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Generic simple coverings of the affine plane

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0. Introduction

In [1, 2, 4, 5, 6] ideas of Deligne are used to prove the factoriality of the surface $Z^p = f(X, Y)$ for a generic choice of polynomial $f(X, Y)$ of arbitrary degree $\geqslant 4$ (with $p \geqslant 3$). In this paper we study the class group of surface $Z^n = f(X, Y)$ for arbitrary positive integer n.

The above mentioned calculation leads us naturally to conjecture that the class group of $Z^n = f(X, Y)$ is factorial for a generic choice of f. To be more precise, let $f = \Sigma\, T_{ij} X^i Y^j$ be a generic polynomial with indeterminate coefficients and let $A_n = K[X, Y, Z]/(Z^n - f)$ where K is the algebraic closure of $\mathbb{F}_p(T_{ij})$ with $\mathbb{F}_p$ the prime field of p elements ($p \geqslant 3$). Assume the degree of f is at least 4. Then we conjecture

0.1. For all $n \in \mathbb{Z}^+$, A_n is factorial.

In this paper we prove that (0.1) reduces to the case $\gcd(p, n) = 1$. We feel that this latter case can be approached by adapting a theorem of Steenbrink [9] from characteristic 0 to characteristic p by systematically replacing singular cohomology by étale cohomology; a project we are currently working on.

In Sections 1 and 2, descent techniques are used to study the class group of arbitrary surfaces $Z^n = f$. Two main results proved are (2.16), which reduces (0.1) to the case $n = pm$ where $\gcd(p, m) = 1$, and (2.5), which shows that if (0.1) is true for some n, then it is true for all divisors of n.

In Section 3 the reduction of (0.1) to the case $\gcd(p, n) = 1$ is accomplished by analyzing the action of $\mathcal{G} = \mathrm{Gal}(K, \mathbb{F}_p(T_{ij}))$ on the divisor class group of $Z^{pm} = f$ (3.8).

1. Galois descent

1.1. NOTATION. If R is a commutative ring with unity and P is a prime ideal of R, denote the residue field of R at P by $k(P) = R_P/PR_P$.

If R is a Krull domain, let $\mathrm{Cl}(R)$ denote the divisor class group of R as defined in P. Samuel's Tata notes [7] (also see [3]).

1.2. DISCUSSION. This section makes use of Galois descent techniques and the next section imploys radical descent methods. Suppose G is a finite group of automorphisms acting on a Krull domain B and A is the fixed subring of B. Denote the multiplicative set of units in B and A by B^* and A^*, respectively. Since G is a finite group, the ring B is integral over A. The inclusion $A \to B$ induces a homomorphism $\varphi\colon \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ by the following theorem.

1.3. THEOREM. *Let $A \subset B$ be Krull rings with B integral over A or with B flat as an A-module. Then there is a well defined group homomorphism $\varphi\colon \mathrm{Cl}(A) \to \mathrm{Cl}(B)$ such that for each height one prime P of A*

$$\varphi(P) = \sum_{P'} e(P', P)P'$$

where the P' are the prime ideals of B lying over P and $e(P', P)$ is the ramification index of P' over P ([7], *pp.* 19–20).

1.4. THEOREM. *Let A and B be as in* (1.2). *Then φ induces an injection $\theta\colon \ker\varphi \to H^1(G, B^*)$. If every prime divisorial ideal of B is unramified over A, then θ is a bijection* ([7], p. 55).

1.5. REMARK. If G in (1.2) is a finite cyclic group generated by an element π, then $H^1(G, B^*)$ is the homology of the complex $B^* \xrightarrow{h} B^* \xrightarrow{N} A^*$ where $h(x) = \pi(x)/x$ for $x \in B^*$ and N is the norm on B^* ([7], p. 57).

1.6. LEMMA. *Assume in* (1.2) *that G is cyclic of order n and B is a unique factorization domain. Assume that for each prime element $b \in B$ either*

(i) *$\pi^s(b)B \neq \pi^t(b)B$ whenever $s \not\equiv t \pmod n$, or*

(ii) *$b \in A$.*

Then $H^1(G, B^*) = 0$.

Proof. By (1.5) $H^1(G, B^*)$ is the homology of the complex $B^* \xrightarrow{h} B^* \xrightarrow{N} A^*$. Assume u is a unit in B and $N(u) = 1$. Let L denote the field of fractions of B. Each element of L^* can be written as a fraction b/a where $b \in B$, $a \in A$. Then by Hilbert's Theorem 90 there exists $x \in B$ such that $h(x) = u$. x can be written as a product $x = wb_1^{e_1} \cdots b_r^{e_r}$ where $w \in B^*$, the b_i are prime elements in B and $e_i \in \mathbb{Z}^+$, $1 \leqslant i \leqslant r$.

