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1. Introduction

Let $G$ be a connected reductive $p$-adic group, and let $\mathcal{V}(G)$ denote the set of tempered virtual characters of $G$, that is the set of finite linear combinations of characters of irreducible, tempered representations of $G$. Every $\Theta \in \mathcal{V}(G)$ satisfies the following weak estimate. Given any Cartan subgroup $T$ of $G$ there is a positive constant $r$ so that

$$\sup_{t \in T} |D_G(t)|^{1/2} |\Theta(t)|(1 + \sigma(t))^{-r} < \infty.$$  

We say that $\Theta$ is supertempered if it satisfies the following strong estimate. For every Cartan subgroup $T$ of $G$ and every positive constant $r$ we have

$$\sup_{t \in T} |D_G(t)|^{1/2} |\Theta(t)|(1 + \sigma_*(t))^r < \infty.$$  

(In the above, $D_G$ is the usual discriminant factor [S, 4.7], $\sigma$ measures polynomial growth on $G$, and $\sigma_*$ measures polynomial growth on $G/Z_G$ [S, 4.1]). If $\Theta$ is the character of an irreducible tempered representation $\pi$, then $\Theta$ is supertempered if and only if $\pi$ is a discrete series representation. In [A], Arthur singled out a set of tempered virtual characters which he conjectured spanned the space of supertempered virtual characters. In this paper we will show that his conjecture is correct.

More precisely, let $P = MN$ be a parabolic subgroup of $G$ and let $\sigma$ be a discrete series representation of $M$. Let $i_{G,M}(\sigma)$ denote the equivalence class of the tempered representation of $G$ parabolically induced from $\sigma$ and let $R$ be the corresponding $R$-group. It is a finite group with the property

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that the commuting algebra of $i_{G,M}(\sigma)$ is naturally isomorphic to $C[R]^\eta$, the complex group algebra of $R$ with multiplication twisted by a cocycle $\eta$. Let $\tilde{R}$ be a central extension

$$1 \to Z \to \tilde{R} \to R \to 1$$

of the $R$-group as in [A]. (When $\eta$ is trivial, we can take $\tilde{R} = R$.) Then there is a character $\chi$ of $Z$ so that the irreducible constituents of $i_{G,M}(\sigma)$ are parameterized by $\Pi(\tilde{R}, \chi)$, the set of equivalence classes of irreducible representations $\rho$ of $\tilde{R}$ with $Z$-character $\chi$. For $\rho \in \Pi(\tilde{R}, \chi)$, let $\pi_{\rho}$ denote the corresponding irreducible constituent of $i_{G,M}(\sigma)$ and let $\Theta_{\rho}$ denote its character. The representation $\pi_{\rho}$ is called elliptic if $\Theta_{\rho}$ is not identically zero on the elliptic set of $G$.

Let $a, z$ denote the real Lie algebras of the split components of $M, G$ respectively. Then $R$ acts on $a$ and for each $r \in R$ we set

$$a_r = \{H \in a: rH = H\}.$$ 

Define

$$R_{\text{reg}} = \{r \in R: a_r = z\}$$

and let $\tilde{R}_{\text{reg}}$ denote the inverse image of $R_{\text{reg}}$ in $\tilde{R}$. Arthur proves in [A, 2.1] that for $\rho \in \Pi(\tilde{R}, \chi)$, $\pi_{\rho}$ is elliptic if and only if the character of $\rho$ does not vanish on $\tilde{R}_{\text{reg}}$. (In particular, for $i_{G,M}(\sigma)$ to have any elliptic constituents it is necessary that $\tilde{R}_{\text{reg}} \neq \emptyset$.) For each $r \in \tilde{R}$, define

$$\Theta(M, \sigma, r) = \sum_{\rho \in \Pi(\tilde{R}, \chi)} \overline{\text{tr}(\rho(r))} \Theta_{\rho}.$$ 

We will prove in Section 3 that, as predicted by Arthur in the introduction to [A], $\Theta(M, \sigma, r)$ is supertempered if $r \in \tilde{R}_{\text{reg}}$. We also prove that every supertempered virtual character is a linear combination of ones of this form.

The method of proof relies on ideas of Harish-Chandra [HC2]. For every $\Theta \in \mathcal{Y}(G)$ and parabolic subgroup $P = MN$, we will define a weak constant term $\Theta_P^w \in \mathcal{Y}(M)$. In the case that $\Theta$ is the character of an irreducible tempered representation $(\pi, V)$ of $G$, then $\Theta_P^w$ is just the (normalized) character of the maximal tempered quotient of the Jacquet module $V/V(\tilde{N})$. In Section 2 we show that $\Theta \in \mathcal{Y}(G)$ is supertempered if and only if $\Theta_P^w = 0$ for all $P \neq G$.

In Section 3 we use the $R$-group machinery developed by Arthur in [A, §2] and the Geometrical Lemma of Bernstein and Zelevinsky [B-Z] to compute the weak constant terms of the elliptic virtual characters $\Theta(M, \sigma, r), r \in \tilde{R}_{\text{reg}}$, and show they are zero. We also prove that any $\Theta \in \mathcal{Y}(G)$ which is supertem-
pered and zero on the elliptic set of $G$ must be zero. This allows us to show that every supertempered virtual character is a linear combination of ones of the form $\Theta(M, \sigma, r), \ r \in \tilde{R}_{\text{reg}}$.

2. Constant terms

Let $F$ be a locally compact, non-discrete, nonarchimedean field of characteristic zero. Let $G$ be the $F$-rational points of a connected, reductive algebraic group over $F$. In this section we will define constant terms and weak constant terms of tempered virtual characters and prove that a tempered virtual character is supertempered if and only if all of its weak constant terms vanish. We use Silberger's book [S] as a convenient reference for Harish-Chandra's theory of constant terms of matrix coefficients. We note however, that it must be used with care since there is an error in the definition of the weak constant term.

For any admissible representation $\pi$ of $G$, let $\mathcal{A}(\pi)$ denote the set of all finite linear combinations of matrix coefficients of $\pi$. Set $\mathcal{A}(G) = \cup \mathcal{A}(\pi)$ where the union is over all admissible representations $\pi$ of $G$. Given any $f \in \mathcal{A}(G)$ and parabolic subgroup $P = MN$ of $G$, there is a constant term $f_P \in \mathcal{A}(M)$ with the property that given $m \in M$, there is $t > 0$ so that

$$\delta_P(ma)^{1/2} f(ma) = f_P(ma)$$

for all $a \in A^+(t)$. Here $\delta_P$ is the modular function of $P$, $A$ is the split component of $M$, and $A^+(t)$ is the set of all $a \in A$ such that $|\alpha(a)| \geq t$ for every simple root $\alpha$ of $P$ with respect to $A$ [S, 2.6].

