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An isoperimetric inequality for the area of plane regions defined by binary forms*

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Abstract. In this paper, it is shown that if F is a binary form with complex coefficients having degree $n \geqslant 3$ and discriminant $D_F \neq 0$, and if A_F is the area of the region $|F(x, y)| \leqslant 1$ in the real affine plane, then $|D_F|^{1/n(n-1)} A_F \leqslant 3B(\frac{1}{3}, \frac{1}{3})$, where $B(\frac{1}{3}, \frac{1}{3})$ denotes the Beta function with arguments of 1/3. Consequently, if F is a form with integer coefficients having non-zero discriminant and degree at least three, then $A_F \leqslant 3B(\frac{1}{3}, \frac{1}{3})$. The value $3B(\frac{1}{3}, \frac{1}{3})$, which numerically approximates to 15.8997, is attained in both inequalities for certain classes of cubic forms.

These inequalities are derived by demonstrating that the sequence $\{M_n\}$ defined by $M_n = \max |D_F|^{1/n(n-1)} A_F$, where the maximum is taken over all forms of degree n with $D_F \neq 0$, is decreasing, and then by showing that $M_3 = 3B(\frac{1}{3}, \frac{1}{3})$. It is conjectured that the limiting value of the sequence $\{M_n\}$ is 2π.

1. Introduction

A *binary form* is a homogeneous polynomial in two variables, that is, a bivariate polynomial of the form

$$F(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ belong to some ring. If the coefficients are complex numbers, then the equation

$$|F(x, y)| = 1, \qquad (x, y) \in \mathbb{R}^2$$

defines an algebraic curve which does not intersect itself. For, on converting to polar coordinates with the substitution

$$x = r \cos \theta, \qquad y = r \sin \theta$$

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this equation becomes

$$r = \frac{1}{|F(\cos\ \theta,\ \sin\ \theta)|^{1/n}}.$$

Hence, the region

$$|F(x, y)| \leqslant 1, \qquad (x, y) \in \mathbb{R}^2$$

has a well-defined area which will be denoted by A_F. The subject of this paper is the estimation of A_F over the class of forms with complex coefficients.

The problem of estimating the quantity A_F arises in the study of Thue equations. A *Thue equation* is a Diophantine equation of the form

$$F(x, y) = h$$

where F is a binary form with rational integer coefficients which is irreducible and has degree $n \geqslant 3$, and h is a non-zero integer. In 1909, Thue [17] showed that the number of integer solutions of such an equation is finite.

In 1933, Mahler [11] gave an estimate for the number, $Z_F(h)$, of solutions of the Thue *inequality*

$$|F(x, y)| \leqslant h$$

in terms of the area, $A_F(h)$, of the plane region $|F(x, y)| \leqslant h$, $(x, y) \in \mathbb{R}^2$. To be specific, he showed that if F is a binary form with rational integer coefficients which is irreducible and has degree $n \geqslant 3$, then

$$|Z_F(h) - A_F(h)| \leqslant ch^{1/(n-1)}$$

where c is a number which depends only on F. Notice, by the homogeneity of F, that

$$A_F(h) = A_F(1)h^{2/n} = A_F h^{2/n}.$$

The number c and the quantity A_F were left unspecified by Mahler. However, he did show that A_F is finite when F is an irreducible binary form with integer coefficients and degree at least three.

More recently, Mueller and Schmidt [12] have given estimates for $Z_F(h)$ which depend only on h and the number of non-zero terms occurring in F. Estimates for A_F also appear in their work. In particular, they show that if F is an irreducible binary form with $s + 1$ non-zero coefficients, then

$$A_F = O((ns^2)^{2s/n})$$

provided that $n \geqslant 4s$. From this, they deduce that A_F is bounded when $n \geqslant s \log s$. Notice that the conditions $n \geqslant 4s$ and $n \geqslant s \log s$ qualitatively mean that F has few coefficients.