Note that since $\pi(x) = ux$, if $\pi(b_i)B = b_jB$, then $\pi(b_i)$ multiplied by a unit must appear in the prime factorization of x in B with the same exponent as b_j. Therefore, in order to show that $u \in h(B^*)$ we may reduce to the case $x = wb\pi(b) \cdots \pi^{m-1}(b)$ where m is the smallest positive integer such that $\pi^m(b)B = bB$. By hypothesis either $b \in A$ and $m = 1$, or $m = n$, in which case $x = wN(b)$. In either case $u = \pi(x)/x = \pi(w)/w$, so that u is a boundary. □

1.7. LEMMA. *Assume in* (1.2) *that G is cyclic of order n and B is a unique factorization domain. Assume for each prime element $b \in B$ either $[k(bB)\colon k(bB \cap A)] = 1$ or $b \in A$. Assume also that B is unramified over A. Then $H^1(G, B^*) = 0$.*

Proof. Let b be a prime element of B and $b \notin A$. Then by hypothesis there are exactly n height one primes of B lying over $bB \cap A$ and each of them is generated by a conjugate of b. Thus b satisfies condition (i) of (1.6). □

1.8. NOTATION. If E is a field, $A = E[X_1, \ldots, X_n]$ is the polynomial ring in s variables over E and $h \neq 0$ is an element of A, let $\deg(h)$ denote the degree of h and h^+ the highest degree form of h. If $g \neq 0$ also belongs to A define $\deg(h/g) = \deg(h) - \deg(g)$.

1.9. ASSUMPTIONS. Throughout K will be an algebraically closed field of characteristic $p \geqslant 3$. Assume $f \in K[X, Y]$ is an irreducible polynomial in two variables X, Y of degree at least 4. We will assume that $\partial f/\partial X$ and $\partial f/\partial Y$ meet transversally and in the maximum possible number of points of K^2. This number is $(\deg f - 1)^2$ if $\deg f \neq 0 \pmod p$ and $(\deg f)^2 - 3\deg f + 3$ otherwise (see [5, pp. 287–288]). Implicit in these assumptions is the fact that $f^+ \notin K[X^p, Y^p]$. We remark that a generic f of degree at least 4 satisfies the conditions stated above.

For each $n \in \mathbb{Z}^+$, let $A_n = K[X, Y, Z]/(Z^n - f)$ and E_n denote the field of fractions of A_n. Let x, y, z denote the images of X, Y, Z in A_n. Then the subring of $K[x, y]$ of A_n is isomorphic to $K[X, Y]$.

Let $W_n = \operatorname{Spec}(A_n)$. Since W_n has only finitely many singular points, A_n is noetherian integrally closed and hence a Krull ring.

1.10. LEMMA. *Assume $n \in \mathbb{Z}^+$ and $\operatorname{Cl}(A_n) = 0$. Then $\operatorname{Cl}(A_m) = 0$ for all $m \in \mathbb{Z}^+$ such that m divides n and $\gcd(p, n/m) = 1$.*

Proof. It's enough to prove the case $n = mq$ where q is a prime number. Let $c \in K$ be a primitive q-th root of unity and let π be the $K(X, Y)$-automorphism on $K(X, Y, Z)$ defined by $\pi(Z) = cZ$. Then π induces an automorphism on A_n. Let G be the cyclic group generated by π and A be the fixed subring of A_n. Then $A = K[x, y, z^q] \cong A_m$.

Let b a prime element of A_n. Then b can be written $b = \Sigma_{i=0}^{q-1} a_i z^i$ for unique $a_i \in A$. Since $[E_n\colon E_m] = q$, $[k(bA_n)\colon k(bA_n \cap A)] = 1$ unless $a_i = 0$ for

$i \geqslant 1$; i.e., unless $b \in A$.

