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The Generalized Hodge Conjecture for stably nondegenerate abelian varieties

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Introduction

The purpose of this paper is to show the validity of the Generalized Hodge Conjecture for stably nondegenerate abelian varieties all of whose simple components are of type I or II (Theorem 5.1). The crucial point is to express the "level" of a given Hodge structure as "the number of boxes" of the Young diagram associated with it (Proposition 4.3). Once this has been done, the Generalized Hodge Conjecture for the abelian varieties mentioned above follows rather easily from the structure theorem proved in [6] on the first cohomology group of such an abelian variety as the representation space of its Hodge group.

The plan of this paper is as follows. In Section 1, we recall the formulation of the Generalized Hodge Conjecture due to Grothendieck. In Section 2, we recall the definition of stably nondegenerate abelian variety and give some examples. In Section 3, we explain the way in which a Young diagram is associated with an irreducible representation of the Lie algebra $\mathrm{sp}(2n, \mathbb{C})$ of the symplectic group of rank n. In Section 4, using this correspondence, we prove a crucial proposition which describes the "level" of a Hodge structure in terms of the associated Young diagram. Finally in Section 5, we state the main theorem and give a proof.

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1. The Generalized Hodge Conjecture

Let X be a non-singular projective variety over $\mathbb{C}$. Then its m-th singular cohomology group $H^m(X, \mathbb{Q})$ has a $\mathbb{Q}$-Hodge structure of weight m. For a $\mathbb{Q}$-sub-Hodge structure W of $H^m(X, \mathbb{Q})$, its *level* is defined to be $\max\{|p-q|; W^{p,q} \neq 0\}$. We denote the level of W by $l(W)$. Note that this number can also be expressed as $\max\{p-q: W^{p,q} \neq 0\}$, since $\dim_{\mathbb{C}} W^{p,q} = \dim_{\mathbb{C}} W^{q,p}$. The Generalized Hodge Conjecture is formulated as follows:

The Generalized Hodge Conjecture ([5], [12]). *Given a non-singular projective variety X over $\mathbb{C}$, suppose that W is a $\mathbb{Q}$-sub-Hodge structure of $H^m(X, \mathbb{Q})$ with $l(W) \leqslant m - 2p$. Then there exists a Zariski closed subset Z of X of codimension p such that W is contained in* $\ker(H^m(X, \mathbb{Q}) \to H^m(X - Z, \mathbb{Q}))$.

For some examples for which this conjecture is known to hold, see [11], [12].

2. Stably nondegenerate abelian varieties

We recall the definition of stable nondegeneracy of an abelian variety. Let A be an abelian variety over $\mathbb{C}$. We say A is stably nondegenerate if, for any $n \geqslant 1$, the Hodge ring of A^n is generated by its divisor classes. In [6], we obtained a characterization of this property. It is formulated in terms of the rank of the Hodge group $\mathrm{Hg}(A)$ and the reduced dimension $\mathrm{rdim}\, A$ of A, which is defined as follows. For a given A, we decompose it to simple components with respect to isogeny:

$$A \sim A_1^{n_1} \times \cdots \times A_k^{n_k},$$

where $A_i \sim A_j$ for $i \neq j$. For each i, we put

$$\mathrm{rdim}\, A_i = \begin{cases} \dim A_i & \text{if } A_i \text{ is of type I,} \\ (1/2)\dim A_i & \text{if } A_i \text{ is of type II,} \\ \dim A_i & \text{if } A_i \text{ is of type III,} \\ (1/d)\dim A_i & \text{if } A_i \text{ is of type IV with} \\ & [\mathrm{End}^0(A_i) : \mathrm{Cent}(\mathrm{End}^0(A_i))] = d^2. \end{cases}$$

and we put $\mathrm{rdim}\, A = \Sigma_{i=1}^k \mathrm{rdim}\, A_i$ (see [8] for the definition of "type"). Then we have the following:

THEOREM 2.1 ([6, Theorem 2.7]). *An abelian variety A is stably nondegenerate if and only if* $\operatorname{rank}(\mathrm{Hg}(A)_{\mathbb{C}}) = \operatorname{rdim} A$.

Remark. (1) Generally, we have $\operatorname{rank}(\mathrm{Hg}(A)_{\mathbb{C}}) \leqslant \operatorname{rdim} A$, hence one can roughly say that A is stably nondegenerate when the rank of $\mathrm{Hg}(A)_{\mathbb{C}}$ is as large as possible.