Even more recently, Mueller and Schmidt [13] have shown that if F is a binary form with integer coefficients, exactly $s + 1$ of which are nonvanishing, such that $a_0 a_n \neq 0$ and $n > 2s$, then

$$A_F \leqslant \begin{cases} 60n^2 s^2 H^{-1/t} & \text{if } t \neq \frac{n}{2} \\ 60n^2 s^2 H^{-1/t}(1 + \frac{4}{n}\log H) & \text{if } t = \frac{n}{2} \end{cases}$$

where H is the maximum of the absolute values of the coefficients of F (often called the *height* of F) and $t = \max(q, n - q)$ with q chosen such that $H = |a_q|$. The condition $n > 2s$ turns out to be essential. Their result shows, in particular, that A_F is quite small for forms F having few coefficients and height which is sufficiently large in terms of the degree.

Mueller and Schmidt considered the Newton polygon of the polynomial $F(x, 1)$ associated with the binary form F. One disadvantage of this approach is that it fails to capture the invariance of A_F under linear transformations of the form. For example, the quantity A_F is invariant under rotations of the region $|F(x, y)| \leqslant 1$ but the form F and hence the polynomial $F(x, 1)$ are not. It is also worth noting that while the estimation of A_F has been restricted to forms having integer coefficients, a more natural class of forms over which A_F ought to be estimated is the class of forms with real coefficients.

In this paper, I will consider the slightly more general class of forms with complex coefficients and non-zero discriminant. It is a consequence of my results that if F is such a form having integer coefficients and degree at least three, then $A_F \leqslant 3B(\frac{1}{3}, \frac{1}{3})$, where $B(\frac{1}{3}, \frac{1}{3})$ denotes the Beta function with arguments of 1/3. It will soon become apparent that this bound is optimal.

2. Statement of Results

The general binary form $F(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$ (with complex coefficients) has a factorization

$$F(x, y) = \prod_{i=1}^{n} (\alpha_i x - \beta_i y)$$

where each of the linear forms $\alpha_i x - \beta_i y$ has complex coefficients. Such a factorization need not be unique; however, the lines $\alpha_i x - \beta_i y = 0$ are uniquely determined by F. For a given factorization, the *discriminant* of F is the quantity

$$D_F = \prod_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i)^2.$$

The discriminant is independent of the factorization chosen for F. Alternatively, if $a_0 \neq 0$ and F has the factorization

$$F(x, y) = a_0 \prod_{i=1}^{n} (x - \gamma_i y)$$

then the γ_i are uniquely determined and

$$D_F = a_0^{2n-2} \prod_{i<j} (\gamma_j - \gamma_i)^2.$$

Let $GL_2(\mathbb{R})$ denote the group of 2×2 real invertible matrices. For each $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, let F_T denote the binary form given by

$$F_T(x, y) = F(ax + by, cx + dy).$$

Two forms F and G are said to be *equivalent under* $GL_2(\mathbb{R})$ if $G = F_T$ for some $T \in GL_2(\mathbb{R})$. Similarly, let $GL_2(\mathbb{Z})$ denote the group of 2×2 invertible matrices having integer coefficients, that is,

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}.$$

Then, the forms F and G are *equivalent under* $GL_2(\mathbb{Z})$ if $G = F_T$ for some $T \in GL_2(\mathbb{Z})$.

Let $B(x, y)$ denote the Beta function of x and y. The Beta function may be defined in terms of the Gamma function by the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

and has the integral representation

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt$$

for $x > 0$ and $y > 0$ (see Abramowitz and Stegun [1]).

I will prove the following result.

THEOREM 1. *Let F be a binary form with complex coefficients having degree $n \geq 3$ and discriminant $D_F \neq 0$. Then*

$$|D_F|^{1/n(n-1)} A_F \leq 3B\left(\frac{1}{3}, \frac{1}{3}\right).$$

This bound is attained precisely when F is a cubic form which, up to multiplication by a complex number, is equivalent under $GL_2(\mathbb{R})$ to the form $xy(x-y)$.