Since f is irreducible in $K[X, Y]$, z is a prime element in A_n. Since $A_n\left[\frac{1}{z}\right]$ is unramified over $A_n\left[\frac{1}{z^q}\right]$, we obtain by (1.7) that $H^1\left(G, A_n\left[\frac{1}{z}\right]^*\right) = 0$. By (1.3) and (1.4) it follows that $\mathrm{Cl}\left(A_m\left[\frac{1}{z}\right]\right) = 0$, which by Nagata's lemma implies $\mathrm{Cl}(A_m) = 0$. □

2. Radical descent

2.1. DISCUSSION. Let B be a Krull ring of characteristic $p \neq 0$, and let L be its quotient field. Let Δ be a derivation of L such that $\Delta(B) \subset B$. Let $L' = \ker(\Delta)$ and $A = L' \cap B$. Then A is a Krull ring and B is integral over A since $B^p \subset A \subset B$. By (1.3) there is a well defined group homomorphism $\varphi\colon \mathrm{Cl}(A) \to \mathrm{Cl}(B)$.

Set $\mathscr{L} = \{t^{-1}\Delta t\colon t^{-1}\Delta t \in B, t \in L^*\}$ and $\mathscr{L}' = \{u^{-1}\Delta u\colon u \in B^*\}$. Then $\mathscr{L}$ is an additive subgroup of B and $\mathscr{L}'$ is a subgroup of $\mathscr{L}$.

2.2. THEOREM. (a) *There exists a canonical monomorphism* $\theta\colon \ker\varphi \to \mathscr{L}/\mathscr{L}'$. (b) *If* $[L\colon L'] = p$ *and if* $\Delta(B)$ *is not contained in any height one prime of* B, *then* θ *is an isomorphism* ([7], p. 62).

2.3. PROPOSITION. *If* $[L\colon L'] = p$ *in* (2.1) *then there exists* $a \in A$ *such that* $\Delta^p = a\Delta$ ([7], p. 63).

2.4. PROPOSITION. *If* $[L\colon L'] = p$ *in* (2.1), *then an element* $x \in L$ *is logarithmic derivative* (*i.e.* $x = t^{-1}\Delta t$ *for some* $t \in L$) *if an only if* $\Delta^{p-1}x - ax + x^p = 0$, *where* $\Delta^p = a\Delta$ ([7], p. 64).

2.5. PROPOSITION. *Assume* $n \in \mathbb{Z}^+$ *and* $\mathrm{Cl}(A_n) = 0$. *Then* $\mathrm{Cl}(A_m) = 0$ *for all positive divisors* m *of* n.

Proof. It's enough to prove the case $n = mq$ where q is a prime number. The case $\gcd(p, q) = 1$ is (1.10). Thus we are left with the case $n = mp$.

The derivation $d = \partial/\partial Z$ defines a derivation on A_n with kernel $K[x, y, z^p] \cong A_m$. By (2.2) $\mathrm{Cl}(A_m) \cong \mathscr{L}/\mathscr{L}'$, where $\mathscr{L} = \{u^{-1}du\colon u \in E_n$ and $u^{-1}du \in A_n\}$ and $\mathscr{L}' = \{u^{-1}du\colon u \in A_n^*\}$. Let $t \in \mathscr{L}\backslash\{0\}$. We have $t = \sum_{i=0}^{n-1} t_i z^i$ for unique $t_i \in k[x, y]$. By (2.4) $d^{p-1}t = -t^p$. If we compare coefficients of $z^{(r-1)p}$ on both sides of this equality, we obtain for each $r = 1, 2, \ldots, m$,

$$t_{rp-1} = \sum_{j=0}^{p-1} t_{r-1+jm}^p z^{nj}. \tag{2.5.1}$$

Since $z^n = f$, we have for each $r = 1, 2, \ldots, m$,

$$t_{rp-1} = \sum_{j=0}^{p-1} t_{r-1+jm}^{p} f^{j}. \tag{2.5.2}$$

Choose s such that $\deg(t_{sp-1}) \geqslant \deg(t_{rp-1})$ for each r. t_{sp-1}^p appears on the right side of one of the equations in (2.5.2). Let t_{up-1} be the element on the left side of this equation. Since $1, f^+, \ldots, (f^+)^{p-1}$ are independent over $K(X^p, Y^p)$, $\deg t_{sp-1} \geqslant \deg(t_{up-1}) \geqslant \deg(t_{sp-1}^p f^j) > p\deg(t_{sp-1})$, which is impossible. Therefore $\mathscr{L} = 0$. □

The next proposition follows easily by (2.2), (2.3) and (2.4). Details are provided in [5]. Also see the proof of (2.13).