Suppose that $(\pi, V)$ is an admissible representation of $G$ and that $P = MN$ is a parabolic subgroup. Write $\tilde{P} = M \tilde{N}$ for the opposite parabolic subgroup. As usual we let $V(\tilde{N})$ be the subspace of $V$ generated by elements of the form $\pi(\tilde{n})v - v, \ \tilde{n} \in \tilde{N}, \ v \in V$, and define $V_{\tilde{N}} = V/V(\tilde{N})$. Let $p: V \to V/V(\tilde{N})$ be the projection map, and for $m \in M, v \in V$, let

$$\pi_{\tilde{N}}(m)p(v) = \delta_P(m)^{-1/2} p(\pi(m)v).$$

Then $(\pi_{\tilde{N}}, V_{\tilde{N}})$ is called the normalized Jacquet module of $(\pi, V)$ corresponding to $\tilde{P}$. It is well known to be an admissible representation of $M$ [S, 2.3.6].

Let $(\tilde{\pi}, \tilde{V})$ denote the contragredient of $(\pi, V)$. For any $\tilde{v} \in \tilde{V}, \ v \in V$, define the matrix coefficient

$$\phi_{r,e}(x) = \langle \tilde{v}, \pi(x)v \rangle, \ x \in G.$$

As in [Ca2, 4.2], the dual of $V_{\tilde{N}}$ is $\tilde{V}_{\tilde{N}}$ with the pairing
\langle p(\tilde{v}), p(v) \rangle = (\phi_{\tilde{v}, v})_P(1), \quad \tilde{v} \in \tilde{V}, \quad v \in V.

(Here \( \phi_{\tilde{v}, v} \in \mathcal{A}(G) \) and \( (\phi_{\tilde{v}, v})_P \in \mathcal{A}(M) \) denotes its constant term as above.) For \( \tilde{v} \in \tilde{V}, \quad v \in V \), define the matrix coefficient

\[ \psi_{\pi(\tilde{v}), \pi(v)}(m) = \langle p(\tilde{v}), \pi(m)p(v) \rangle, \quad m \in M. \]

Then it is easy to check using [S, 2.7.1] that

\[ \psi_{\pi(\tilde{v}), \pi(v)}(m) = (\phi_{\tilde{v}, v})_P(m) \]

for all \( m \in M, \quad \tilde{v} \in \tilde{V}, \quad v \in V \). Thus we have

\[ \mathcal{A}(\pi_\tilde{v}) = \{ f_P : f \in \mathcal{A}(\pi) \}. \]

Suppose that \( \Theta = \Theta_{\pi} \) is the character of \( \pi \). Then define \( \Theta_P = \Theta_{\pi_\tilde{v}} \), the character of \( \pi_\tilde{v} \), and call \( \Theta_P \) the constant term of \( \Theta \) along \( P \). Let \( G' \) denote the set of regular semisimple elements of \( G \), \( M' = M \cap G' \).

**Lemma 2.1.** Given any \( m \in M' \), there is \( t > 0 \) so that

\[ \delta_p(ma)^{1/2} \Theta(ma) = \Theta_P(ma) \]

for all \( a \in A^+(t) \). Further, \( a \mapsto \Theta_P(ma) \) is the only \( A \)-finite function with this property.

**Proof.** The equality is a rephrasing of Casselman's theorem [Ca1, 5.2]. The uniqueness follows from [S, 2.6.2]. \( \square \)

Let \( f \in \mathcal{A}(G) \) and let \( P \) be a parabolic subgroup of \( G \). Then as in [S, 3.1] we can write

\[ f_P = \sum_{\chi} f_{P, \chi} \]

where the \( \chi \) are quasi-characters of \( A \), and define

\[ X_f(P, A) = \{ \chi : f_{P, \chi} \neq 0 \}. \]

Now if \( \pi \) is an admissible representation of \( G \) we set

\[ X_\pi(P, A) = \bigcup_{f \in \mathcal{A}(\pi)} X_f(P, A). \]

As in [S, 3.3.1], we can decompose \( V_\tilde{\pi} \) as an \( M \)-module direct sum

\[ V_\tilde{\pi} = \sum_{z \in X_\pi(P, A)} V_{\tilde{\pi}, z}. \]

Let \( \Theta_{P, \chi} \) denote the character of the restriction of \( \pi_\tilde{v} \) to \( V_{\tilde{\pi}, z} \). Then

\[ \Theta_P = \sum_{\chi \in X_\pi(P, A)} \Theta_{P, \chi} \]
and

$$\Theta_{P,\chi}(ma) = \chi(a)\Theta_{P,\chi}(m)$$

for all $m \in M$, $a \in A$.

Given parabolic pairs $(P, A)$ and $(P_1, A_1)$ we write $(P, A) \prec (P_1, A_1)$ if $P \subset P_1$ and $A \subset A_1$. In this case, if $P_1 = M_1 N_1$, we write $P^* = P \cap M_1$. Given $\chi_1 \in \chi_*(P_1, A_1)$, let $X_\chi(P, A, \chi_1) = \{\chi \in X_\chi(P, A) : |\chi|_{A_1} = |\chi_1|\}$. The following lemma is an easy consequence of the definition of the constant terms as characters of Jacquet modules.

**LEMMA 2.2.** Suppose that $(P, A) \prec (P_1, A_1)$. Then

(i) $\Theta_P = (\Theta_{P_1})_{P^*};$

(ii) For all $\chi_1 \in X_\chi(P_1, A_1)$,

$$(\Theta_{P,\chi_1})_{P^*} = \sum_{\chi \in X_\chi(P, A, \chi_1)} \Theta_{P,\chi}.$$

For any continuous function $f$ on $G$, we say $f$ satisfies the weak inequality if there exists a positive constant $r$ so that

$$\sup_{x \in G} |f(x)| \Xi(x)^{-1}(1 + \sigma(x))^{-r} < \infty.$$ 

Here $\sigma$ is the usual polynomial growth factor defined in [S, 4.1] and $\Xi$ is the usual spherical function defined in [S, 4.2]. Let $\mathcal{A}_T(G)$ denote the set of functions in $A(G)$ which satisfy the weak inequality.

