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The nonlinear superposition operator acting on Bergman spaces

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1. Introduction

Let Δ denote the unit disc $\{z:|z|<1\}$ in the complex plane and let $H(\Delta)$ denote the space of analytic functions in Δ with the topology of the uniform convergence in compact subsets of Δ. Given a function $f:\mathbb{C}\to\mathbb{C}$ we associate to it the operator F_f defined by

$$F_f(u)(z) = f(u(z)), \quad u\in H(\Delta).$$

This operator is known as the autonomous nonlinear superposition (or composition) operator [1]. If A and B are linear subspaces of $H(\Delta)$ and $F_f(u)\in B$ whenever $u\in A$ we shall say that F_f acts from A to B. It is easy to see that if F_f acts from $H(\Delta)$ to $H(\Delta)$, then f must be an entire function and conversely. In this case mere action implies the continuity and the boundedness of the operator [2]. That mere action implies continuity has already been proved for various spaces of real functions, for instance L^p spaces [6] and Sobolev spaces [7]. Necessary and sufficient conditions have been given in [2] in order that F_f acts from H^p to H^q, $0<p,q\leqslant+\infty$, where H^p denotes the classical Hardy space in the unit disc. It is also true in this case that mere action implies continuity [2]. If N denotes the Nevanlinna space of functions in $H(\Delta)$ of bounded characteristic then the actions from $\bigcup_{p<q} H^q$ to N and from N to N have been studied in [3].

In this note we shall consider the problem of action and continuity between the Bergman space B_p defined by

$$B_p = \{u\in H(\Delta): u\in L^p(\mathrm{d}x\,\mathrm{d}y)\}, \quad 0<p<\infty.$$

The space B_∞ is the usual one of bounded analytic functions in Δ. The topology in these spaces is given by the metric induced (when $p\geqslant 1$) by

$$\|u\|_{B_p} = \left(\frac{1}{\pi}\iint_\Delta |u(z)|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

If $p < 1$ the topology induced by the metric $\|u\|_{B_p}^p$ is used. We also consider the action between B_p and the Hardy space H^q and vice versa. For functions in H^p we use the standard notation

$$M_p(r, u) = \left(\frac{1}{2\pi}\int_0^{2\pi} |u(r\,e^{i\theta})|^p\,d\theta\right)^{1/p} \quad \text{and} \quad \|u\|_p = \lim_{r\to 1} M_p(r, u).$$

The symbol BN (which stands for Bergman-Nevanlinna) shall denote the set of functions u in $H(\Delta)$ such that

$$\iint_\Delta \log^+|u(z)|\,dx\,dy < \infty.$$

Clearly $H^p \subset B_p$ and $B_p \subset BN$ for all p. Finally, we study the action between Hardy functions and Bergman-Nevanlinna functions.

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2. The action in B_p

We shall need the following lemma.

LEMMA 1. *Let* $0 < p < \infty$. *If* $u \in B_p$ *then*

$$|u(z)| \leqslant \frac{\|u\|_{B_p}}{(1 - |z|)^{2/p}}, \quad z \in \Delta.$$

Proof. This is an easy consequence of the subharmonicity of $|u|^p$.

Next we are ready to prove the following result. In what follows the symbol "$[s]$" denotes the integer part of s.

THEOREM 1. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire function. Then* F_f *acts from* B_p *to* B_q, $0 < p, q \leqslant \infty$ *if and only if* f *is a polynomial of degree less than or equal to* $\left[\frac{p}{q}\right]$.

Proof. If f is a polynomial of degree $n \leqslant \left[\frac{p}{q}\right]$ then $f \circ u \in B_q$, $\forall u \in B_p$. In fact, it is enough to see that if $k \leqslant \left[\frac{p}{q}\right]$, $k \in \mathbb{N}$, then $u^k \in B_q$. This is true since

$$\|u^k\|_{B_q} = \left(\frac{1}{\pi}\iint_\Delta |u(z)|^{kq}\,dx\,dy\right)^{1/q} = \left(\left(\frac{1}{\pi}\iint_\Delta (|u(z)|)^{kq}\,dx\,dy\right)^{1/kq}\right)^k$$

$$\leqslant \left(\frac{1}{\pi}\iint_{\Delta}(|u(z)|)^p \,\mathrm{d}x\,\mathrm{d}y\right)^{k/p} = \|u\|_{B_p}^k.$$