2.6. PROPOSITION. *Let D be the derivation on $K(X, Y)$ defined by*

$$D = \frac{\partial f}{\partial Y}\frac{\partial}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial}{\partial Y}.$$

(a) $\ker D \cap K[X, Y] = K[X^p, Y^p, f]$;
(b) *A_p is isomorphic to $K[X^p, Y^p, f]$;*
(c) *$\mathrm{Cl}(A_p)$ is isomorphic to $\mathscr{L}_0 = \{u^{-1}Du\colon u \in K(X, Y)$ and $u^{-1}Du \in K[X, Y]\}$;*
(d) *There exists $a_0 \in K[X^p, Y^p, f]$ such that $D^p = a_0 D$ and* $\deg(a_0) \leqslant (p-1)(\deg(f) - 2)$ ([5], pp. 616–622).

2.7. THEOREM. *Let Φ be an algebraically closed field of characteristic $p \neq 0$. Let $g \in \Phi[X, Y]$, $D = g_X \frac{\partial}{\partial Y} - g_Y \frac{\partial}{\partial X}$ and a be such that $D^p = aD$. Let $Q \in \Phi^2$ be such that $g_X(Q) = g_Y(Q) = 0$ and $\sqrt{H(Q)}$ a root of $T^2 = H(Q)$, where $H = g_{XX}g_{YY} = g_{XY}^2$. Then $a(Q) = (\sqrt{H(Q)})^{p-1}$ (see* [4, Theorem 1.5]).

2.8. NOTATION. Let $S = \{Q \in K^2\colon f_X(Q) = f_Y(Q) = 0\}$.

2.9. LEMMA. *If $t \in K[X, Y]$, then $\{Q \in S\colon t(Q) = 0\}$ has less than or equal to* $\deg(t)\cdot(\deg(f) - 1)$ *elements.*

Proof. Let $t = t_1^{e_1} \cdots t_s^{e_s}$ be the prime factorization of t in $K[X, Y]$. Since f_X and f_Y have no common factors, t_i is relatively prime to either f_X or f_Y, $1 \leqslant i \leqslant s$. By Bezout's Theorem [8] the number of points $Q \in S$ such that $t_i(Q) = 0$ is at most $(\deg t_i)(\deg f - 1)$. It then follows that the number of $Q \in S$ such that $t(Q) = 0$ is at most $(\Sigma \deg t_i)\cdot(\deg f - 1) \leqslant \deg t(\deg f - 1)$. □

2.10. LEMMA. *If $t \in K[X, Y]$ and $t(Q) = 0$ for each $Q \in S$, then either $t = 0$ or* $\deg t > \deg f - 2$.

Proof. Assume $t \neq 0$ and $\deg t \leqslant \deg f - 2$. By (2.9), the number of points $Q \in S$ such that $t(Q) = 0$ is at most $(\deg f - 2)(\deg f - 1)$. By (1.9), there is at least one point $Q \in S$ such that $t(Q) \neq 0$. □

2.11. LEMMA. *Assume* $a_0 \in K[X^p, Y^p, f]$ *is such that* $D^p = a_0 D$. *If* $t \in K[X, Y]$, $\deg t \leqslant \deg f - 2$ *and* $D^{p-1}t - a_0 t = 0$, *then* $t = 0$.

Proof. Given $Q \in S$, $(D^{p-1}t)(Q) = 0$ and $a_0(Q) \neq 0$ by (1.9) and (2.7) (recall that $\partial f/\partial X$ and $\partial f/\partial Y$ meet transversally at Q). Therefore $t(Q) = 0$. By (2.10) we obtain $t = 0$. □

2.12. NOTATION. The derivation D on $K(X, Y)$ extends to a derivation on $K(X, Y, Z)$ with $Z^n - f$ in its kernel. Thus D induces a derivation on E_n which we denote by D_n. $\mathscr{L}_n$ will denote the additive group of logarithmic derivatives of D_n in A_n, $\mathscr{L}_n = \{u^{-1}D_n u : u \in E_n \text{ and } u^{-1}D_n u \in A_n\}$. $\mathscr{L}'_n$ will denote the subgroup of $\mathscr{L}_n$ of logarithmic derivatives of units in A_n.

2.13. PROPOSITION. (a) A_{np} *is isomorphic to* $\ker D_n \cap A_n$; (b) *there is a well defined group homomorphism* $\varphi_n: \mathrm{Cl}(A_{np}) \to \mathrm{Cl}(A_n)$ *with* $\ker \varphi_n \cong \mathscr{L}_n/\mathscr{L}'_n$.