Since the discriminant of a form with integer coefficients is an integer, Theorem 1 immediately provides the following estimate for A_F.

COROLLARY 1. *If F is a binary form with integer coefficients having non-zero discriminant and degree at least three, then*

$$A_F \leqslant 3B\left(\frac{1}{3}, \frac{1}{3}\right).$$

This bound is attained for forms with integer coefficients which are equivalent under $GL_2(\mathbb{Z})$ to $xy(x-y)$.

The approximate numerical value of $3B(\frac{1}{3}, \frac{1}{3})$ is 15.8997. Notice that Theorem 1 cannot be extended to quadratic forms since $|D_F|^{1/2}A_F$ is infinite for the form $F(x, y) = x^2 - y^2$. In fact, if F is a quadratic form with real coefficients, then $|D_F|^{1/2}A_F$ is infinite when $D_F > 0$ but equals 2π when $D_F < 0$. Notice further, that the condition $D_F \neq 0$ in Theorem 1 is required to exclude pathological examples where $D_F = 0$ and A_F is infinite.

The quantity $|D_F|^{1/n(n-1)}A_F$ is natural to consider since it is absolutely invariant with respect to $GL_2(\mathbb{R})$ (while the quantity A_F is not). Indeed, if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, then

$$\begin{aligned} A_F &= \iint_{|F(x,y)| \leqslant 1} dx\, dy \\ &= \iint_{|F(au+bv, cu+dv)| \leqslant 1} |\det T|\, du\, dv \\ &= |\det T| A_{F_T} \end{aligned}$$

and

$$D_{F_T} = (\det T)^{n(n-1)} D_F;$$

hence

$$|D_{F_T}|^{1/n(n-1)} A_{F_T} = |D_F|^{1/n(n-1)} A_F$$

for all $T \in GL_2(\mathbb{R})$. The quantity $|D_F|^{1/n(n-1)}A_F$ is also invariant with respect to replacing F by kF for any complex number k since

$$|D_{kF}| = |k|^{2(n-1)}|D_F| \quad \text{and} \quad A_{kF} = \frac{1}{|k|^{2/n}} A_F.$$

On the other hand, $|D_F|^{1/n(n-1)}A_F$ is *not* invariant with respect to $GL_2(\mathbb{C})$ since, for example, the forms $x^2 + y^2$ and $x^2 - y^2$ are equivalent under $GL_2(\mathbb{C})$ but the area of the region $|x^2 + y^2| \leqslant 1$ is finite while the area of the region $|x^2 - y^2| \leqslant 1$ is infinite.

The proof of Theorem 1 relies on reducing the estimation of $|D_F|^{1/n(n-1)}A_F$ for a general form to the estimation of $|D_F|^{1/6}A_F$ over cubic forms and on demonstrating that the quantity $|D_F|^{1/6}A_F$ is maximized over the cubic forms by a form F for which the polynomial $F(x, 1)$ has three distinct real roots. It is straightforward to show that, up to multiplication by a complex number, any such form is equivalent under $GL_2(\mathbb{R})$ to $xy(x - y)$. The inequality

$$|D_F|^{1/n(n-1)}A_F \leqslant 3B\left(\frac{1}{3}, \frac{1}{3}\right)$$

then follows from a routine area calculation.

The principal ideas can be generalized to give the proof a more inductive flavour and to provide more insight into the nature of $|D_F|^{1/n(n-1)}A_F$. This is the content of Theorem 2 and Theorem 3 below.