Next, we assume that F_f acts from B_p to B_q, $0 < p, q < \infty$. Let $\varepsilon > 0$ and define $u_\varepsilon^{(1)}(z) = \left(\frac{1}{1-z} - \frac{1}{2}\right)^{2/p+\varepsilon}$. Clearly $u_\varepsilon^{(1)}$ belongs to B_p. Therefore $f \circ u_\varepsilon^{(1)} \in B_q$ and by Lemma 1 one can write

$$|f(u_\varepsilon^{(1)}(z))| \leqslant \frac{\|f \circ u_\varepsilon^{(1)}\|_{B_q}}{(1-|z|)^{2/q}}, \quad z \in \Delta. \tag{2.1}$$

Set $w_1 = u_\varepsilon^{(1)}(z)$. Assume first that $p < 1$ and take $\varepsilon < 1 - p$. Then the range set of $u_\varepsilon^{(1)}$ is $\mathbb{C}\backslash 0$. Given $w_1 \in \mathbb{C}\backslash 0$ let $z \in \Delta$ such that $w_1 = u_\varepsilon^{(1)}(z)$ and

$$|z| = \frac{|w_1^{p+\varepsilon/2} - \frac{1}{2}|}{|w_1^{p+\varepsilon/2} + \frac{1}{2}|}.$$

Thus, from (2.1) we get

$$|f(w_1)| \leqslant \frac{\|f \circ u_\varepsilon^{(1)}\|_{B_q}}{\left(1 - \frac{|w_1^{p+\varepsilon/2} - \frac{1}{2}|}{|w_1^{p+\varepsilon/2} + \frac{1}{2}|}\right)^{2/q}}$$

$$= \frac{\|f \circ u_\varepsilon^{(1)}\|_{B_q} |w_1^{p+\varepsilon/2} + \frac{1}{2}|^{2/q}}{(|w_1^{p+\varepsilon/2} + \frac{1}{2}| - |w_1^{p+\varepsilon/2} - \frac{1}{2}|)^{2/q}}.$$

If w_1 is such that $|w_1| > \mathrm{e}^{1/2(p+\varepsilon)}$ then $z_1 = w_1^{p+\varepsilon/2}(-\pi \leqslant \operatorname{Arg} w_1 < \pi)$ satisfies $\operatorname{Re} z_1 > \mathrm{e}^{1/4}\cos(-\pi/2(p+\varepsilon)) > 0$. In fact, since $\operatorname{Arg} w_1 \geqslant -\pi$ then $((p+\varepsilon)/2)\operatorname{Arg} w_1 \geqslant -\pi/2(p+\varepsilon)$. On the other hand

$$\operatorname{Re} z_1 = |w_1|^{p+\varepsilon/2}\cos\left(\frac{p+\varepsilon}{2}\operatorname{Arg} w_1\right) > |w_1|^{p+\varepsilon/2}\cos\left(-\frac{\pi}{2}(p+\varepsilon)\right).$$

Hence, if $|w_1| > \mathrm{e}^{1/2(p+\varepsilon)}$ then $\operatorname{Re} z_1 > \mathrm{e}^{1/4}\cos(\pi/2(p+\varepsilon)) > 0$. Therefore, one can find a positive constant c (depending on ε and p) such that

$$|w_1^{p+\varepsilon/2} + \tfrac{1}{2}| - |w_1^{p+\varepsilon/2} - \tfrac{1}{2}| > c, \quad |w_1| > \mathrm{e}^{1/2(p+\varepsilon)}.$$

If we use this inequality in (2.2) we obtain

$$|f(w_1)| \leqslant C(p, q, \varepsilon)(|w_1|^{p+\varepsilon/q} + 2^{-2/q})$$

for all w_1 such that $|w_1| > e^{1/2(p+\varepsilon)}$. Thus f is a polynomial of degree less than or equal to $p + \varepsilon/q$. By letting $\varepsilon \to 0$ we obtain the desired result when $p < 1$.