Proof. $\ker D_n \cap A_n \supseteq K[x^p, y^p, z]$, the latter is clearly isomorphic to $K[X, Y, Z]/(Z^{np} - f^{(p)})$, where $f^{(p)}$ is obtained from f by raising each coefficient of f to the p-th power. Since K is perfect, the automorphism $\alpha \to \alpha^p$ of K induces an isomorphism $A_{np} \to K[x^p, y^p, z]$. It follows that $K[x^p, y^p, z]$ is integrally closed. Since $[E_n : K(x^p, y^p, z)] = p$, $\ker D_n \cap A_n$ and $K[x^p, y^p, z]$ have the same field of fractions. Since $\ker D_n \cap A_n$ is integral over $K[x^p, y^p, z]$, we obtain (a). (b) is an immediate consequence of (a) and (2.2). □

2.14. PROPOSITION. *Let* $t = \Sigma_{i=0}^{n-1} t_i z^i \in A_n$, *where* $t_i \in K[x, y]$, $0 \leqslant i < n$. *For each* $i = 0, 1, \ldots, n-1$, *let* $J(i) = \{j : 0 \leqslant j < n \text{ and } pj \equiv i (\mathrm{mod}\, n)\}$. *Then* $t \in \mathscr{L}_n$ *if and only if for each* $i = 0, 1, \ldots, n-1$,

$$D^{p-1}t_i - a_0 t_i = -\sum_{j \in J(i)} t_j^p f^{(pj-i)/n},$$

where a_0 *is such that* $D^p = a_0 D$.

Proof. By (2.4), $t \in \mathscr{L}_n$ if and only if $D_n^{p-1}t - a_0 t = -t^p$; which holds if and only if $\Sigma (D^{p-1}t_i - a_0 t_i) z^i = -\Sigma t_i^p z^{ip}$. Since $1, z, \ldots, z^{n-1}$, is a basis for E_n over $K(x, y)$ and since $Z^n = f$ we obtain the desired result by comparing powers of z on both sides of the above equation.

2.15. LEMMA. *Let* $t = \Sigma_{i=0}^{n-1} t_i z^i \in A_n$, *where* $t_i \in K[x, y]$, $0 \leqslant i < n$. *If* $t \in \mathscr{L}_n$, *then* $\deg t_i \leqslant \deg f - 2$ *for each* i.

Proof. Let r be such that $\deg t_r \geqslant \deg t_i$ for each i. We consider two cases.

Case 1. $\gcd(p, n) = 1$.

We have $pr = nq + s$ for $q, s \in \mathbb{Z}$ with $q \geqslant 0$, $0 \leqslant s < n$. *By* (2.14),

$D^{p-1}t_s - a_0 t_s = -t_r^p f^q$. By (2.6), $\deg a_0 \leqslant (\deg f - 2)(p-1)$. A simple induction shows that $\deg(D^{p-1}t_s) \leqslant \deg t_s + (\deg f - 2)(p-1)$. Thus $p \deg t_r \leqslant \deg(D^{p-1}t_s - a_0 t_s) \leqslant \deg t_s + (\deg f - 2)(p-1) \leqslant \deg t_r + (\deg f - 2)(p-1)$. Hence $\deg t_r \leqslant \deg f - 2$.

Case 2. $p \mid n$.

Again $pr = nq + s$ as in Case 1. By (2.14),

$D^{p-1}t_s - a_0 t_s = -\sum_{j \in J(s)} t_j^p f^{(pj-s)/n}$. Since p divides n and each $j \in J(s)$ is less than n, the integers $(pj - s)/n$ are distinct modulo n. Since $f^+ \notin K(x^p, y^p)$ by (1.9) and since $r \in J(s)$ it follows

$$\begin{aligned} p \deg t_r = \deg(t_r^p) &\leqslant \deg(t_r^p f^q) \leqslant \deg(\Sigma\, t_j^p f^{(pj-s)/n}) \\ &= \deg(D^{p-1}t_s - a_0 t_s) \leqslant \deg t_s + (\deg f - 2)(p-1) \\ &\leqslant \deg t_r + (\deg f - 2)(p-1). \end{aligned}$$

Hence $\deg t_r \leqslant \deg f - 2$. □

2.16. THEOREM. *Let* $m \in \mathbb{Z}^+$ *such that* $\gcd(p, m) = 1$. *If* $\mathrm{Cl}(A_{pm}) = 0$ *then* $\mathrm{Cl}(A_{p^r m}) = 0$ *for all* $r \geqslant 0$.

Proof. The case $r = 0$ follows by (2.5). The case $r = 1$ is by hypothesis. To prove the remaining cases we need to establish the below claim.