THEOREM 2. *Suppose that*

$$|D_F|^{1/(n-1)(n-2)}A_F \leqslant C$$

for all forms F of degree $n - 1$ with $D_F \neq 0$. Then

$$|D_F|^{1/n(n-1)}A_F < C$$

for all forms F of degree n with $D_F \neq 0$. Hence, if

$$M_n = \max|D_F|^{1/n(n-1)}A_F$$

where the maximum is taken over all forms F of degree n with $D_F \neq 0$, then

$$M_3 > M_4 > M_5 > \cdots$$

THEOREM 3. *The quantity $|D_F|^{1/n(n-1)}A_F$ is maximized over the class of forms of degree n with complex coefficients and non-zero discriminant by a form F with real coefficients for which the polynomial $F(x, 1)$ has n distinct real roots. In fact, if F is a form of degree n for which the polynomial $F(x, 1)$ has at least one non-real root, then*

$$|D_F|^{1/n(n-1)}A_F < M_n$$

where M_n is as defined in the statement of Theorem 2.

It is convenient to adopt the convention that if F is a form which has y as a factor, then the polynomial $F(x, 1)$ has a *root at infinity* (denoted ∞); similarly, if F has x as a factor, then $F(1, y)$ has a root at infinity. Throughout this paper, a root at infinity will be considered a real root. With this convention, the slopes of the asymptotes of the curve $|F(x, y)| = 1$ are the real roots of the polynomial $F(1, y)$.

I originally established the monotonicity of the sequence $\{M_n\}$ by applying the generalized form of Hölder's inequality to a certain integral representation of A_F. Enrico Bombieri has since suggested to me that this result may be established in a slightly simpler way by appealing to the inequality between arithmetic and geometric means. The details of both approaches will be given in Section 4.

Theorem 3 will be established in Section 5 by considering the quantity $|D_F|^{1/n(n-1)}A_F$ as a function of n complex variables and then appealing to an appropriate maximum principle. I am very grateful to Professor Bombieri for suggesting this approach.

In light of Theorem 3, it is natural to wonder whether the assumption in Theorem 1 that F have *complex* coefficients is an unnecessary complication. However, it will soon become apparent that this assumption is required for the induction in Theorem 2 to work.

In a subsequent paper, I will examine more closely the nature of the sequence $\{M_n\}$. Based on that work, I believe that the following is true.

CONJECTURE 1. The sequence $\{M_n\}$ defined by

$$M_n = \max|D_F|^{1/n(n-1)}A_F$$

where the maximum is taken over all forms of degree n with complex coefficients and discriminant $D_F \neq 0$, decreases monotonically to the value 2π.

Coincidentally, the conjectured limiting value of this sequence is equal to the value of $|D_F|^{1/2}A_F$ when F is a quadratic form with real coefficients and negative discriminant.

3. An integral representation for A_F

As mentioned in the Introduction, the curve $|F(x, y)| = 1$ may be expressed in polar form as

$$r = \frac{1}{|F(\cos\ \theta,\ \sin\ \theta)|^{1/n}}.$$

Hence, from Calculus,

$$A_F = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$
$$= \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{|F(\cos\theta, \sin\theta)|^{2/n}}.$$

Now the curve $|F(x, y)| = 1$ is symmetric about the origin, and so using an appropriate substitution we have

$$A_F = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{|F(\cos\theta, \sin\theta)|^{2/n}}$$
$$= \int_{-\pi/2}^{\pi/2} \frac{d\theta}{|(\cos\theta)^n F(1, \tan\theta)|^{2/n}}$$
$$= \int_{-\infty}^{\infty} \frac{dv}{|F(1, v)|^{2/n}}.$$

Similarly,

$$A_F = \int_{-\infty}^{\infty} \frac{du}{|F(u, 1)|^{2/n}}.$$

This representation for A_F reveals several of the difficulties to be overcome when estimating the quantity $|D_F|^{1/n(n-1)} A_F$. To see this, suppose that

$$F(x, y) = (\alpha_1 x - y) \cdots (\alpha_n x - y)$$

where $\alpha_1, \ldots, \alpha_n$ are distinct real numbers, and consider

$$A_F = \int_{-\infty}^{\infty} \frac{dv}{|(v - \alpha_1) \cdots (v - \alpha_n)|^{2/n}}.$$