Now, we assume that $p \geqslant 1$. Let

$$S_1 = \left\{w_1 : \frac{-\pi}{2(p+\varepsilon)} \leqslant \operatorname{Arg} w_1 < \frac{\pi}{2(p+\varepsilon)}, \; |w_1| > 1\right\}.$$

If $w_1 \in S_1$ we choose z_1 such that $w_1 = z_1^{2/p+\varepsilon}$ with $-\pi/4 < \operatorname{Arg} z_1 < \pi/4$. Thus $(|w_1^{p+\varepsilon/2} + \frac{1}{2}| - |w_1^{p+\varepsilon/2} - \frac{1}{2}|)^{2/q} \geqslant c > 0$, for some constant c. Combining this inequality with (2.2) we obtain

$$\begin{aligned} |f(w_1)| &\leqslant \frac{\|f \circ u_\varepsilon^{(1)}\|_{B_q}}{c} (|w_1|^{p+\varepsilon/2} + \tfrac{1}{2})^{2/q} \\ &\leqslant c(\varepsilon, q)\left(|w_1|^{p+\varepsilon/q} + \frac{1}{2^{2/q}}\right). \end{aligned} \tag{2.3}$$

Let

$$w_2 \in S_2 = \left\{w_2 : \frac{\pi}{2(p+\varepsilon)} \leqslant \operatorname{Arg} w_2 < \frac{3\pi}{2(p+\varepsilon)}, \; |w_2| > 1\right\}$$

and

$$u_\varepsilon^{(2)}(z) = \left(\frac{1}{1-z} - \frac{1}{2}\right)^{2/p+\varepsilon} e^{i\pi/p+\varepsilon}.$$

Clearly $u_\varepsilon^{(2)} \in B_p$. By hypothesis $f \circ u_\varepsilon^{(2)} \in B_q$ and by Lemma 1

$$|f(u_\varepsilon^{(2)})(z)| \leqslant \frac{\|f \circ u_\varepsilon^{(2)}\|_{B_q}}{(1-|z|)^{2/q}}, \quad z \in \Delta. \tag{2.4}$$

Given $w_2 \in S_2$ we choose z_2 such that $|z_2| > 1$ and $w_2 = z_2^{2/p+\varepsilon} e^{i\pi/p+\varepsilon}$ with $-\pi/4 \leqslant \operatorname{Arg} z_2 \leqslant \pi/4$. From (2.4) we obtain

$$\begin{aligned} |f(w_2)| &\leqslant \frac{\|f \circ u_\varepsilon^{(2)}\|_{B_q}}{\left(1 - \dfrac{|w_2^{p+\varepsilon/2} e^{i\pi/2} - \frac{1}{2}|}{|w_2^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2}|}\right)^{2/q}} \\ &= \frac{\|f \circ u_\varepsilon^{(2)}\|_{B_q} |w_2^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2}|^{2/q}}{(|w_2^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2}| - |w_2^{p+\varepsilon/2} e^{i\pi/2} - \frac{1}{2}|)^{2/q}}. \end{aligned} \tag{2.5}$$

Since $-\pi/4 \leqslant \operatorname{Arg} z_2 \leqslant \pi/4$ then

$$|w_2^{p+\varepsilon/2}\,\mathrm{e}^{-i\pi/2}+\tfrac{1}{2}|-|w_2^{p+\varepsilon/2}\,\mathrm{e}^{-i\pi/2}-\tfrac{1}{2}|\geqslant c^{q/2}>0.$$

Hence, from (2.5) we obtain

$$|f(w_2)|\leqslant c^{-1}\|f\circ u_\varepsilon^{(2)}\|_{B_q}|w_2^{p+\varepsilon/2}\,\mathrm{e}^{-i\pi/2}+\tfrac{1}{2}|^{2/q}.$$

By repeating the same argument n times, where n is such that $\bar{\Delta}=\mathbb{C}\backslash\bigcup_{i=1}^{n}S_i$, we obtain

$$|f(w)|\leqslant c_1(|w|^{p+\varepsilon/2}+\tfrac{1}{2})^{2/q},\quad w\in\mathbb{C}\backslash\bar{\Delta},$$

where c_1 depends on f, ε and q. This proves that f is an entire function of order at most $p+\varepsilon/q$. Letting ε tend to zero gives the desired result for $p\geqslant 1$.