CLAIM. If p divides n, then the composition $A_{n/p} \xrightarrow{\cong} K[x, y, z^p] \hookrightarrow A_n$ maps $\mathscr{L}_{n/p}$ isomorphically onto $\mathscr{L}_n$.

Proof of Claim. Let $t = \Sigma_{i=0}^{n-1} t_i z^i \in \mathscr{L}_n$ where $t_i \in K[x, y]$ and $n = p^s m$. Since $s \geqslant 1$, we have that if $\gcd(i, p) = 1$, then by (2.14), $D^{p-1}t_i - a_0 t_i = 0$; which by (2.11) and (2.15) implies $t_i = 0$. Thus $t \in K[x, y, z^p] \cong A_{n/p}$. Therefore the isomorphism that maps $A_{n/p}$ onto $K[x, y, z^p]$ maps $\mathscr{L}_{n/p}$ onto $\mathscr{L}_n$.

Now $\mathrm{Cl}(A_{pm}) = 0$ and (2.13) imply $\mathscr{L}_m / \mathscr{L}'_m = 0$. Then the claim shows that $\mathscr{L}_{p^r m} / \mathscr{L}'_{p^r m} = 0$ for all $r \geqslant 1$. The remaining cases of the theorem follow by (2.13) and a simple induction. □

2.17. PROPOSITION. *The kernel of* $\varphi_n : \mathrm{Cl}(A_{np}) \to \mathrm{Cl}(A_n)$ *is finite p-group of type* $(p, \ldots, p)$ *of order* p^M, *where* $M \leqslant n \deg f (\deg f - 1)/2$.

Proof. By (2.13) we need only show that $\mathscr{L}_n$ has the stated properties. By the claim in the proof of (2.16) we may reduce to the case $\gcd(p, n) = 1$.

Let $t = \Sigma_{i=0}^{n-1} t_i z^i \in \mathscr{L}_n$, where $t_i \in K[x, y]$, $0 \leqslant i < n$. By (2.15), each $t_i = \Sigma\, \alpha_{rs}^{(i)} x^r y^s$ where each $\alpha_{rs}^{(i)} \in K$ and $\deg t_i \leqslant \deg f - 2$. $pi = nq + j$ for

$q, j \in \mathbb{Z}$, $q \geqslant 0$, $0 \leqslant j < n$. $\gcd(p, n) = 1$ implies $J(i) = \{i\}$; which by (2.14) yields

$$D^{p-1}t_j - a_0 t_j = -t_i^p f^q. \tag{2.17.1}$$

Comparing the coefficients of $x^{ap}y^{bp}$ on both sides of (2.17.1) we see that for each triple of nonnegative integers (e, a, b) with $e < n$ and $a + b \leqslant \deg f - 2$, $\alpha_{ab}^{(e)}$ must satisfy an equation of the form

$$L_{(e,a,b)} = (\alpha_{ab}^{(e)})^p, \tag{2.17.2}$$

where L_{ab} is a linear expression in the $\alpha_{rs}^{(i)}$ with coefficients in K. There are a total of $n \deg f(\deg f - 1)/2$ such equations. The ring $R = K[\ldots, \alpha_{rs}^{(i)}, \ldots]$ with these relations is a finite dimensional K-vector space spanned by all monomials in the $\alpha_{rs}^{(i)}$ of degree $\leqslant (p-1)n \deg f(\deg f - 1)/2$. This shows R is Artinian and has a finite number of maximal ideals. Thus the equations in (2.17.2) have only a finite number of solutions in K, which by Bezout's theorem [8, p. 198] is at most $p^{n \deg f(\deg f - 1)/2}$.

Since $\mathscr{L}_n \subset K[x, y, z]$, each element of $\mathscr{L}_n$ has p-torsion. □

2.18. REMARK. Our main objective is to reduce conjecture (0.1) to the case $\gcd(p, n) = 1$. Theorem (2.16) allows us to reduce to the case $n = pm$ where $\gcd(p, m) = 1$. In the next section we use results concerning $\mathrm{Gal}(\overline{K(T_{ij})}/K(T_{ij}))$ to complete the project. Proposition (2.5) gives us some flexibility when attempting (0.1). For example, we may reduce (0.1) to the case $n \equiv 1 \pmod{p}$.