Notice that this integral has singularities at $\alpha_1, \ldots, \alpha_n$ corresponding to the asymptotes

$$y - \alpha_1 x = 0, \ldots, y - \alpha_n x = 0$$

of the curve $|F(x, y)| = 1$. The behaviour of A_F depends on the relative separation of the roots $\alpha_1, \ldots, \alpha_n$. For example, if all the α's were close to zero, then the resulting integral would be close to

$$\int_{-\infty}^{\infty} \frac{dv}{|v|^2}$$

and so A_F would become arbitrarily large. In fact, if at least half the roots cluster to a point, the resulting integral has an accumulated singularity with exponent at least one, and hence is divergent. Notice, however, that when the α's are close together, the quantity

$$|D_F|^{1/n(n-1)} = \left|\prod_{i<j} (\alpha_j - \alpha_i)^2\right|^{1/n(n-1)}$$

is close to zero (as must be the case if $|D_F|^{1/n(n-1)}A_F$ is to remain bounded). On the other hand, the squares of the differences $(\alpha_j - \alpha_i)^2$ could be quite large resulting in an arbitrarily large discriminant (and an arbitrarily small A_F, although this is not immediately obvious).

4. Proof of Theorem 2

In view of the integral representation for A_F given in the previous section and the invariance of $|D_F|^{1/n(n-1)}A_F$ with respect to $GL_2(\mathbb{R})$ and with respect to replacing F by kF for any complex number k, it is apparent that the analysis of $|D_F|^{1/n(n-1)}A_F$ over the class of forms of degree n with complex coefficients and non-zero discriminant is equivalent to the analysis of the quantity

$$\prod_{i>j} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|(v-\alpha_1)\cdots(v-\alpha_n)|^{2/n}}$$

over all n-tuples $(\alpha_1, \ldots, \alpha_n)$ of distinct complex numbers. In this section, I will demonstrate that the sequence $\{M_n\}$ defined by

$$M_n = \max|D_F|^{1/n(n-1)}A_F,$$

where the maximum is taken over all forms of degree n with $D_F \neq 0$, is decreasing, by applying Hölder's inequality to the integral

$$\int_{-\infty}^{\infty} \frac{dv}{|(v-\alpha_1)\cdots(v-\alpha_n)|^{2/n}}.$$

Put

$$f(z) = (z - \alpha_1) \cdots (z - \alpha_n),$$

$$f_i(z) = \frac{f(z)}{z - \alpha_i},$$

and let D_f and D_{f_i} denote the discriminants of f and f_i respectively. Notice that

$$\prod_{i=1}^{n} f_i(z) = f(z)^{n-1}$$

and

$$\prod_{i=1}^{n} |D_{f_i}| = |D_f|^{n-2}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{dv}{|f(v)|^{2/n}} = \int_{-\infty}^{\infty} \frac{dv}{\prod_{i=1}^{n} |f_i(v)|^{2/n(n-1)}}.$$

Applying the generalized form of Hölder's inequality to the latter integral, with each exponent equal to n, we have

$$\int_{-\infty}^{\infty} \frac{dz}{\prod_{i=1}^{n} |f_i(z)|^{2/n(n-1)}} < \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} \frac{dz}{|f_i(z)|^{2/(n-1)}} \right\}^{1/n}.$$

This inequality is strict since for $i \neq j$, there is no constant k for which $|f_i(v)| = k|f_j(v)|$ for almost all v.