Fix a parabolic pair $(P, A)$ and let $a$ be a variable element of $A$. Following Harish-Chandra [HC1, Section 21], we say that $a \rightarrow_p \infty$ if there exists a number $\epsilon > 0$ so that (1) $\log_a |a(a)| \geq \epsilon \sigma(a)$ and (2) $|a(a)| \rightarrow \infty$ for every root $\alpha$ of $(P, A)$. For any $f \in \mathcal{A}_T(G)$ there is a weak constant term $f^w_p \in \mathcal{A}_T(M)$ defined as in [S, 4.5.5]. It is the unique element of $\mathcal{A}_T(M)$ such that

$$\lim_{a \rightarrow_p \infty} |\delta_p(ma)^{1/2} f(ma) - f^w_p(ma)| = 0$$

for every $m \in M$. Note Silberger’s definition of $a \rightarrow_p \infty$ in [S, p. 101] does not have property (1). This is needed for the validity of [S, 4.5.5]. For $f \in \mathcal{A}_T(G)$, set $X_f(P, A) = X_f(P, A) \cap A$. Then by [S, 4.5.5],

$$f^w_p = \sum_{\chi \in X_f(P, A)} f_{P,\chi}.$$

Now let $\pi$ be an irreducible tempered representation of $G$ and let $P = MN$
be a parabolic subgroup of $G$. Decompose the Jacquet module

$$V_{\tilde{N}} = \sum_{\chi \in \text{Jac}(P, A)} V_{\tilde{N}, \chi}.$$ 

Define $X^w_{\pi}(P, A) = X_{\pi}(P, A) \cap \hat{A}$. Then as in [S, 5.4.1.3], the maximal tempered quotient of $\pi_{\tilde{N}}$ is

$$(V_{\tilde{N}})^w = \sum_{\chi \in X^w_{\pi}(P, A)} V_{\tilde{N}, \chi}.$$ 

If $\Theta = \Theta_{\pi}$, we define $\Theta^w = \Theta_{(V_{\tilde{N}})^w}$, the character of $(V_{\tilde{N}})^w$. Thus we have

$$\Theta^w = \sum_{\chi \in X^w_{\pi}(P, A)} \Theta_{P, \chi}.$$ 

Let $\phi: A \to \mathbb{C}$ be an $A$-finite function. Then we can write $\phi = \sum_{\chi} \phi_{\chi}$. We say a quasi-character $\chi$ of $A$ is an exponent of $\phi$ if $\phi_{\chi} \neq 0$, and say $\phi$ is a tempered $A$-finite function if all of its exponents are unitary. Fix $m \in M'$. Then $a \mapsto \Theta^w_P(ma)$ is a tempered $A$-finite function.

**Lemma 2.3.** Let $m \in M'$. Then

$$\lim_{a \to \infty} |\delta_P^{1/2}(ma)\Theta(ma) - \Theta^w_P(ma)| = 0.$$ 

Further, $a \mapsto \Theta^w_P(ma)$ is the only tempered $A$-finite function with this property.

*Proof.* Using Lemma 2.1 it is enough to show that

$$\lim_{a \to \infty} |\Theta_P(ma) - \Theta^w_P(ma)| = 0.$$ 

Write $X^w_{\pi}(P, A) = X_{\pi}(P, A) \cap X^w_{\pi}(P, A)^\circ$. Then

$$|\Theta_P(ma) - \Theta^w_P(ma)| = \left| \sum_{\chi \in X^w_{\pi}(P, A)} \Theta_{P, \chi}(ma) \right| \leq \sum_{\chi \in X^w_{\pi}(P, A)} |\chi(a)\Theta_{P, \chi}(m)|.$$ 

But by [S, 4.5.3], for all $\chi \in X^w_{\pi}(P, A)$, we have $\lim_{a \to \infty} |\chi(a)| = 0$. The uniqueness follows from [S, 4.1.6].

Let $\mathcal{E}_1(G)$ denote the set of equivalence classes of irreducible tempered representations of $G$. For $\pi \in \mathcal{E}_1(G)$, write $\Theta_{\pi}$ for the character of $\pi$. We will say that $\Theta$ is a tempered virtual character of $G$, and write $\Theta \in \mathcal{V}'(G)$, if there are
\[ \pi_1, \ldots, \pi_k \in \mathcal{E}_k(G), \quad c_1, \ldots, c_k \in \mathbb{C}, \] 

such that 

\[ \Theta = \sum_{i=1}^{k} c_i \Theta_{\pi_i}. \]

For any \( \Theta = \sum_i c_i \Theta_{\pi_i} \in \mathcal{V}(G) \) we can define constant terms \( \Theta_P \) and \( \Theta_P^\wedge \) by

\[ \Theta_P = \sum_i c_i (\Theta_{\pi_i})_P, \quad \Theta_P^\wedge = \sum_i c_i (\Theta_{\pi_i})_P^\wedge. \]

Define \( X_\phi(P, A) \) to be the set of all quasicharacters \( \chi \) of \( A \) such that \( \Theta_{P, \chi} \neq 0 \).

Let \( (P, A) \) be a parabolic pair with simple roots \( \alpha_1, \ldots, \alpha_l \). Let \( A^+ \) denote the positive chamber of \( A \) with respect to \( P \) and let \( a \) denote the real Lie algebra of \( A \). Finally, let \( \chi \) be a quasicharacter of \( A \) which is unitary when restricted to \( Z \), the split component of the center of \( G \). Then as in [S, 4.5.10], we can define

\[ \gamma = -\sum_{i=1}^l c_i \alpha_i \in a^* \]

by setting

\[ |\chi(a)| = q^{\langle \gamma, H(a) \rangle}, \quad a \in A. \]

We will say that \( \chi \) is rapidly decreasing on \( A^+ \) if \( c_i > 0 \) for all \( 1 \leq i \leq l \). For \( x \in G \) define

\[ \sigma_\phi(x) = \inf_{z \in \mathbb{Z}} \sigma(xz). \]

Then \( \chi \) is rapidly decreasing on \( A^+ \) implies that for every \( t > 0 \) we have

\[ \sup_{a \in A^+} |\chi(a)|(1 + \sigma_\phi(a))^t < \infty. \]

**Lemma 2.4.** Let \( \Theta \in \mathcal{V}(G) \). The following are equivalent.

(i) \( \Theta_P^\wedge = 0 \) for all \( P \neq G \);

(ii) For every parabolic pair \( (P, A) \), every \( \chi \in X_\phi(P, A) \) is rapidly decreasing on \( A^+ \).

**Proof.** Let \( \Theta \in \mathcal{V}(G) \). Using [S, 4.5.2, 4.5.3], we see that for every \( (P, A) \) and \( \chi \in X_\phi(P, A) \) we have \( |\chi(a)| \leq 1 \) for all \( a \in A^+ \). Thus if we define \( \gamma = -\sum_{i=1}^l c_i \alpha_i \) as above associated to \( \chi \), we have \( c_i \geq 0 \) for all \( i \). Further, \( \Theta_P^\wedge \) is the sum of the \( \Theta_{P, \chi} \) where \( \chi \) runs over the unitary characters \( \chi \in X_\phi(P, A) \), that is the \( \chi \) for which \( c_i = 0 \) for all \( i \). Thus (ii) clearly implies (i).