(2) The following abelian varieties are known to be stably nondegenerate (see [6, section 4] and [10]):

(2.1) products of elliptic curves,
(2.2) simple abelian varieties of odd dimension without complex multiplication,
(2.3) $J_0(N)$, the jacobian variety of the modular curve $X_0(N)$,
(2.4) generic abelian varieties.

Moreover we know that, under a certain condition, the product of stably nondegenerate abelian varieties is also stably nondegenerate ([7]).

3. Young diagram

Let g denote the Lie algebra $\mathrm{sp}(2n, \mathbb{C})$ of the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. Let ω_i $(i = 1, \dots, n)$ denote its fundamental weights. As in the case of the representations of the Lie algebra of the special linear group, we can attach the Young diagram to each of the irreducible representations of g as follows (see [4]). Let $\lambda_1, \dots, \lambda_n$ be non-negative integers with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Such an n-tuple $(\lambda_1, \dots, \lambda_n)$ is called a Young diagram of length n. With a Young diagram $(\lambda_1, \dots, \lambda_n)$, we associate the irreducible representation of g with the highest weight

$$(\lambda_1 - \lambda_2)\omega_1 + \cdots + (\lambda_{n-1} - \lambda_n)\omega_{n-1} + \lambda_n \omega_n.$$

By this way we obtain a one-to-one correspondence between the Young diagrams of length n and the irreducible representations of g.

4. Hodge structure and Young diagram

Let A be an abelian variety over $\mathbb{C}$. By definition, the Hodge group $\mathrm{Hg}(A)$ of A is a subgroup of $\mathrm{Aut}(H^1(A, \mathbb{Q}))$, hence it also acts on $H^m(A, \mathbb{Q}) \simeq \Lambda^m(H^1(A, \mathbb{Q}))$. Recall that $\mathrm{Hg}(A^k) \simeq \mathrm{Hg}(A)$ and that the representation of $\mathrm{Hg}(A^k)$ into $H^1(A^k, \mathbb{Q}) \simeq \oplus^k H^1(A, \mathbb{Q})$ is equivalent to the diagonal representation of $\mathrm{Hg}(A)$ (see [6, section 1]).

PROPOSITION 4.1. *For a given sub-vector space W of $H^m(A^k, \mathbb{Q})$, the following conditions are equivalent:*

(4.1.1) *W is a $\mathbb{Q}$-sub-Hodge structure of $H^m(A^k, \mathbb{Q})$,*
(4.1.2) *W is stable under the action of* $\mathrm{Hg}(A)$ *on $H^m(A^k, \mathbb{Q})$.*

Proof. This is a direct consequence of the fact that the Hodge group $\mathrm{Hg}(A)$ is the special Mumford-Tate group associated with the Tannaka category generated by $H^1(A, \mathbb{Q})$ (see [3, section 3]).

In view of this proposition, we see that any assertions about the Hodge structure of $H^*(A^*, \mathbb{Q})$ should be expressed in terms of the representation of $\mathrm{Hg}(A)$. Therefore, if the Lie algebra $\mathrm{Lie}(\mathrm{Hg}(A)_{\mathbb{C}})$ of the Hodge group tensored by $\mathbb{C}$ is isomorphic to $\mathrm{sp}(2n, \mathbb{C})$, then the notion of "level" of a sub-Hodge structure may be formulated in terms of the Young diagram. In order to give such a dictionary, we introduce the following:

DEFINITION 4.2. *A $\mathbb{Q}$-sub-Hodge structure W* of *$H^m(A^k, \mathbb{Q})$ is said to be irreducible if it is irreducible as a representation space of $Hg(A)$.*

Then we can state the following crucial:

PROPOSITION 4.3. *Let A be an abelian variety such that* $\mathrm{Lie}(\mathrm{Hg}(A)_{\mathbb{C}}) \simeq \mathrm{sp}(2n, \mathbb{C})$. *Let W be an irreducible $\mathbb{Q}$-sub-Hodge structure of $H^m(A^k, \mathbb{Q})$ and let $(\lambda_1, \ldots, \lambda_n)$ be the Young diagram associated with it. Then $l(W) = \Sigma_{i=1}^n \lambda_i$.*

Remark. The quantity $\Sigma_{i=1}^n \lambda_i$ in this proposition is equal to "the number of boxes" of the Young diagram when it is expressed in the usual way by arranging λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. Hence it is very easy to detect the level from the associated Young diagram.