3. Continuity of F_f

We shall prove in this section that if F_f acts from B_p to B_q then it is necessarily continuous. We also prove local Lipschitzness.

First of all let us prove the following lemma.

LEMMA 2. *If $u_k\to u$ in B_p, $n\in\mathbb{N}$ and $n\leqslant\left[\dfrac{p}{q}\right]$, then $u_k^n\to u^n$ in B_q.*

Proof. The proof is similar to the analogue lemma given in [2]. We give it here somewhat simplified. The case $n=1$ is obvious. Let us assume that $n>1$. The functions u_k^n and u^n belong to B_q. In fact, since $nq\leqslant p$ then for every $u\in B_p$ we have

$$\|u^n\|_{B_q}\leqslant\|u\|_{B_p}^n.$$

On the other hand,

$$\begin{aligned}\|u_k^n-u^n\|_{B_q}&=\left(\frac{1}{\pi}\iint_\Delta|u_k^n-u^n|^q\,\mathrm{d}x\,\mathrm{d}y\right)^{1/q}\\&=\left(\left(\frac{1}{\pi}\iint_\Delta(|u_k^n-u^n|^{1/n})^{qn}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/nq}\right)^{n}\\&\leqslant\left(\frac{1}{\pi}\iint_\Delta|u_k^n-u^n|^{p/n}\,\mathrm{d}x\,\mathrm{d}y\right)^{n/p}\\&=\left(\frac{1}{\pi}\iint_\Delta|u_k-u|^{p/n}|u_k^{n-1}+\cdots+u^{n-1}|^{p/n}\,\mathrm{d}x\,\mathrm{d}y\right)^{n/p}.\end{aligned}$$

Now we use Hölder inequality

$$\left|\int fg\right| \leqslant \left(\int f^r\right)^{1/r}\left(\int g^s\right)^{1/s},$$

with $f = |u_k - u|^{p/n}$, $g = |u_k^{n-1} + \cdots + u^{n-1}|^{p/n}$, $r = n > 1$ and

$$\frac{1}{n} + \frac{1}{s} = 1 \left(s = \frac{n}{n-1}\right),$$

and obtain

$$\|u_k^n - u^n\|_{B_q} \leqslant \left(\frac{1}{\pi}\iint_\Delta |u_k - u|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$
$$\left(\frac{1}{\pi}\iint_\Delta |u_k^{n-1} + \cdots + u^{n-1}|^{p/n-1}\,\mathrm{d}x\,\mathrm{d}y\right)^{n-1/p}$$
$$\leqslant c\|u_k - u\|_{B_p}\sum_{l=0}^{n-1}\|u_k^{n-1-l}u^l\|_{B_{p/n-1}}, \tag{3.1}$$

where c is a constant. Again by a refined version of Hölder inequality we get that $u_k^{n-1-l}u^l \in B_{p/n-1}$, $l = 0, 1, \ldots, n-1$ and

$$\|u_k^{n-1-l}u^l\|_{B_{p/n-1}} \leqslant \|u_k^{n-1-l}\|_{B_{p/n-1-l}}\|u^l\|_{B_{p/l}} = u_k\|_{B_p}^{n-1-l}\|u\|_{B_p}^l \tag{3.2}$$

This inequality implies that all summands on the right-hand side of (3.1) are bounded (for all k). Hence $u_k^n \to u^n$ in B_q as required.

THEOREM 2. *If F_f acts from B_p to B_q then it is necessarily continuous, bounded and locally Lipschitz.*

Proof. Since F_f acts from B_p to B_q then, by Theorem 1, f is a polynomial of degree $n \leqslant \left[\frac{p}{q}\right]$. Set $f(z) = a_n z^n + \cdots + a_0$. Let $u_k(z) \to u(z)$, as $k \to \infty$, in B_p. Then

$$F_f(u_k)(z) - F_f(u)(z) = a_n(u_k^n(z) - u^n(z)) + \cdots + a_1(u_k(z) - u(z)).$$

Thus

$$\|F_f(u_k) - F_f(u)\|_{B_q} \leqslant C(\|u_k^n - u^n\|_{B_q} + \cdots + \|u_k - u\|_{B_q}) \to 0 \quad \text{as } k \to \infty$$

by Lemma 2. The boundedness of F_f comes from the inequality

$$\|F_f(u)\|_{B_q} \leqslant C(|a_n|\,\|u\|^n_{B_p} + \cdots + |a_1|\,\|u\|_{B_p} + |a_0|), \quad u \in B_p$$

which can be deduced from $\|u^n\|_{B_q} \leqslant \|u\|^n_{B_p}$, for all $u \in B_p$ and all $n \leqslant \left[\frac{p}{q}\right]$.