3. The action of the Galois group

3.1. NOTATION. In this section $\mathbb{F}_p$ is the prime field of characteristic $p \geqslant 3$, T_{ij} are indeterminates algebraically independent over $\mathbb{F}_p$ where $0 \leqslant i + j \leqslant M$ with M a positive integer greater than or equal to 4. We denote the following:

$f = \Sigma\, T_{ij} X^i Y^j$

$H = f_{XX} f_{YY} - f_{XY}^2$, the hessian of f,

$K = \overline{\mathbb{F}_p(T_{ij})}$, the algebraic closure of $\mathbb{F}_p(T_{ij})$,

$\mathscr{G} = \mathrm{Gal}(K, \mathbb{F}_p(T_{ij}))$,

$S = \{Q \in K^2 \colon f_X(Q) = f_Y(Q) = 0\}$, For $n \in \mathbb{Z}^+$, let $\bar{S}_n = \{(\alpha, \beta, \gamma) \in K^3: (\alpha, \beta) \in S \text{ and } \gamma^n = f(\alpha, \beta)\}$.

In [1, 4] it is shown that S has the maximum possible number of elements as described in (1.9). Let $Q_1, \ldots, Q_I$ be a listing of the elements of S. Then we can list the elements of $\bar{S}_n$ as Q_{ij}, where if $Q_{ij} = (\alpha, \beta, \gamma)$, then $(\alpha, \beta) = Q_i$. Finally, for each i, let $\sqrt{H(Q_i)}$ denote a fixed root of the equation $T^2 = H(Q_i)$.

The next two theorems are proved in [2] and [4].

3.2. THEOREM. *$\mathcal{G}$ acts on S as the full symmetric group* (see [4, p. 353] and [2, p. 296]).

3.3. THEOREM. *For every pair $Q_i \neq Q_j \in S$, there exists $\sigma \in \mathcal{G}$ such that σ acts as the identity on S, and*

$$\sigma(\sqrt{H(Q_e)}) = \begin{cases} -\sqrt{H(Q_e)}, & \text{if } e = i, j \\ \sqrt{H(Q_e)}, & \text{otherwise.} \end{cases}$$

([4, p. 354] and [2, p. 297]).

3.4. REMARK. Assume $n \in \mathbb{Z}^+$ such that $\gcd(p, n) = 1$. Let $c \in K$ be a primitive n-th root of unity. Let π be the $K(X, Y)$-automorphism on $K(X, Y, Z)$ defined by $\pi(Z) = cZ$. Then π induces an automorphism on A_n and let $T: A_n \to K[x, y]$ denote the trace map.

Since the points $Q_{ij} \in \bar{S}_n$ lie on the surface $Z^n = f$, we may define $t(Q_{ij})$ for $t \in A_n$ by evaluating any preimage of t in $K[X, Y, Z]$ at Q_{ij}. Observe that if for a fixed i, $t(Q_{ij}) = 0$ for all j, then for each j, $T(t)(Q_{ij}) = 0$, which yields $T(t)(Q_i) = 0$.

3.5. LEMMA. *Assume* $\gcd(p, n) = 1$ *and* $t = \Sigma_{r=0}^{n-1} t_r z^r \in A_n$. *If for a fixed i, $t(Q_{ij}) = 0$ for each j, then $t_r(Q_i) = 0$ for each $r = 0, 1, \ldots, n-1$.*

Proof. It is well known that $f(Q_i) \neq 0$ for each i (it also follows by (3.2)). Let s be a nonnegative integer less than n. Then $t(Q_{ij}) = 0$ for each j implies $z^{n-s} t(Q_{ij}) = 0$ for each j. As we saw in (3.4) we obtain $T(z^{n-s} t)(Q_i) = n z^n t_s(Q_i) = n f(Q_i) t_s(Q_i) = 0$; hence $t_s(Q_i) = 0$.

3.6. LEMMA. *Assume* $\gcd(p, n) = 1$. *For each $t \in \mathcal{L}_n$ and $Q_i \in S$, there is an $r_{ij} \in \mathbb{F}_p$ such that $t(Q_{ij}) = r_{ij}\sqrt{H(Q_i)}$. Furthermore, the map*

$$\Phi: \mathcal{L} \to \bigoplus_{i,j} \mathbb{F}_p \cdot \sqrt{H(Q_{ij})}$$

defined by $\Phi(t) = (t(Q_{ij}))$ is an injection of groups.

Proof. Given $t \in \mathcal{L}_n$, $D_n^{p-1} t - a_0 t = -t^p$ where $a_0 \in K[x^p, y^p, f]$ such that $D^p = a_0 D$ by (2.4). Evaluate both sides of this equality at Q_{ij} to obtain

$a_0(Q_i)t(Q_{ij}) = t^p(Q_{ij})$. Now use (2.7) to obtain the first statement of the lemma.