Now assume that \( \Theta_P^\wedge = 0 \) for all \( P \neq G \). Fix a parabolic pair \( (P, A) \) and \( \chi \in X_\phi(P, A) \). If \( (P, A) = (G, Z) \) there is nothing to check. Assume that \( (P, A) \) is proper. Define the constants \( c_i \geq 0 \) associated to \( \chi \) as above. Suppose that
Let \( (P_1, A_1) \) be a proper parabolic pair such that \( (P, A) \neq (P_1, A_1) \) and \( \langle a_i, H_p(a) \rangle = 0 \) for all \( a \in A_1, 2 \leq i \leq l \). Note that if \( l = 1 \) we take \( P_1 = P \). Let \( \chi_1 \) be the restriction of \( \chi \) to \( A_1 \). Then \( \chi_1 \) is unitary since \( c_1 = 0 \). Since \( P_1 \neq G \), we have \( \Theta_{P, z_i} = 0 \) so that \( \Theta_{P, z_i} = 0 \) for every unitary character \( \chi_1 \) of \( A_1 \). Now using Lemma 2.2 we see that \( \Theta_{P, z'} = 0 \) for every quasicharacter \( \chi' \) of \( A \) such that the restriction of \( \chi' \) to \( A_1 \) is \( \chi_1 \). But this contradicts the assumption that \( \chi \in X_{\Theta}(P, A) \).

Let \( \pi \) be an irreducible, admissible representation of \( G \). By [Cl, 3.4] we know that \( \pi \) is tempered if and only if given any Cartan subgroup \( T \) of \( G \) there is a positive constant \( r \) so that

\[
\sup_{t \in T} |D_G(t)|^{1/2} |\Theta_{\pi}(t)(1 + \sigma(t))^{-r}| < \infty.
\]

Here \( D_G \) is the standard discriminant factor defined in [S, 4.7]. Let \( \Theta \in \mathcal{Y}(G) \). Then we say that \( \Theta \) is supertempered if and only if for every Cartan subgroup \( T \) of \( G \) and every positive constant \( r \) we have

\[
\sup_{t \in T} |D_G(t)|^{1/2} |\Theta(t)(1 + \sigma_{\ast}(t))^{r}| < \infty.
\]

**Theorem 2.5.** Let \( \Theta \in \mathcal{Y}(G) \). Then \( \Theta \) is supertempered if and only if \( \Theta_{\ast} = 0 \) for all \( P \neq G \).

**Proof.** Suppose that \( \Theta_{\ast} = 0 \) for \( P \neq G \). We will use the argument of Clozel in [Cl, 3.4] to show that \( \Theta \) is supertempered. Let \( T \) be a Cartan subgroup of \( G \). As in [Cl, 3.4] we can write \( T \) as a finite union of subsets \( T(M^\pm_e) \) where \( P = MN \) runs over a set of representatives for conjugacy classes of parabolic subgroups of \( G \). Here \( T(M^\pm_e) = 0 \) unless \( T \subset M \) modulo conjugation. Thus we assume that the \( P \) are chosen so that \( T(M^\pm_e) = 0 \) unless \( T \subset M \). For regular \( t \in T(M^\pm_e) \) we have

\[
|D_G(t)|^{1/2} \Theta(t) = |D_M(t)|^{1/2} \Theta_P(t).
\]

If \( P = M = G \), then \( T(G^\pm) \) is compact modulo center so that we know that

\[
\sup_{t \in T(G^\pm) \cap G} |D_G(t)|^{1/2} |\Theta(t)(1 + \sigma_{\ast}(t))^{r}| < \infty.
\]

Now suppose that \( P \neq G \). Now \( T(M^\pm_e) \) can be written as a finite union of sets of the form \( A^\dagger T_e t_i \), where \( T_e \) is the maximal compact subgroup of \( T \) and \( t_i \in T \). Now for \( t = act_i \in A^\dagger T_e t_i \) we can write
Now since $T_c$ is compact,

$$\sup_{\alpha \in T_c} |D_M(\alpha t_i)|^{1/2} |\Theta_{P}(\alpha t_i)|(1 + \sigma_*(\alpha t_i))^r < \infty$$

for all $\chi$. But using Lemma 2.4, every $\chi \in X_{\omega}(P, A)$ is rapidly decreasing on $A^+$ so that

$$\sup_{\alpha \in A^+} |\chi(\alpha) (1 + \sigma_*(\alpha))^r < \infty$$

for any $r > 0$.

Conversely, suppose that $\Theta$ is supertempered. Let $P = MN \neq G$ be a parabolic subgroup of $G$ and fix $m \in M'$. Let $T = Z_G(m)$. Then for all $a \in A$, $ma \in T$ so that for any $r > 0$ there is $C_r > 0$ so that

$$|D_G(ma)|^{1/2} |\Theta(ma)| < C_r (1 + \sigma_*(ma))^{-r}$$

for all $a \in A$ with $ma \in T'$. Thus

$$\lim_{a \to \infty} \delta_\pi(ma)^{1/2} \Theta(ma) = 0.$$

Thus by Lemma 2.3, $\Theta_\pi(ma) = 0$ for all $a \in A$.

**COROLLARY 2.6.** Let $\pi \in S_\xi(G)$. Then $\Theta_\pi$ is supertempered if and only if $\pi$ is a discrete series representation.

**Proof.** Let $\pi \in S_\xi(G)$ and write $\Theta = \Theta_\pi$. Then for any parabolic pair $(P, A)$, we have $X_\Theta(P, A) = \bigcup_{f \in \mathcal{A}(\pi)} X_f(P, A)$. Suppose that $\pi$ is a discrete series representation. Then using [S, 4.5.10], for every $f \in \mathcal{A}(\pi)$, every $\chi \in X_f(P, A)$ is rapidly decreasing on $A^+$. Thus using Lemma 2.6 and Theorem 2.7 we see that $\Theta$ is supertempered. Conversely, if $\Theta$ is supertempered, again using Lemma 2.6, Theorem 2.7, and [S, 4.5.10], we see that every $f \in \mathcal{A}(\pi)$ is square integrable mod center so that $\pi$ is discrete series.