In order to prove that F_f is locally lipschitz we must see that if u, $v \in B(0, R) \subset B_p$ then there exists a constant $C = C(p, q, R, f)$ such that $\|F_f(u) - F_f(v)\|_{B_q} \leqslant C\|u - v\|_{B_p}$.

On the one hand,

$$\|F_f(u) - F_f(v)\|_{B_q} \leqslant C(\|u^n - v^n\|_{B_q} + \cdots + \|u - v\|_{B_q}).$$

On the other hand, in the same way that we deduced (3.1) and (3.2) we obtain

$$\begin{aligned}\|u^n - v^n\|_{B_q} &\leqslant C_1\|u - v\|_{B_p} \sum_{l=0}^{n-1} \|u^{n-1-l} - v^l\|_{B_{p/n-1}} \\ &\leqslant C_1\|u - v\|_{B_p}\left(\sum_{l=0}^{n-1} \|u\|^{n-1-l}_{B_p}\|v\|^l_{B_p}\right) \leqslant C(p, q, R, f)\|u - v\|_{B_p},\end{aligned}$$

for all $n \leqslant \left[\frac{p}{q}\right]$.

4. The action from B_p to the Bergman-Nevanlinna space

The Bergman-Nevanlinna space is defined by

$$BN = \left\{u \in H(\Delta) : \|u\|_{BN} = \frac{1}{\pi}\iint_\Delta \log^+|u(z)|\,\mathrm{d}x\,\mathrm{d}y < \infty\right\}.$$

It is easy to see that $B_p \subset BN$, $\forall p > 0$. The following result is an easy consequence of Theorem 1.

COROLLARY 1. *Let f be an entire function such that F_f acts from BN to B_q, $0 < q \leqslant \infty$. Then f is constant.*

Proof. In particular F_f acts from B_p to B_q, for all $p > 0$. Then from Theorem 1 we conclude that f is a polynomial of degree at most $\left[\frac{p}{q}\right]$. Taking p less than q one obtains the desired conclusion.

LEMMA 3. *If $u \in BN$ then*

$$\log^+|u(z)| \leqslant \frac{\|u\|_{BN}}{(1 - |z|)^2}, \quad z \in \Delta.$$

This lemma is a consequence of the subharmonicity of $\log^+|u|$.

THEOREM 3. *Let f be an entire function. Then F_f acts from $\bigcup_{p<q} B_q$ to BN $(0 < p < \infty)$ if and only if f has order at most p.*

Proof. Assume that F_f acts from $\bigcup_{p<q} B_q$ to BN. Let $\varepsilon > 0$. The functions

$$w_1 = u_\varepsilon(z) = \left\{\frac{1}{1-z} - \frac{1}{2}\right\}^{2/p+\varepsilon}$$

belong to $\bigcup_{p<q} B_q$. Hence, $f \circ u_\varepsilon \in BN$ and so, by Lemma 3

$$|f(w_1)| \leqslant e^{C/(1-|z|)^2}. \tag{4.1}$$

If $p < 1$ we take ε so that $p + \varepsilon < 1$. Then the range of u_ε is $\mathbb{C}\backslash 0$. Thus, given $w_1 \in \mathbb{C}\backslash 0$ we take $z \in \Delta$ such that

$$|z| = \left|\frac{w_1^{p+\varepsilon/2} - \frac{1}{2}}{w_2^{p+\varepsilon/2} + \frac{1}{2}}\right|.$$

From (4.1) we get

$$\log|f(w_1)| \leqslant \frac{C}{(1-|z|)^2}$$

$$= \frac{C}{\left(1 - \left|\dfrac{w_1^{p+\varepsilon/2} - \frac{1}{2}}{w_1^{p+\varepsilon/2} + \frac{1}{2}}\right|\right)^2} = \frac{C|w_2^{p+\varepsilon/2} + \frac{1}{2}|^2}{(|w_1^{p+\varepsilon/2} + \frac{1}{2}| - |w_1^{p+\varepsilon/2} - \frac{1}{2}|)^2}. \tag{4.2}$$