Now suppose that

$$\int_{-\infty}^{\infty} \frac{dz}{|f_i(z)|^{2/(n-1)}} \leqslant \frac{C}{|D_{f_i}|^{1/(n-1)(n-2)}}$$

for $i = 1, 2, \ldots, n$. Then

$$\prod_{i=1}^{n}\left\{\int_{-\infty}^{\infty}\frac{dz}{|f_i(z)|^{2/(n-1)}}\right\}^{1/n}\leqslant\prod_{i=1}^{n}\left\{\frac{C}{|D_{f_i}|^{1/(n-1)(n-2)}}\right\}^{1/n}$$
$$=\frac{C}{\prod_{i=1}^{n}|D_{f_i}|^{1/n(n-1)(n-2)}}$$
$$=\frac{C}{|D_f|^{1/n(n-1)}}$$

and so

$$\int_{-\infty}^{\infty}\frac{dv}{|f(v)|^{2/n}}<\frac{C}{|D_f|^{1/n(n-1)}}.$$

Consequently,

$$M_3>M_4>M_5\cdots$$

as required.

The monotonicity of the sequence $\{M_n\}$ can also be established by appealing to the inequality between arithmetic and geometric means, in the form

$$x_1x_2\cdots x_n\leqslant\frac{1}{n}(x_1^n+x_2^n+\cdots+x_n^n).$$

Indeed, let

$$F(x, y)=(\alpha_1x-y)\cdots(\alpha_nx-y)$$

and put

$$F_i(x, y)=\frac{F(x, y)}{\alpha_ix-y}.$$

Then

$$\prod_{i=1}^{n}F_i(x, y)=F(x, y)^{n-1},$$

$$\prod_{i=1}^{n}|D_{F_i}|=|D_F|^{n-2}$$

and so

$$|D_F|^{1/n(n-1)}|F(1, v)|^{-2/n} = \prod_{i=1}^{n} |D_{F_i}|^{1/n(n-1)(n-2)}F_i(1, v)^{-2/n(n-1)}$$
$$\leqslant \frac{1}{n}\sum_{i=1}^{n} |D_{F_i}|^{1/(n-1)(n-2)}F_i(1, v)^{-2/(n-1)}.$$

The latter inequality is strict for all but finitely many v since for $i \neq j$,

$$|D_{F_i}|^{1/(n-2)}|F_i(1, v)|^{-2} = |D_{F_j}|^{1/(n-2)}|F_j(1, v)|^{-2}$$

for at most two values of v. Hence,

$$|D_F|^{1/n(n-1)}A_F < \frac{1}{n}\sum_{i=1}^{n} |D_{F_i}|^{1/(n-1)(n-2)}A_{F_i}$$

and the inequality $M_n < M_{n-1}$ follows as before.

5. Proof of Theorem 3

As noted in the previous section, the analysis of the quantity $|D_F|^{1/n(n-1)}A_F$ is equivalent to the analysis of the quantity

$$Q(\alpha_1, \ldots, \alpha_n) = \prod_{i<j} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|(v - \alpha_1) \cdots (v - \alpha_n)|^{2/n}}$$

where $\alpha_1, \ldots, \alpha_n$ are distinct complex numbers. In this section, I will show that if at least one of the α_i is non-real, then

$$Q(\alpha_1, \ldots, \alpha_n) < M_n.$$

It will then follow that Q is maximized at a point $(\alpha_1, \ldots, \alpha_n)$ for which each α_i is real (or ∞).

Throughout this section, I will adopt the convention that if one of the α's is infinite, say α_n, then

$$Q = \prod_{1 \leqslant i \leqslant j \leqslant n-1} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|(v - \alpha_1) \cdots (v - \alpha_{n-1})|^{2/n}}.$$

Before discussing the details of the proof of Theorem 3, let us recall the following terminology from the theory of functions.

A continuous real-valued function u of a single complex variable $z = x + iy$ is *harmonic* if it has continuous partial derivatives of the second order and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A continuous real-valued function v of a single complex variable is said to be *subharmonic* if, in any region of the complex plane, v is less than or equal to the harmonic function u which coincides with v on the boundary of the region. A subharmonic function need not be continuous; however, this assumption allows one to simplify the definition to some extent. There are several equivalent definitions of subharmonicity; the one given here highlights the property of convexity.