**3. Supertempered characters**

Suppose that $P = MN$ is any parabolic subgroup of $G$ and let $A$ be the split component of $M$. Let $W(G/A) = N_G(A)/M$ be the Weyl group of $A$. Elements
of \( W(G/A) \) normalize \( M \) and act on representations of \( M \). Let \( \sigma \in \mathcal{S}_2(M) \) and define \( W_\sigma = \{ w \in W(G/A) : w\sigma \simeq \sigma \} \). As in [A, §2], corresponding to each \( w \in W_\sigma \) there is an intertwining operator for the representation \( \text{Ind}_G^G(\sigma \otimes 1) \) of \( G \) unitarily induced from \( \sigma \). Write \( W_0^0 \) for the subgroup of all \( w \in W_\sigma \) such that the corresponding intertwining operator is scalar. Let \( \Sigma_0 \) be the set of reduced roots \( \alpha \) of \( (G, A) \) such that the corresponding reflection \( w_\alpha \in W_0^0 \). Then it is known that \( \Sigma_0 \) is a root system. Let \( \Delta_0 \) be the set of simple roots for a choice of positive roots in \( \Sigma_0 \) and define \( R_\sigma = \{ w \in W_\sigma : w\Delta_0 = \Delta_0 \} \). Then \( W_\sigma \) is the semidirect product of \( W_0^0 \) and \( R_\sigma \) and \( R = R_\sigma \) is the \( R \)-group for \( \text{Ind}_G^G(\sigma \otimes 1) \).

\( R \) has the property that the commuting algebra \( C(\sigma) \) of the induced representation is naturally isomorphic to the complex group algebra \( C[R] \) with multiplication twisted by a cocycle \( \eta \). Fix a finite central extension

\[ 1 \to Z \to \tilde{R} \to R \to 1 \]

going over which the cocycle \( \eta \) splits and a character \( \chi \) of \( Z \) as in [A, §2]. Then the irreducible constituents of \( \text{Ind}_G^G(\sigma \otimes 1) \) are naturally parameterized by \( \Pi(\tilde{R}, \chi) \), the set of equivalence classes of irreducible representations of \( \tilde{R} \) whose central character on \( Z \) is \( \chi \). For each \( \rho \in \Pi(\tilde{R}, \chi) \) we will write \( \pi_\rho \) for the corresponding irreducible constituent of \( \text{Ind}_G^G(\sigma \otimes 1) \) and \( \Theta_\rho \) for its character. Corresponding to each \( r \in \tilde{R} \) we can define a virtual character as in [A, 2.3] by

\[ \Theta(M, \sigma, r) = \sum_{\rho \in \Pi(\tilde{R}, \chi)} \frac{\text{tr}(\rho(r))}{\text{tr}(\rho(1))} \Theta_\rho. \]

Let \( \mathfrak{a} \) denote the real Lie algebra of \( A \) and let \( \mathfrak{z} \) denote the real Lie algebra of \( Z \), the split component of \( G \). For each \( r \in R \), let

\[ a_r = \{ H \in \mathfrak{a} : rH = H \}. \]

Set

\[ R_{\text{reg}} = \{ r \in R : a_r = \mathfrak{z} \} \]

and let \( \tilde{R}_{\text{reg}} \) denote the inverse image of \( R_{\text{reg}} \) in \( \tilde{R} \). If \( r \in \tilde{R}_{\text{reg}} \), we say \( (M, \sigma, r) \) is an elliptic triple. Arthur says in the introduction of [A] that when \( (M, \sigma, r) \) is an elliptic triple, then the virtual character \( \Theta(M, \sigma, r) \) should be super tempered, and that virtual characters of this type should span the set of super tempered virtual characters. In this section we will prove the following theorems.

**THEOREM 3.1.** For every elliptic triple \( (M, \sigma, r) \), the virtual character \( \Theta(M, \sigma, r) \) is super tempered. Conversely, given \( \Theta \in \mathcal{Y}(G) \) super tempered, there are finitely many elliptic triples \( (M_1, \sigma_1, r_1) \) and complex numbers \( c_i \) so that
$\Theta = \sum_i c_i \Theta(M_i, \sigma_i, r_i)$.

**Theorem 3.2.** Suppose that $\Theta \in \mathcal{Y}(G)$ is super tempered and that $\Theta$ is zero on the elliptic set of $G$. Then $\Theta = 0$.

In order to prove these two theorems we will first need some lemmas. We first recall a result of Bernstein and Zelevinsky [B-Z] on Jacquet modules of induced representations. Fix a minimal parabolic subgroup $P_0 = M_0 N_0$ of $G$ and let $A_0$ be the split component of $M_0$. Let $\mathcal{L}(G)$ denote the finite set of Levi subgroups of parabolic subgroups $P$ of $G$ containing $P_0$. For each $M \in \mathcal{L}(G)$, set $\mathcal{L}(M) = \{M' \in \mathcal{L}(G) : M' \subset M\}$ and let $W_{M} = N_{M}(A_0)/M_0$. Each $M \in \mathcal{L}(G)$ is the Levi component of a unique parabolic subgroup $P_{M} = P_0 M$ containing $P_0$.

Let $M \in \mathcal{L}(G)$. Given any admissible representation $\tau$ of $M$ we write $i_{G,M}(\tau)$ for the equivalence class of the admissible representation $\text{Ind}_{P_0}^{G}(\tau \otimes 1)$ of $G$. If $\Theta_{\tau}$ is the character of $\tau$, we also write $i_{G,M}(\Theta_{\tau})$ for the character of $i_{G,M}(\tau)$. Given any admissible representation $\pi$ of $G$, we write $r_{M,G}(\pi)$ for the equivalence class of the admissible representation $\pi_{M}$ of $M$ where as in Section 2, $\pi_{M}$ denotes the normalized Jacquet module of $\pi$ corresponding to $P_{M} = MN$. If $\Theta_{\pi}$ is the character of $\pi$, we also write $r_{M,G}(\Theta_{\pi})$ for the character of $r_{M,G}(\pi)$. It is the constant term $(\Theta_{\pi})_{P_0}$ of $\Theta_{\pi}$ with respect to $P_{M}$. We will also write $(\Theta_{\pi})_{M} = (\Theta_{\pi})_{P_0}$ and $(\Theta_{\pi})_{M} = (\Theta_{\pi})_{P_0}$.

