C}^{n-1} \bigcup_{i=1}^{n-1} \{ (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1} : x_i \in \mathbb{R} \}.
\]

Hence, arguing as before, we have

\[
Q(x_1, \ldots, x_n) \leq Q(x_1, \ldots, x_{n-2}, r_{n-1}, r_n)
\]

for some real number \( r_{n-1} \) (possibly \( \infty \)) distinct from \( r_n \). Continuing in this way, we find that

\[
Q(x_1, \ldots, x_n) \leq Q(r_1, \ldots, r_n)
\]
for some \( n \)-tuple \( (r_1, \ldots, r_n) \) of distinct real numbers (possibly including the point at infinity). In fact, if at least one of the \( x \)'s is non-real, then this inequality is strict.

Therefore, if \( F \) is a form of degree \( n \) for which the polynomial \( F(x, 1) \) has at least one non-real root, then

\[
|D_F|^{1/(n-1)} A_F < M_n.
\]

This completes the proof of Theorem 3.

6. Proof of Theorem 1

In view of Theorem 2 and Theorem 3, it suffices to prove that every form \( F \) for which the polynomial \( F(x, 1) \) has three distinct real roots is (up to multiplication by a complex number) equivalent under \( GL_2(\mathbb{R}) \) to \( xy(x - y) \) and that

\[
|D_F|^{1/6} A_F = 3B \left( \frac{1}{3}, \frac{1}{3} \right)
\]

for any such form. Since \( |D_F|^{1/6} A_F \) is invariant with respect to replacing \( F \) by \( kF \) for any complex number \( k \), there is no loss of generality in assuming that \( F \) has real coefficients in this case. Hence, let \( F \) be a cubic form with real coefficients for which \( D_F > 0 \).

Notice that a linear substitution applied to \( F \) induces a fractional linear transformation of the roots of the polynomial \( F(1, y) \). Indeed, if \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \), then the roots are transformed according to the rule

\[
t \mapsto \frac{at - c}{d - bt}.
\]

Since every fractional linear transformation with real coefficients may be given by the rule

\[
\frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}
\]

where the \( z \)'s and \( w \)'s are real numbers such that \( z_1, z_2, z_3 \) are mapped to \( w_1, w_2, w_3 \) respectively, it follows that \( F \) is equivalent under \( GL_2(\mathbb{R}) \) to the form \( F_1(x, y) = xy(x - y) \). Hence
\[ |D_F|^{1/6} A_F = A_F, \]
\[ = \int_{-\infty}^{\infty} \frac{dv}{|F_1(1, v)|^{2/3}} \]
\[ = \int_{-\infty}^{\infty} \frac{dv}{|v(1 - v)|^{2/3}} \]
\[ = 3B \left( \frac{1}{3}, \frac{1}{3} \right) \]
as required.

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References