Proof. Let us put $V = H^1(A, \mathbb{Q})$ and let h denote the Lie algebra of the Hodge group $\mathrm{Hg}(A)$. It follows from the definition of the Hodge group ([14]) that h is the smallest $\mathbb{Q}$-Lie subalgebra of $\mathrm{gl}(V)$ such that $h_{\mathbb{C}}$ contains

$$H_0 = \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix},$$

where E_n denotes the identity matrix of degree n. We call H_0 the *Hodge element of h* as in [loc. cit.]. Here we need some notation. Let $e_1, \ldots, e_n$ denote a basis of $H^0(A, \Omega^1) = H^{1,0}(A) \subset V_{\mathbb{C}}$ and let $e_{-1}, \ldots, e_{-n}$ denote its dual basis under the given Riemann form, so that we have

$$V_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}, \quad \text{where } H^{1,0} = \langle e_1, \ldots, e_n \rangle_{\mathbb{C}},$$
$$H^{0,1} = \langle e_{-1}, \ldots, e_{-n} \rangle_{\mathbb{C}},$$
$$\langle e_i, e_{-j} \rangle = \delta_{ij}.$$

Let $E_{a,b}$ (a, $b = \pm 1, \ldots, \pm n$) denote the $2n$-by-$2n$ matrix such that

$$(E_{a,b})_{k,l} = \begin{cases} 1 & \text{if } (k, l) = (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

Let us put $H_i = E_{i,i} - E_{-i,-i}$ for $1 \leqslant i \leqslant n$ and let $(\varepsilon_i)_{1 \leqslant i \leqslant n}$ be the dual basis of (H_i). Then the fundamental weights are given by

$$\omega_1 = \varepsilon_1,$$
$$\omega_2 = \varepsilon_1 + \varepsilon_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\omega_n = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$$

(see [1, Chapitre VIII, section 13]). In this notation, the Hodge element H_0 introduced above is expressed as

$$H_0 = \sum_{i=1}^{n} H_i.$$

Since $H^m(A^k, \mathbb{C}) \simeq \Lambda^m(H^1(A^k, \mathbb{C})) \simeq \Lambda^m(\oplus^k V_{\mathbb{C}}) \simeq \Lambda^m(\oplus^k(H^{1,0} \oplus H^{0,1}))$, the elements of its (p, q)-component are of the form

$$\sum (v_1 * v_2 * \cdots * v_p) \otimes (w_1 * w_2 * \cdots * w_q),$$

where $v_i \in H^{1,0}$, $w_j \in H^{0,1}$ for any $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ and "$*$" represents "$\otimes$' or "Λ". Hence we see that

$$(H^m(A^k, \mathbb{C}))^{(p,q)} \simeq \left\{ v \in \Lambda^m \left(\bigoplus^k V_{\mathbb{C}} \right); \ H_0 v = (p - q)v \right\}.$$

Therefore, if W is an irreducible $\mathbb{Q}$-sub-Hodge structure of $H^m(A^k, \mathbb{Q})$, then its level $l(W)$ is equal to the maximum of the eigenvalues of H_0 on $W_{\mathbb{C}}$. On the other hand, it follows from Proposition 3.1 that W corresponds to an irreducible representation of $\mathrm{Hg}(A)$, hence $W_{\mathbb{C}}$ is defined by a dominant weight

$$\omega_W = k_1 \omega_1 + \cdots + k_n \omega_n$$

for some non-negative integers $k_1, \ldots, k_n$. Let v_W denote one of its dominant vectors. Then for any H in the Cartan subalgebra of g, we have

$$Hv_W = \omega_W(H)v_W.$$

Moreover we know that the weights of $W_{\mathbb{C}}$ are of the form

$$\omega_W - \sum_{i=1}^{n} p_i\alpha_i,$$

where α_i $(i = 1, \ldots, n)$ are the simple roots of g and p_i $(i = 1, \ldots, n)$ are certain nonnegative integers. Hence we have the following equalities:

$$\begin{aligned}
l(W) &= \{\text{the maximum of the eigenvalues of } H_0 \text{ on } W_{\mathbb{C}}\} \\
&= \max\left\{\left(\omega_W - \sum_{i=1}^{n} p_i\alpha_i\right)(H_0);\ p_1, \ldots, p_n \geqslant 0\right\} \\
&= \omega_W(H_0) \\
&= (k_1\omega_1 + \cdots + k_n\omega_n)\left(\sum_{i=1}^{n} H_i\right) \\
&= ((k_1 + \cdots + k_n)\varepsilon_1 + (k_2 + \cdots + k_n)\varepsilon_2 + \cdots + k_n\varepsilon_n)\left(\sum_{i=1}^{n} H_i\right) \\
&= (k_1 + \cdots + k_n) + (k_2 + \cdots + k_n) + \cdots + k_n \\
&= \{\text{the number of boxes in the Young diagram associated with } W_{\mathbb{C}}\}.
\end{aligned}$$

This completes the proof of Proposition 4.3.

5. Main theorem and its proof

The purpose of this section is to give a proof of the following:

THEOREM 5.1. *Let A be a stably nondegenerate abelian variety all of whose simple components are of type I or II. Then the Generalized Hodge Conjecture holds for A.*

Proof. First, note that, in order to show the validity of the Generalized Hodge Conjecture, it suffices to prove the following statement:

If W is an *irreducible* $\mathbb{Q}$-sub-Hodge structure of $H^m(A, \mathbb{Q})$ of level *equal* to $m - 2p$, then there exists a Zariski closed subset Z of A of codimension

p such that $W \subset \ker(H^m(A, \mathbb{Q}) \to H^m(A - Z, \mathbb{Q}))$.

We divide the proof into three steps.

Case 1: $A = B^k$ $(k \geqslant 1)$, where B is an n-dimensional abelian variety such that $\mathrm{Lie}(\mathrm{Hg}(B)_{\mathbb{C}}) \simeq \mathrm{sp}(2n, \mathbb{C})$ and that $H^1(B, \mathbb{C})$ is isomorphic to $\mathbb{C}^{2n}$ as the representation space of $\mathrm{Lie}(\mathrm{Hg}(B)_{\mathbb{C}}) \simeq \mathrm{sp}(2n, \mathbb{C})$.

In this case, if we put $V = H^1(B, \mathbb{C}) \simeq \mathbb{C}^{2n}$, then $H^m(A, \mathbb{C})$ is decomposed as

$$\bigoplus_{m_1 + \cdots + m_l = m} (\Lambda^{m_1} V \otimes \cdots \otimes \Lambda^{m_l} V)$$

with the natural action of $\mathrm{sp}(2n, \mathbb{C})$. Here we recall the following:

LEMMA 5.1.1 ([1, chapitre VIII, §13]). *Let $V = \mathbb{C}^{2n}$ with the natural action of $g = \mathrm{sp}(2n, \mathbb{C})$. Then, for any $i = 1, \ldots, n$, the i-th exterior product $\Lambda^i V$ is decomposed into irreducible representations as*

$$\omega_i \oplus \omega_{i-2} \oplus \cdots \oplus \omega_{i-2[i/2]}, \tag{1}$$

where the inclusion of ω_a into $\Lambda^i V$ $(a = i-2, i-4, \ldots)$ is defined by taking the exterior product with $\Omega = \Sigma_{j=1}^n e_j \wedge e_{-j}$ $(i-a)/2$ times.

Note that Ω corresponds to a divisor class $D \in H^2(B, \mathbb{C}) \simeq \Lambda^2 H^1(B, \mathbb{C})$ and that taking exterior product with Ω corresponds to intersecting with D. Also note that, in terms of Young diagrams, (1) is expressed as

$$(1^i) \oplus (1^{i-2}) \oplus \cdots \oplus (1^{i-2[i/2]}),$$

where (1^a) means the Young diagram $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 = \cdots = \lambda_a = 1$, $\lambda_{a+1} = \cdots = \lambda_n = 0$. As for the irreducible decomposition of the tensor product $(1^a) \otimes (1^b)$, we know the following:

LEMMA 5.1.2 ([4], [9]). *Let a, b be nonnegative integers with $a \geqslant b$. Then*

$$\begin{aligned}(1^a) \otimes (1^b) = {} & \{(1^{a+b}) \oplus (2, 1^{a+b-2}) \oplus \cdots \oplus (2^b, 1^{a-b})\} \\ & \oplus \{(1^{a+b-2}) \oplus (2, 1^{a+b-4}) \oplus \cdots \oplus (2^{b-1}, 1^{a-b})\} \\ & \oplus \cdots \\ & \oplus \{(1^{a-b})\},\end{aligned}$$

where we understand that the following conventions are assumed:

(1) *If a Young diagram which appears on the right-hand side has more than n rows, then it is discarded.*
(2) $(2^c, 1^d)$ *represents the Young diagram* (λ_i) *with* $\lambda_1 = \cdots = \lambda_c = 2$, $\lambda_{c+1} = \cdots = \lambda_{c+d} = 1$.
(3) (1^0) *represents the trivial representation.*

Note that the reduction of the number of boxes on the right-hand side is caused by the "contraction" of tensors, namely by taking the tensor product with the alternating tensor Ω (see [4]). These lemmas already enable us to show the Generalized Hodge Conjecture for Case 1 as follows. Let W be an irreducible $\mathbb{Q}$-sub-Hodge structure of $H^m(B^k, \mathbb{Q})$ of level $l(W) = m - 2p$. Then by Proposition 4.1, it corresponds to an irreducible component $W_{\mathbb{C}}$ which appears in a direct summand $\Lambda^{m_1} V \otimes \cdots \otimes \Lambda^{m_l} V$ of $H^m(B^k, \mathbb{C})$. Let $D(W)$ denote the Young diagram associated with it. Then by Proposition 4.3, the number of boxes in $D(W)$ is equal to $m - 2p$. But it follows from Lemma 5.1.1 and Lemma 5.1.2 that the reduction of the number of boxes is caused only by taking the exterior (or tensor) products with the alternating tensor Ω p-times. As is noted above, this means that W arises by wedging a sub-Hodge structure of $H^{m-2p}(B^k, \mathbb{Q})$ with D^p. This implies that W is supported on some Zariski closed subset of B^k of codimension p. This completes the proof of the Generalized Hodge Conjecture in Case 1.

Case 2: $A = B_1^{k_1} \times B_2^{k_2}$, where both B_1 and B_2 satisfy the same condition as B in Case 1 and B_1 is not isogenous to B_2.

Put $V_1 = H^1(B_1, \mathbb{C})$ and $V_2 = H^1(B_2, \mathbb{C})$. In this case it follows from [6, §3] that the Hodge group $\mathrm{Hg}(A)$ is isomorphic to $\mathrm{Hg}(B_1) \times \mathrm{Hg}(B_2)$ and that the representation of $\mathrm{Hg}(A)_{\mathbb{C}}$ in $H^1(A, \mathbb{C}) \simeq (\oplus^{k_1} V_1) \oplus (\oplus^{k_2} V_2)$ is equivalent to the direct sum of the diagonal action of $\mathrm{Hg}(B_1)_{\mathbb{C}}$ on $\oplus^{k_1} V_1$ and that of $\mathrm{Hg}(B_2)_{\mathbb{C}}$ on $\oplus^{k_2} V_2$. Let us put $g_i = \mathrm{Lie}(\mathrm{Hg}(B_i)_{\mathbb{C}})$ $(i = 1, 2)$ and $g = \mathrm{Lie}(\mathrm{Hg}(A)_{\mathbb{C}}) \simeq g_1 \times g_2$. Then it follows from Proposition 4.1 that an irreducible $\mathbb{Q}$-sub-Hodge structure of $H^m(A, \mathbb{Q})$ gives rise to an irreducible representation of $g = g_1 \times g_2$. The latter must be of the form $W_1 \otimes W_2$, where $W_i \subset H^{m_i}(B_i^{k_i}, \mathbb{C})$ is an irreducible representation of g_i $(i = 1, 2)$ for some pair (m_1, m_2) of nonnegative integers with $m_1 + m_2 = m$. Further its level is equal to the sum of the numbers of boxes in $D(W_1)$ and $D(W_2)$. (Note that this follows from a similar argument as in the proof of Proposition 4.3.) But as we have already seen in the proof for Case 1, there exist nonnegative integers p_1, p_2 such that $\{$the number of boxes in $D(W_i)\} = m_i - 2p_i$ for $i = 1, 2$. Moreover it follows that each W_i is supported on a Zariski closed subset Z_i of $B_i^{k_i}$ of codimension p_i. Hence $W_1 \otimes W_2$ is supported on the Zariski closed subset $Z_1 \times Z_2$ of

$B_1^{k_1} \times B_2^{k_2} = A$ of codimension $p_1 + p_2$. This completes the proof of the Generalized Hodge Conjecture for Case 2.