Write $t = \Sigma_s t_s z^s \cdot \Phi(t) = 0$ implies $t_s(Q_i) = 0$ for each i by (3.5). By (2.10) and (2.15), each $t_s = 0$. □

3.7. THEOREM. *Assume* $\gcd(p, n) = 1$. *Then the map* $\mathrm{Cl}(A_{np}) \to \mathrm{Cl}(A_n)$ *is an injection.*

Proof. By (2.13) it's enough to show $\mathscr{L}_n = 0$. Let $t \in \mathscr{L}_n$ and suppose $t \neq 0$. Assume $\Phi(t) = (r_{ij}\sqrt{H(Q_i)})$. If $\sigma \in \mathscr{G}$ then $\sigma(t) \in \mathscr{L}_n$ and the action of σ on t is compatible with the action of σ on $\Phi(t)$. By (3.2) we may assume that $r_{11} \neq 0$. By (3.3), there is $\sigma', \sigma'' \in \mathscr{G}$ such that

$$\sigma'(\sqrt{H(Q_i)}) = \begin{cases} -\sqrt{H(Q_i)}, & i = 1,\ 2 \\ \sqrt{H(Q_i)}, & \text{otherwise} \end{cases}$$

$$\sigma''(\sqrt{H(Q_i)}) = \begin{cases} -\sqrt{H(Q_i)}, & i = 1,\ 3 \\ \sqrt{H(Q_i)}, & \text{otherwise.} \end{cases}$$

Then $\hat{t} = t - \sigma'(t) - \sigma''(t) + \sigma''\sigma'(t) \in \mathscr{L}_n$ and has the property that $\hat{t}(Q_{ij}) = 0$ for all $i \geqslant 2, 0 \leqslant j < n$, and $\hat{t} \neq 0$ since the first coordinate of $\Phi(\hat{t})$ is $4r_{11}\sqrt{H(Q_1)} \neq 0$.

We have $\hat{t} = \Sigma_{s=0}^{n-1} t_s z^s$, where $t_s \in K[x, y]$, $0 \leqslant s < n$. By (3.5) $t_s(Q_i) = 0$ for each s and each $i \geqslant 2$. We now show that this implies each $t_s = 0$; thus obtaining a contradiction.

If $\deg f \neq 0 \pmod p$, then S has $(\deg f - 1)^2$ *distinct points. By* (2.15), *deg* $t_s \leqslant \deg f - 2$. *If* $t_s \neq 0$ then $t_s(Q) = 0$ at most $(\deg f - 2)(\deg f - 2)$ points $Q \in S$ by (2.9). Hence $t_s = 0$.

The case $\deg f \equiv 0 \pmod p$ requires a bit more effort. For each $s = 1, \dots, n-1$, let $m(s)$ be the smallest positive integer m such that $p^m s > n$. We proceed by induction to show that $t_s = 0$.

If $m = 1$, then $ps = nq + r$ where $q, r \in \mathbb{Z}^+, r < n$. By (2.14) $D^{p-1}t_r - a_0 t_r = -t_s^p f^q$. The degree of the left side of the equality is at most $p(\deg f - 2)$ by (2.6) and (2.15). Since $q \geqslant 1$, we obtain $\deg t_s \leqslant \deg f - 3$. By (2.9) and the fact that S has $(\deg f)^2 - 3\deg f + 3$ points, we have $t_s = 0$.

Assume that $t_s = 0$ whenever $m(s) < d$ and $1 \leqslant s_0 < n$ with $m(s_0) = d \geqslant 2$. By (2.14), $D^{p-1}t_{ps_0} - a_0 t_{ps_0} = -t_{s_0}^p$. Since $m(ps_0) = m(s_0) - 1$, $t_{ps_0} = 0$; hence $t_{s_0} = 0$. From this it follows that $\hat{t} = t_0 \in K[x, y]$. In the introduction we mentioned that $\mathrm{Cl}(A_p) = 0$ for a generic g of degree $\geqslant 4$, which shows $t_0 = 0$ by (2.13). □

3.8. THEOREM. *For a generic f of degree at least 4 the following two statements are equivalent*:

(1) $\mathrm{Cl}(A_n) = 0$ *for all* $n \in \mathbb{Z}^+$;

(2) $\mathrm{Cl}(A_n) = 0$ *for all* $n \in \mathbb{Z}^+$ *where* $\gcd(p, n) = 1$.

Proof. By (2.16) and (3.7). □

4. References

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