Given $M$, $L \in \mathcal{L}(G)$, set

$$W_{M,L} = \{w \in W_{G} : w(M \cap P_0) \subset P_0, w^{-1}(L \cap P_0) \subset P_0\}.$$ 

Then $W_{M,L}$ gives a complete set of coset representatives for the double cosets $W_{L} \setminus W_{G}/W_{M}$ and for each $w \in W_{M,L}$, $w M \cap L \in \mathcal{L}(L)$ and $M \cap w^{-1}L \in \mathcal{L}(M)$. Now the Geometrical Lemma [B-Z, 2.12] implies the following character formula. Let $M, L \in \mathcal{L}(G)$ and let $\tau$ be an admissible representation of $M$. For each $w \in W_{M,L}$, write $L_{w} = L \cap w M$. Then as above we can define $r_{w^{-1}L_{w},M}(\tau)$. It is an admissible representation of $w^{-1}L_{w}$. Now $w r_{w^{-1}L_{w},M}(\tau)$ is an admissible representation of $L_{w}$ and $i_{L_{w},L}(w r_{w^{-1}L_{w},M}(\tau))$ is an admissible representation of $L$. We will also denote this representation by $i_{L_{w},L}(r_{L_{w},L}(w\tau))$. Then the character formula can be written as

$$r_{L_{w},L}(i_{G,M}(\Theta_{\tau})) = \sum_{w \in W_{M,L}} i_{L_{w},L}(r_{L_{w},L}(w\Theta_{\tau})). \quad (3.3)$$

Let $L \in \mathcal{L}(G)$ and suppose that $\Theta' \in \mathcal{Y}(L)$, the set of tempered virtual characters of $L$. Then by linearity we can define $\Theta = i_{G,L}(\Theta') \in \mathcal{Y}(G)$. Let $L^{\text{ell}}$ denote the set of regular elliptic elements of $L$. Thus $x \in L^{\text{ell}}$ just in case $x$ is a
regular semisimple element of \( L \) and the centralizer of \( x \) in \( L \) is compact modulo the center of \( L \).

**Lemma 3.4.** Let \( \Theta' \in \mathcal{Y}(L) \), \( \Theta = i_{G,L}(\Theta') \). Then for all \( x \in L^{\text{ell}} \),

\[
\Theta^v_L(x) = \sum_{s \in W^{L,L}_0} s \Theta'(x).
\]

Here \( W^{L,L}_0 = \{ s \in W^{L,L} : sL = L \} \).

**Proof.** First, using (3.3) which clearly extends by linearity to virtual characters, we have

\[
r_{L,G}(i_{G,L}(\Theta')) = \sum_{s \in W^{L,L}} i_{L,L_s}(r_{L_s,sL}(s \Theta')).
\]

Now suppose that \( s \in W^{L,L} \) and \( L_s = L \cap sL \) is a proper Levi subgroup of \( L \). Then \( i_{L,L_s}(r_{L_s,sL}(s \Theta')) \) is a properly induced character of \( L \) and hence is zero on \( L^{\text{ell}} \). But when \( L_s = L = sL \), then \( i_{L,L_s}(r_{L_s,sL}(s \Theta')) = s \Theta' \). Thus for all \( x \in L^{\text{ell}} \) we have

\[
\Theta_L(x) = r_{L,G}(i_{G,L}(\Theta'))(x) = \sum_{s \in W^{L,L}_0} s \Theta'(x).
\]

But since \( \Theta' \) is tempered, so is \( s \Theta' \) for all \( s \in W^{L,L}_0 \), and so

\[
\Theta^v_L(x) = \Theta_L(x) = \sum_{s \in W^{L,L}_0} s \Theta'(x). \qedhere
\]

**Proof of Theorem 3.2.** Let \( \Theta \in \mathcal{Y}(G) \) such that the restriction of \( \Theta \) to \( G^{\text{ell}} \) is zero, but \( \Theta \neq 0 \). Then using Theorem D and Proposition 1 of Kazhdan \([K]\), there are proper Levi subgroups \( M_i \in \mathcal{L}(G) \) and tempered virtual characters \( \Theta_i \in \mathcal{Y}(M_i) \) such that \( \Theta = \sum_i i_{G,M_i}(\Theta_i) \). Let \( d_i \) be the dimension of \( M_i \) and let \( d \) be the maximum of the \( d_i \). The expression of \( \Theta \) as a sum of induced virtual characters is not unique, but we can assume that we have chosen the \( M_i, \Theta_i \), so that \( d \) is as small as possible. We can also assume that \( M_i \) is not conjugate to \( M_j \) for \( i \neq j \).

Now assume that \( \Theta \) is supertempered. Pick \( M_1 \) such that \( d_1 = d \) is maximal and let \( x \in M_1^{\text{ell}} \). Using (3.3), for all \( i \) we have

\[
i_{G,M_1}(\Theta_i)_{M_1} = \sum_{s \in W^{M_i,M_i}} i_{M_1,M_i}(r_{M_i,sM_i}(s \Theta_i)).
\]

Let \( s \in W^{M_i,M_i} \) and suppose that \( M_{i,s} = M_1 \cap sM_i = M_1 \). Then \( M_1 \subset sM_i \). But \( \dim M_1 \geq \dim sM_i \) so that \( M_1 = sM_i \). But we assumed that \( M_i \) is not conjugate
to $M_1$ for $i \neq 1$. Thus for all $i \neq 1$ and $s \in W_{M_i,M_1}, M_{i,s}$ is a proper Levi subgroup of $M_1$, so that the induced characters are all zero on elliptic elements of $M_1$. Thus for $i \neq 1$ we have

$$i_{G,M_1}(\Theta_i)^{M_1}_s(x) = i_{G,M_1}(\Theta_i)_{M_1}(x) = 0.$$ 

But $\Theta$ is supertempered, so this implies that

$$i_{G,M_1}(\Theta_i)^{M_1}_s(x) = \Theta_s^{M_1}(x) = 0.$$ 

Define $\Theta'_1 \in \mathcal{Y}(M_1)$ by

$$\Theta'_1 = k^{-1} \sum_{s \in W_0^{M_1,M_1}} s \Theta_1$$

where $k$ is the cardinality of $W_0^{M_1,M_1}$. Then $i_{G,M_1}(\Theta'_1) = i_{G,M_1}(\Theta_1)$. But, by Lemma 3.4, we see that for $x \in M_1^{\text{ell}}, \Theta'_1(x) = k^{-1} i_{G,M_1}(\Theta_1)^{M_1}_s(x) = 0$. Thus $\Theta'_1$ is zero on the elliptic set of $M_1$. Now using Kazhdan’s theorem [K], there are proper Levi subgroups $L_j$ of $M_1$ and $\Theta'_j \in \mathcal{Y}(L_j)$ so that $\Theta'_1 = \sum_j i_{M_1,L_j}(\Theta'_j)$. Thus $i_{G,M_1}(\Theta_1) = \sum_j i_{G,L_j}(\Theta'_j)$. We have seen that for any $M_1$ such that $d_1 = d$ is maximal, $i_{G,M_1}(\Theta_1)$ can be written as a sum of virtual characters induced from Levi subgroups of strictly smaller dimension. This contradicts our assumption about the expression of $\Theta$. Thus $\Theta = 0$. 