If $|w_1| > e^{1/2(p+\varepsilon)}$ then $z_1 = w_1^{p+\varepsilon/2} (-\pi \leqslant \operatorname{Arg} w < \pi)$ satisfies

$$\operatorname{Re} z_1 > e^{1/4} \cos\left(-\frac{\pi}{2}(p+\varepsilon)\right) > 0.$$

Therefore, there is a positive constant C such that

$$|w_1^{p+\varepsilon/2} + \tfrac{1}{2}| - |w_1^{p+\varepsilon/2} - \tfrac{1}{2}| > C, \; |w_1| > e^{1/2(p+\varepsilon)}. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\log|f(w_1)| \leqslant C_1 |w_1^{p+\varepsilon/2} + \tfrac{1}{2}|^2,$$

for a suitable positive constant C_1 and all w_1 such that $|w_1| > e^{1/2(p+\varepsilon)}$. This shows that f is of order at most $p + \varepsilon$. Letting ε tend to zero permits us to conclude that f is of order at most p.

Next, we assume that $p \geqslant 1$. In this case we argue as in Theorem 1 and obtain from (4.2) that

$$\log |f(w)| = O(|w|^{p+\varepsilon})$$

for all w outside a ball. Hence f is an entire function of order at most p.

Let us suppose now that f is an entire function of order less than or equal to p and $u \in \bigcup_{p<q} B_q$. Take $\varepsilon > 0$ such that $u \in B_{p+\varepsilon}$. There is a constant C so that

$$\log^+ M(r, f) \leqslant r^{p+\varepsilon} + C, \quad \forall r \geqslant 0.$$

To prove that $f \circ u \in BN$ we write

$$\iint_\Delta \log^+|f(u(z))| \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \log^+|f(u(r\,\mathrm{e}^{i\theta}))| \,\mathrm{d}\theta$$

$$\leqslant \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \log^+ M(|u(r\,\mathrm{e}^{i\theta})|, f)\,\mathrm{d}\theta \leqslant \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} |u(r\,\mathrm{e}^{i\theta})|^{p+\varepsilon}\,\mathrm{d}\theta + \pi C$$

$$= \iint_\Delta |u(z)|^{p+\varepsilon}\,\mathrm{d}x\,\mathrm{d}y + \pi C < \infty,$$

since $u \in B_{p+\varepsilon}$.

THEOREM 4. *Let f be an entire function of order less than p or of order p and finite type* $(0 < p < \infty)$. *Then F_f acts from B_p to BN.*

Proof. We may assume that f is of order p and finite type $\sigma - \delta > 0$. Hence, there is a constant C such that

$$\log^+ M(r, f) \leqslant \sigma r^p + C, \quad r \geqslant 0.$$

If $u \in B_p$ then

$$\iint_\Delta \log^+|f(u(z))| \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \log^+|f(u(r\,\mathrm{e}^{i\theta}))|\mathrm{d}\theta$$

$$\leqslant \int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \log^+ M(|u(r\,\mathrm{e}^{i\theta})|, f)\,\mathrm{d}\theta$$

$$\leqslant \sigma \iint_\Delta |u(z)|^p\,\mathrm{d}x\,\mathrm{d}y + \pi C < \infty,$$

as required.

5. Transforming Hardy functions into Bergman functions and vice versa

We shall begin by stating the following classical results by Hardy and Littlewood [5].

LEMMA 4. *Let u be analytic in Δ and*

$$M_p(r, u) \leqslant \frac{C}{(1-r)^{\beta}}, \quad 0 < p < \infty, \quad \beta \geqslant 0.$$

Then there is a constant $K = K(p, \beta)$ such that

$$M_{q_1}(r, u) \leqslant \frac{KC}{(1-r)^{\beta + 1/p - 1/q_1}}, \quad p < q_1 \leqslant \infty.$$

A proof of this result can be found in [4, p. 84].