An important property of subharmonic functions is that they satisfy the maximum principle. The *maximum principle for subharmonic functions* states that a non-constant subharmonic function has no maximum in its region of definition. Consequently, the maximum of a subharmonic function on a closed bounded set is attained on the boundary of the set.

The generalizations of these concepts to functions of several complex variables are respectively the notions of pluriharmonicity and plurisubharmonicity. A continuous real-valued function of several complex variables is said to be *plurisubharmonic* if its restriction to any complex line is subharmonic on that line. The function is *pluriharmonic* if its restriction to any complex line is harmonic on that line. A *complex line* in $\mathbb{C}^n$ is a set of the form

$$\{a + b\zeta : \zeta \in \mathbb{C}\}$$

where a, $b \in \mathbb{C}^n$. Notice that any positive linear combination of plurisubharmonic functions is plurisubharmonic. Further, the composition of a plurisubharmonic function and a monotonically increasing convex function is plurisubharmonic (see Gunning [8] or Hormander [10]).

Now consider the quantity

$$Q(\alpha_1, \ldots, \alpha_n) = \prod_{i<j} |\alpha_j - \alpha_i|^{2/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|(v - \alpha_1) \cdots (v - \alpha_n)|^{2/n}}$$

as a function of the complex variables $\alpha_1, \ldots, \alpha_n$. This function is plurisubharmonic on the region

$$\mathscr{R} = \mathbb{C}^n \Big\backslash \bigcup_{i=1}^{n} \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n : \alpha_i \in \mathbb{R}\}.$$

To see this, it suffices, by linearity, to show that each of the functions q_v given by

$$q_v(\alpha_1,\ldots,\alpha_n) = \frac{\Pi_{i<j}|\alpha_j - \alpha_i|^{2/n(n-1)}}{|(v-\alpha_1)\ \cdots\ (v-\alpha_n)|^{2/n}}$$

with v a real number, is plurisubharmonic on $\mathscr{R}$. Since the exponential function is convex and monotonically increasing, it suffices to demonstrate this for

$$\log q_v(\alpha_1,\ldots,\alpha_n) = \frac{2}{n(n-1)} \sum_{i<j} \log|\alpha_j - \alpha_i| - \frac{2}{n}\sum_{i=1}^{n} \log|v - \alpha_i|.$$

Now, for $i \neq j$, the function $(\alpha_1,\ldots,\alpha_n) \mapsto \log|\alpha_j - \alpha_i|$ is plurisubharmonic on $\mathbb{C}^n$; in fact, it is pluriharmonic on $\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n : \alpha_i \neq \alpha_j\}$. Further, the function $(\alpha_1,\ldots,\alpha_n) \mapsto -\log|v-\alpha_i|$ with v a real number, is pluriharmonic except on the hyperplane $\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n : \alpha_i = v\}$. Hence, Q is plurisubharmonic on $\mathscr{R}$ as claimed.

In particular, for fixed values of $\alpha_1,\ldots,\alpha_{n-1}$, the quantity $Q(\alpha_1,\ldots,\alpha_n)$, when viewed as a function of α_n, is subharmonic in the upper and lower half planes. Moreover, it is continuous and non-constant on $\mathbb{C}$. Hence, by the maximum principle for subharmonic functions,

$$Q(\alpha_1,\ldots,\alpha_n) \leqslant Q(\alpha_1,\ldots,\alpha_{n-1}, r_n)$$

for some real number r_n (possibly ∞). Moreover, this inequality is strict for non-real values of α_n.