Let $M \in \mathcal{L}(G)$, $\sigma \in \mathcal{E}_2(M)$, and define $W_\sigma, R = R_\sigma,$ and $\Pi(\tilde{R}, \chi)$ as in the beginning of this section. Define $\pi = i_{G,M}(\sigma)$ and for each $\rho \in \Pi(\tilde{R}, \chi)$, let $\pi_\rho$ be the irreducible constituent of $\pi$ corresponding to $\rho$ and $\Theta_\rho$ its character. Let $L \in \mathcal{L}(G)$ such that $M \subset L$ and assume that $L$ satisfies the compatibility condition in [A, §2]. Then the $R$-group for $i_L,M(\sigma)$ is $R(L) = R \cap W(L/A)$. Let $\tilde{R}(L)$ denote the inverse image of $R(L)$ in $\tilde{R}$. Then $\Pi(\tilde{R}(L), \chi)$ parameterizes the irreducible constituents of $i_L,M(\sigma)$. Given $\rho \in \Pi(\tilde{R}, \chi), \rho' \in \Pi(\tilde{R}(L), \chi)$, write $\text{Res}(\rho)$ for the restriction of $\rho$ to $\tilde{R}(L)$ and $\text{Ind}(\rho')$ for the representation of $\tilde{R}$ induced from $\rho'$. It follows easily from the remarks of Arthur in [A, §2] explaining the compatibility of formula [A, 2.4] with induction, that if $\tau_\rho$ is the irreducible constituent of $i_L,M(\sigma)$ corresponding to $\rho' \in \Pi(\tilde{R}(L), \chi)$, then

$$m(\pi_\rho, i_{G,L}(\tau_\rho)) = m(\rho, \text{Ind}(\rho')) = m(\rho', \text{Res}(\rho)).$$

LEMMA 3.5. Suppose that $R_\sigma = W_\sigma$ and let $L \in \mathcal{L}(G)$. Then given $\rho \in \Pi(\tilde{R}, \chi)$,

$$(\Theta_\rho)^L_{\tilde{R}} = \sum_{\tau \in \mathcal{E}_2(L)} m(\pi_\rho, i_{G,L}(\tau)) \Theta_\tau.$$ 

Proof. Using (3.3) we can compute
where the $\gamma_v$ are quasicharacters of the split component $A_v$ of $L_v$. Since $v\sigma \in \mathcal{O}_2(vM)$, $\Theta_{v\sigma}$ is super tempered by Corollary 2.6. Now as in Lemma 2.4, every exponent $\gamma_v$ of $(\Theta_{v\sigma})_{L_v}$ is unitary on $A_{vM} \subset A_v$, and is rapidly decreasing in the appropriate chamber of $A_v/A_{vM}$. But the central character of $\iota_{L,L_v}(\Theta_{v\sigma})_{L_v,\gamma_v}$ is the restriction of $\gamma_v$ to $A_L$. Thus $\iota_{L,L_v}(\Theta_{v\sigma})_{L_v,\gamma_v}$ can have unitary central character only if $A_L \subset A_{vM}$ so that $vM \subset L$. In this case $L_v = vM$. Thus we see that

$$(\Theta_{v\sigma})_{L_v} = \sum_{\gamma_v} (\Theta_{v\sigma})_{L_v,\gamma_v}$$

where $W_0^{M,L} = \{v \in W^{M,L}; vM \subset L\}$. Fix $v \in W_0^{M,L}$. Then $i_{L,vM}(v\sigma) = v\iota_{v^{-1}L,M}(\sigma)$. There is no compatibility condition since $R_\sigma = W_\sigma$, and we can write the irreducible constituents of $i_{v^{-1}L,M}(\sigma)$ as $\tau_\rho$, $\rho' \in \Pi(\tilde{R}(v^{-1}L), \chi)$. Further, $\tau_\rho'$ occurs in $i_{v^{-1}L,M}(\sigma)$ with multiplicity deg $\rho'$. Thus $i_{L,vM}(v\sigma)$ has irreducible constituents $v\tau_\rho'$ occuring with multiplicity deg $\rho'$.

Given $v_1, v_2 \in W_0^{M,L}$, $i_{L,v_1M}(v_1\sigma)$ and $i_{L,v_2M}(v_2\sigma)$ have constituents in common if and only if they are equal. In this case there is $s \in W_L$ such that $v_2M = sv_1M$ and $v_2\sigma \simeq sv_1\sigma$. Thus $v_2^{-1}sv_1 \in W_\sigma = \{w \in W_G; wM = M, w\sigma \simeq \sigma\}$ so that $v_1$ and $v_2$ are in the same double coset of $W_L \setminus W_G/W_\sigma$. Recall $W_\sigma \simeq W_\sigma/W_M$ and write $W_\sigma(v^{-1}L) = W_\sigma \cap W(v^{-1}L/A)$. Then we see that $i_{L,vM}(\Theta_{v\sigma})$ occurs in $(\Theta_{\sigma})_{L_v}^L$ exactly $[W_\sigma/W_\sigma(v^{-1}L)]$ times. Thus for $v \in W_0^{M,L}$, each $\Theta_{v\sigma}$, $\rho' \in \Pi(\tilde{R}(v^{-1}L), \chi)$, occurs in $(\Theta_{\sigma})_{L_v}^L$ with multiplicity deg $\rho'[W_\sigma/W_\sigma(v^{-1}L)] = deg \rho'[R/R(v^{-1}L)]$ since $R = W_\sigma$ by hypothesis.

Let $\rho \in \Pi(\tilde{R}, \chi)$. By the standard Frobenius reciprocity result [Ca2, 3.2.4], for any $\tau \in \mathcal{O}_1(L)$,

$$m(\pi_\rho, i_{G,L}(\tau)) = \text{dim} \text{Hom}_L((\pi_\rho)_{N_1}, \tau)$$

where we use normalized induction and the normalized Jacquet module. However, since $(\pi_\rho)_{N_1}^w$ is the maximal tempered quotient of $(\pi_\rho)_{N_1}$ and we are assuming that $\tau$ is tempered,

$$\text{Hom}_L((\pi_\rho)_{N_1}, \tau) = \text{Hom}_L((\pi_\rho)^w_{N_1}, \tau).$$
Thus $\Theta_\tau$ occurs in $(\Theta_{\rho})^*_L$ with multiplicity $m(\rho, \tau) \geq m(\pi_\rho, i_{G,L}(\tau))$.