LEMMA 5. *If $0 < p < q_1 \leqslant \infty$, $u \in H^p$, $\lambda \geqslant p$, and $\alpha = 1/p - 1/q_1$ then*

$$\int_0^1 (1-r)^{\lambda\alpha - 1} M_{q_1}(r, f)^{\lambda} \, \mathrm{d}r < \infty.$$

The reader can find a proof of this result in [4, p. 87].

The next result shows that one cannot transform Bergman functions into Hardy functions by means of nonlinear superposition. In case $p = \infty$ it is trivial that F_f acts for any f.

THEOREM 5. *Let f be an entire function. If $p \neq \infty$ then F_f acts from B_p to H^q if and only if f is constant.*

Proof. If F_f acts from B_p to H^q then, by Theorem 1, f is a polynomial. Now we get the desired conclusion by noting that for a non-constant polynomial f it is not true that $f \circ u \in H^q$, $\forall u \in B_p$. If this were true then the zeros of all Bergman functions would have to satisfy the Blaschke condition, and this is false.

THEOREM 6. *Let f be an entire function. Then F_f acts from H^p to B_q if and only if f is a polynomial of degree at most $\left[\frac{2p}{q}\right]$.*

If $p = \infty$ then F_f acts from H^p to B_q for any f. In the proof of this theorem we shall rule out this case.

COROLLARY 2. *The operator F_f acts from H^p to B_p if and only if f is a polynomial of degree one or two.*

Proof of Theorem 6. The proof that f must be a polynomial of degree less than or equal to $\left[\frac{2p}{q}\right]$ can be done as in Theorem 1. Let us assume now that f is a polynomial of degree $n \leqslant \left[\frac{2p}{q}\right]$. We shall prove that if $u \in H^p$ then $u^n \in B_q$. Let us suppose, first of all, that $n < \frac{2p}{q}$. Then

$$\begin{aligned} \iint_\Delta |u^n(z)|^q \, dx \, dy &= \int_0^1 r \, dr \int_0^{2\pi} |u(r e^{i\theta})|^{nq} \, d\theta \\ &\leqslant 2\pi \int_0^1 r \, dr \left(\frac{1}{2\pi} \int_0^{2\pi} |u(r e^{i\theta})|^{2p} \, d\theta \right)^{nq/2p} \\ &= 2\pi \int_0^1 r M_{2p}(r, u)^{nq} \, dr. \end{aligned} \tag{5.1}$$

Now, using Lemma 4 with $\beta = 0$, $q_1 = 2p$ we obtain

$$M_{2p}(r, u) \leqslant \frac{C}{(1-r)^{1/p - 1/2p}} = \frac{C}{(1-r)^{1/2p}},$$

for some constant C. Combining this inequality with (5.1) one gets

$$\iint_\Delta |u^n(z)|^q \, dx \, dy \leqslant 2\pi C \int_0^1 \frac{r \, dr}{(1-r)^{nq/2p}} < \infty.$$

Thus $u^n \in B_q$, as required.

Next we assume that $n = 2p/q$. In this case we use Lemma 5 with $q_1 = 2p$, $\alpha = 1/2p$, and $\lambda = 2p$ to conclude that

$$\int_0^1 M_{2p}(r, u)^{2p} \, dr < \infty$$

as required.

6. The action from H^p to BN

If $p = \infty$ and f is any entire function then F_f acts from H^∞ to H^∞ and consequently it acts from H^∞ to BN. When $p < \infty$ we have the following result.

THEOREM 7. *Let* $f:\mathbb{C}\to\mathbb{C}$. *Then* F_f *acts from* $\bigcup_{p<q} H^q$ *to* BN *if and only if* f *is an entire function of order at most* $2p$.

Proof. Let us assume that F_f acts from $\bigcup_{p<q} H^q$ to BN. Clearly f must be an entire function. On the other hand, given $\varepsilon > 0$, the function

$$u_\varepsilon(z) = \left\{\frac{1}{1-z} - \frac{1}{2}\right\}^{1/p+\varepsilon}$$

belongs to $\bigcup_{p<q} H^q$. Using these functions and Lemma 3 and arguing as in Theorem 1 we deduce that f has order at most $2p$. Conversely, let us suppose that f has order at most $2p$. Let $u \in \bigcup_{p<q} H^q$ and $\varepsilon > 0$ such that $u \in H^{p+\varepsilon}$. Next we take a constant C such that

$$\log^+ M(r, f) \leqslant r^{2(p+\varepsilon)} + C, \quad \forall r \geqslant 0,$$

and use this inequality to get

$$\iint_\Delta \log^+|f(u(z))|\, \mathrm{d}x\, \mathrm{d}y \leqslant \iint_\Delta |u(z)|^{2(p+\varepsilon)}\, \mathrm{d}x\, \mathrm{d}y + \pi C < \infty,$$

since $u^2 \in B_{p+\varepsilon}$ in view of Corollary 2.