Now the quantity $Q(\alpha_1,\ldots,\alpha_{n-1}, r_n)$ when viewed as a function of the complex variables $\alpha_1,\ldots,\alpha_{n-1}$ is plurisubharmonic on

$$\mathbb{C}^{n-1} \Big\backslash \bigcup_{i=1}^{n-1} \{(\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{C}^{n-1} : \alpha_i \in \mathbb{R}\}.$$

Hence, arguing as before, we have

$$Q(\alpha_1,\ldots,\alpha_n) \leqslant Q(\alpha_1,\ldots,\alpha_{n-2}, r_{n-1}, r_n)$$

for some real number r_{n-1} (possibly ∞) distinct from r_n. Continuing in this way, we find that

$$Q(\alpha_1,\ldots,\alpha_n) \leqslant Q(r_1,\ldots,r_n)$$

for some n-tuple $(r_1, \ldots, r_n)$ of distinct real numbers (possibly including the point at infinity). In fact, if at least one of the α's is non-real, then this inequality is strict.

Therefore, if F is a form of degree n for which the polynomial $F(x, 1)$ has at least one non-real root, then

$$|D_F|^{1/n(n-1)} A_F < M_n.$$

This completes the proof of Theorem 3.

6. Proof of Theorem 1

In view of Theorem 2 and Theorem 3, it suffices to prove that every form F for which the polynomial $F(x, 1)$ has three distinct real roots is (up to multiplication by a complex number) equivalent under $GL_2(\mathbb{R})$ to $xy(x - y)$ and that

$$|D_F|^{1/6} A_F = 3B\left(\frac{1}{3}, \frac{1}{3}\right)$$

for any such form. Since $|D_F|^{1/6} A_F$ is invariant with respect to replacing F by kF for any complex number k, there is no loss of generality in assuming that F has real coefficients in this case. Hence, let F be a cubic form with real coefficients for which $D_F > 0$.

Notice that a linear substitution applied to F induces a fractional linear transformation of the roots of the polynomial $F(1, y)$. Indeed, if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, then the roots are transformed according to the rule

$$t \mapsto \frac{at - c}{d - bt}.$$

Since every fractional linear transformation with real coefficients may be given by the rule

$$\frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$$

where the z's and w's are real numbers such that z_1, z_2, z_3 are mapped to w_1, w_2, w_3 respectively, it follows that F is equivalent under $GL_2(\mathbb{R})$ to the form $F_1(x, y) = xy(x - y)$. Hence

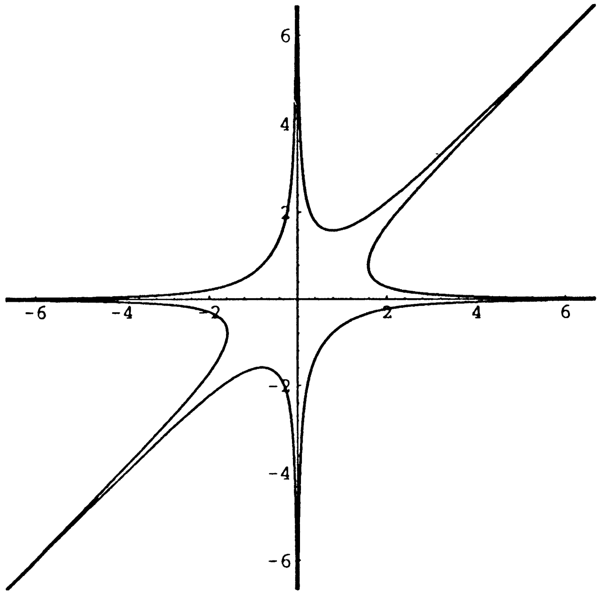

Fig. 1. $|xy(x-y)| = 1$.

$$
\begin{aligned}
|D_F|^{1/6} A_F &= A_{F_1} \\
&= \int_{-\infty}^{\infty} \frac{dv}{|F_1(1, v)|^{2/3}} \\
&= \int_{-\infty}^{\infty} \frac{dv}{|v(1-v)|^{2/3}} \\
&= 3B\left(\frac{1}{3}, \frac{1}{3}\right)
\end{aligned}
$$

as required.

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