Fix $\tau \in \mathfrak{g}_L^*(M)$. If $\tau$ is not of the form $v\tau_{\rho'}$ for some $v \in W^M_{0,L}$, $\rho' \in \Pi(\check{\rho}(v^{-1}M), \chi)$, then $\Theta_\tau$ does not occur in $(\Theta_{\rho})^*_L$ and so $m(\rho, \tau) = 0$ for all $\rho$. It is an easy consequence of [A, 1.1], that $m(\pi_\rho, i_{G,M}(\tau)) = 0$ for all $\rho$ also. Thus $m(\rho, \tau) = m(\pi_\rho, i_{G,L}(\tau)) = 0$ in this case. Now let $\tau = v\tau_{\rho'}$ for some $v \in W^M_{0,L}$, $\rho' \in \Pi(\check{\rho}(v^{-1}M), \chi)$ and suppose that $m(\rho, v\tau_{\rho'}) = m(\pi_\rho, i_{G,L}(v\tau_{\rho'})) = m(\rho, \text{Ind}(\rho'))$ for some $\rho$. Now since the multiplicity of $\pi_\rho$ in $\pi$ is deg $\rho$, we see that the multiplicity of $\Theta_{v\tau_{\rho'}}$ in $(\Theta_{\rho})^*_L$ is strictly greater than

$$\sum_{\rho} \deg \rho m(\rho, \text{Ind}(\rho')) = \deg \text{Ind}(\rho') = \deg \rho'[R/R(v^{-1}L)].$$

But this contradicts the above calculation using the Geometric Lemma. Thus $m(\rho, v\tau_{\rho'}) = m(\pi_\rho, i_{G,L}(v\tau_{\rho'}))$ for all $\rho$.

**Lemma 3.6.** Suppose that $R_{\text{reg}} \neq \emptyset$. Then $R_\sigma = W_\sigma$.

**Proof.** Suppose that $R_\sigma \neq W_\sigma$. Then $\Delta_0 \neq \emptyset$. For each $\alpha \in \Delta_0$, define $H_\alpha \in \mathfrak{a}$ dual to $\alpha$. Set $H_0 = \sum_{\alpha \in \Delta_0} H_\alpha$. Then for any $r \in R$, $r\Delta_0 = \Delta_0$ so that $rH_0 = H_0$. Thus $H_0 \in \mathfrak{a}_\sigma$. But $\alpha(H_0) > 0$ for all $\alpha \in \Delta_0$ so that $H_0 \notin \mathfrak{z}$. Thus $R_{\text{reg}} = \emptyset$. □

**Proof of Theorem 3.1**

Let $(M, \sigma, r)$ be an elliptic triple. Thus $M \in \mathcal{L}(G)$ is a Levi subgroup of $G, \sigma \in \mathfrak{e}_2(M)$, and $r \in \check{R}_{\text{reg}}$. We must show that $\Theta = \Theta(M, \sigma, r) = \sum_{\rho \in \Pi(\check{\rho}, \chi)} \text{tr}(\rho(r)) \Theta_\rho$ is super tempered. Since $R_{\text{reg}} \neq \emptyset$, we know from Lemma 3.6 that $R_\sigma = W_\sigma$. Now from Lemma 3.5 we know that for any $L \in \mathcal{L}(G)$,

$$\Theta^*_L = \sum_{\rho} \frac{\text{tr}(\rho(r))}{\sum_{\tau \in \mathfrak{g}_L^*(L)} m(\pi_\rho, i_{G,L}(\tau))} \Theta_\tau = \sum_{\tau \in \mathfrak{g}_L^*(L)} \Theta_\tau \sum_{\rho} \frac{\text{tr}(\rho(r))}{m(\pi_\rho, i_{G,L}(\tau))}.$$ 

But we know that $m(\pi_\rho, i_{G,L}(\tau)) = 0$ for all $\rho$ unless there is $v \in W_G$ so that $M \subset v^{-1}L$ and $\tau = v\tau_{\rho'}$ for some $\rho' \in \Pi(\check{\rho}(v^{-1}L), \chi)$. Assume that this is the case. Then $m(\pi_\rho, i_{G,L}(v\tau_{\rho'})) = m(\rho, \text{Ind}(\rho'))$. Thus

$$\sum_{\rho} \frac{\text{tr}(\rho(r))}{m(\pi_\rho, i_{G,L}(v\tau_{\rho'}))} \sum_{\rho} m(\rho, \text{Ind}(\rho')) \text{tr}(\rho(r)) = \text{tr}(\text{Ind}(\rho'))(r).$$

But if $L \neq G$, $r \in \check{R}_{\text{reg}}$ cannot be conjugate to an element of $\check{R}(v^{-1}L)$, and so $\text{tr}(\text{Ind}(\rho')(r)) = 0$. Thus $\Theta^*_L = 0$.

We have proven that $\Theta^*_L = 0$ for every $L \in \mathcal{L}(G)$, $L \neq G$. But $\mathcal{L}(G)$ contains
representatives for all conjugacy classes of Levi subgroups of \( G \). Further, although the definition of \( \Theta^* = \Theta^*_P \) depends on the choice of the parabolic-subgroup \( \bar{P}_L \) with Levi component \( L \), the formula from Lemma 3.5 shows that it is independent of all choices. Thus \( \Theta \) is supertempered.

Conversely, suppose that \( \Theta \in \mathcal{Y}(G) \) is supertempered. As in [A, §3], virtual characters of the form \( \Theta(M, \sigma, r) \), \( M \in \mathcal{L}(G) \), \( \sigma \in \mathcal{S}(M) \), \( r \in \mathcal{R}_\sigma \), span \( \mathcal{Y}(G) \). Thus there are triples \((M_i, \sigma_i, r_i)\), \( 1 \leq i \leq k \), as above and complex numbers \( c_i \) so that

\[
\Theta = \sum_{i=1}^{k} c_i \Theta(M_i, \sigma_i, r_i).
\]

Now by the results of [A, §3], \( \Theta(M_i, \sigma_i, r_i) \) is zero on \( G^{\text{all}} \) unless \( i \in I_{\text{reg}} = \{1 \leq i \leq k; r_i \in (\mathcal{R}_\sigma)_{\text{reg}}\} \). But for all \( i \in I_{\text{reg}} \), \( \Theta(M_i, \sigma_i, r_i) \) is supertempered. Thus

\[
\Theta' = \Theta - \sum_{i \in I_{\text{reg}}} c_i \Theta(M_i, \sigma_i, r_i)
\]

is supertempered and zero on the elliptic set of \( G \). Thus \( \Theta' = 0 \) by Theorem 3.2. \( \square \)

References

[Ca2] W. Casselman, Introduction to the theory of admissible representations of \( p \)-adic reductive groups, unpublished notes.