As a corollary we have

COROLLARY 3. *If* $u \in N$, $u(z) \neq 0$ *in* Δ, *and* $\log u(z) \neq 0$ *in* Δ, *then* $\mathrm{e}^{(\log u)2-\varepsilon} \in BN$, $\forall \varepsilon$, $0 < \varepsilon \leqslant 2$.

Proof. For $\varepsilon = 0$ the result breaks down as it is shown by the example

$$u(z) = \exp\left\{\frac{1+z}{1-z}\right\}.$$

To prove the corollary we proceed as follows. Since u does not vanish in Δ then $\log u$ is analytic there. Moreover $\log |u| \in h^1$, where h^1 is the space of harmonic functions in Δ which satisfy the Riesz-Herglotz representation. This can be seen from the following relations

$$\int_0^{2\pi} |\log |u(r\,\mathrm{e}^{i\theta})||\, \mathrm{d}\theta = \int_0^{2\pi} \log^+|u(r\,\mathrm{e}^{i\theta})|\, \mathrm{d}\theta + \int_0^{2\pi} \log^-|u(r\,\mathrm{e}^{i\theta})|\, \mathrm{d}\theta$$

$$= 2\int_0^{2\pi} \log^+|u(r\,\mathrm{e}^{i\theta})|\, \mathrm{d}\theta - 2\pi \log |u(0)|,$$

and the fact that $u \in N$. Then $\log u \in H^p$, $\forall p < 1$. Thus $(\log u)^{2-\varepsilon} \in \bigcup_{1/2<q} H^q$,

$\forall\varepsilon$, $0 < \varepsilon \leqslant 2$. Since $f(z) = e^z$ is an entire function of order 1 then, by Theorem 7, $e^{(\log u)^{2-\varepsilon}} \in BN$, as required.

Finally, we have

THEOREM 8. *Let f be an entire function of order less than $2p$ or of order $2p$ and finite type. Then F_f acts from H^p to BN.*

Proof. We may assume that f is of order $2p$ and finite type $\sigma - \delta > 0$. There is a constant C such that

$$\log M(r, f) \leqslant \sigma r^{2p} + C, \quad r \geqslant 0.$$

If $u \in H^p$ we get from the last inequality

$$\iint_\Delta \log^+|f(u(z))|\, dx\, dy \leqslant \sigma \iint_\Delta |u(z)|^{2p}\, dx\, dy + \pi C < \infty,$$

since $u^2 \in B_p$ in view of Corollary 2.

7. Some open questions

We finish this article by posing some questions. (1) If F_f acts from B_p to BN then, in particular, it acts from $\bigcup_{p<q} B_q$ to BN and, by Theorem 3, it has order at most p. In case that f has order p is it true that f has finite type? (2) One may ask the corresponding question for the action between H^p and BN.

References

1. Appel, J., Zabrejko, P. P.: *Nonlinear Superposition Operators.* Cambridge University Press, 1990.
2. Cámera, G. A.: The autonomous nonlinear superposition operator in H^p (to appear).
3. Cámera, G. A.: On entire functions acting via nonlinear superposition on Nevanlinna functions (to appear).
4. Duren, P.: *Theory of H^p Spaces.* Academic Press, New York, 1970.
5. Hardy, G. H. and Littlewood, J. E.: Some properties of fractional integrals. II. *Math. Z.* 34 (1932) 403–439.
6. Krasnosel'skij, M. A.: On the continuity of the operator $Fu(x) = f(x, u(x))$ (Russian). *Doklady Akad. Nauk SSSR* 77(2) (1951) 185–188.
7. Marcus, M. and Mizel, V. J.: Every superposition operator mapping one Sobolev space into another is continuous. *J. Funct. Anal.* 33 (1979) 217–229.