Case 3: A is an arbitrary abelian variety which satisfies the conditions in Theorem 5.1.

In this case, it follows from [6, §3] that $\mathrm{Lie}(\mathrm{Hg}(A)_{\mathbb{C}})$ is isomorphic to a product $sp(2n_1, \mathbb{C}) \times \cdots \times sp(2n_l, \mathbb{C})$ for some positive integers $n_1, \ldots, n_l$ and that $H^1(A, \mathbb{C})$ is, as the representation space of $\mathrm{Lie}(\mathrm{Hg}(A)_{\mathbb{C}})$, equivalent to the direct sum $(\oplus^{k_1} V_1) \oplus \cdots \oplus (\oplus^{k_l} V_l)$ *of the diagonal representation of* $sp(2n_i, \mathbb{C})$ into $\oplus^{k_i} V_i$ with $V_i \simeq \mathbb{C}^{2n_i}$. (The latter fact is a direct consequence of a theorem of Deligne on microweights. See [2], [13, Theorem 0.5.1].) In each V_i, the fundamental alternating form $\Omega_i \in \Lambda^2 V_i$ is invariant under the action of $sp(2n_i, \mathbb{C})$, hence it corresponds to a linear combination of divisor classes with complex coefficients on A by the Lefschetz theorem. Therefore we see that the Generalized Hodge Conjecture holds for A, by a similar argument as in Case 2. This completes the proof of Theorem 5.1.

References

1. N. Bourbaki, *Groupes et Algèbres de Lie*, chaps. VII–VIII, Hermann, Paris, 1975.
2. P. Deligne, "Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques," Automorphic forms, Representations and *L*-functions, *Proc. Sympos. Pure Math.* vol. 33, part 2, Amer. Math. Soc., Providence, R.I. 1979, pp. 247–289.
3. P. Deligne, "Hodge cycles on abelian varieties," Hodge Cycles, Motives, and Shimura Varieties, Lect. Notes in Math. 900, Springer-Verlag, Berlin-Heidelberg, 1982, pp. 9–100.
4. M. Fischler, Young-tableau methods for Kronecker products of representations of the classical groups, *J. Math. Phys.* 22 (1981), 637–648.
5. A. Grothendieck, Hodge's general conjecture is false for trivial reasons, *Topology* 8 (1969), 299–303.
6. F. Hazama, Algebraic cycles on certain abelian varieties and powers of special surfaces, *J. Fac. Sci. Univ. Tokyo* 31 (1985), 487–520.
7. F. Hazama, Algebraic cycles on nonsimple abelian varieties, *Duke Math. J.* 58 (1989), 31–37.
8. D. Mumford, *Abelian varieties*, Tata Inst. and Oxford Univ. Press, 1970.
9. A. L. Onishchik, E. B. Vinberg, *Lie groups and algebraic groups*, Springer-Verlag, Berlin-Heidelberg, 1990.
10. J.-P. Serre, *Résumè des cours de 1984–85*, Collège de France.
11. T. Shioda, "What is known about the Hodge conjecture?" Algebraic varieties and analytic varieties, *Advanced Studies in Pure Mathematics* 1, Kinokuniya, Tokyo, 1983, pp. 55–68.
12. J. H. M. Steenbrink, "Some remarks about the Hodge conjecture," Hodge Theory, *Lect. Notes in Math.* 1246, Springer-Verlag, Berlin-Heidelberg, 1987, pp. 165–175.
13. Y. G. Zarkhin, Weights of simple Lie algebras in the cohomology of algebraic varieties, *Math. USSR, Izv.*, 24 (1985), 245–281.
14. Y. G. Zarkhin, "Linear irreducible Lie algebras and Hodge structures," Algebraic geometry, *Lect. Notes in Math.* 1479, Springer-Verlag, Berlin-Heidelberg, 1991, pp. 281–297.