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M. DE WILDE
P. B. A. LECOMTE

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An homotopy formula for the Hochschild cohomology

M. DE WILDE and P.B.A. LECOMTE
Université de Liège, Institut de Mathématique, Avenue des Tilleuls 15, B-4000 Liège, Belgium, dewilde@math.ulg.ac.be, lecomte@math.ulg.ac.be

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1. Introduction

In [6], S. Gutt describes an explicit right inverse for the Hochschild coboundary acting on the space of 2-cochains of the algebra of smooth functions of a manifold. It allows her to construct a star-product on the cotangent bundle of a Lie group.

In this note, we produce a general explicit homotopy formula for the Hochschild cohomology of the algebra of smooth functions. We restrict to local cochains, i.e. locally multidifferential operators (see Section 2 for more details). The inclusion of that complex in the full Hochschild complex induces an isomorphism in cohomology. This cohomology is of course well known since a long time, see for instance [2, 10]. It can be identified canonically with the skew multiderivations, i.e. cochains which are skew and which are derivations as a function of each single argument (see [7] for a similar result in homology). But to our knowledge, no such formula were available although it could be very useful to have in hand an explicit homotopy operator when constructing star-products on particular symplectic or Poisson manifolds.

We illustrate this by using our formula to produce a star-product on the dual of a Lie algebra in such a way that the powers of the elements of the algebra are the same as their deformed powers.

This property characterizes the deformation. Indeed, we show under quite general conditions that if a graded algebra is generated by its elements of degree $\leqslant 1$, then a deformation of that algebra that does not alter the powers of the generators is completely determined by its values on them.

This allows us to recover in particular the fact that the universal enveloping algebra of a Lie algebra is the evaluation of a star-product on its dual at an appropriate value of the deformation parameter as follows for instance from [6].

For general facts about star-product, the reader is referred to [1, 3, 5].

We are indebted to the referee for useful remarks and references to earlier results and algebraic literature on cohomology.

2. The graded Lie algebra of the space of smooth functions

Let M be a smooth, Hausdorff second countable manifold. Denote by N the space of smooth complex valued functions on M. We set $\mathcal{M} = \oplus_{a\in\mathbb{Z}}\mathcal{M}^a$ where $\mathcal{M}^a$ denotes the space of $(a+1)$-linear local maps from N into itself.

Recall that a multilinear map A from N into itself is *local* if

$$\operatorname{supp} A(u_0,\ldots,u_a) \subset \bigcap_{0\leqslant i\leqslant a} \operatorname{supp} u_i$$

for all $u_0,\ldots,u_a \in N$, supp denoting the support. From the multilinear version [2] of a well-known theorem of Peetre [9], if A is local, then it is locally a multi-differential operator. Thus, $\mathcal{M}$ is the space of cochains of N the restrictions of which to the relatively compact open subsets of M are differential. The space $\mathcal{M}$ is equipped with a $\mathbb{Z}$-graded Lie algebra structure $\triangle$. It has been first introduced by M. Gerstenhaber and rediscovered by the authors [3, 4, 5]. We recall its definition.

For A, $B \in \mathcal{M}$ of degree a and b respectively

$$A \triangle B = i_B A - (-1)^{ab} i_A B,$$

where $i\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is defined by $i_A B = 0$ if $b=-1$ and by

$$\begin{aligned}&(i_A B)(u_0,\ldots,u_{a+b})\\ &\quad= \sum_{j=0}^{b}(-1)^{ja}B(u_0,\ldots,u_{j-1},A(u_j,\ldots,u_{j+a}),\ldots,u_{a+b})\end{aligned}$$

otherwise. Using $\triangle$, it is easy to express the Hochschild coboundary operator $\delta\colon\mathcal{M}\to\mathcal{M}$ [4]. Indeed, as easily seen, for $A\in\mathcal{M}^a$

$$\delta A = (-1)^a m \triangle A,$$

where $m\colon N\times N\to N$ is the usual multiplication of functions.

We shall use in the sequel the cup-product of $\mathcal{M}$. We denote it by $(A, B)\to AB$. It is defined by

$$(AB)(u_0,\ldots,u_{a+b+1}) = A(u_0,\ldots,u_a)B(u_{a+1},\ldots,u_{a+b+1}).$$

It is associative and homogeneous of weight 0 with respect to the natural grading of $\mathcal{M}$ in which the degree of A is the number of its arguments.

The maps $i_A (A\in\mathcal{M}^a)$ and δ are easily seen to be graded derivations of the cup-product of degree a and 1 respectively. This means that

$$i_A(BC) = (i_A B)C + (-1)^{a(b+1)} B i_A C,$$

and

$$\delta(BC) = (\delta B)C + (-1)^{b+1} B\delta C,$$

for $B\in\mathcal{M}^b$ and $C\in\mathcal{M}$.

3. Reduction to the cochains vanishing on the constants

Recall that $A \in \mathcal{M}^a$ is said to be vanishing on the constants (nc in short) if $A(u_0, \dots, u_a)$ vanishes whenever one of the u_i's is constant or if $a = -1$. The space of nc elements of $\mathcal{M}$ will be denoted by $\mathcal{M}_{nc}$. It is obvious that it is a graded Lie subalgebra of $\mathcal{M}$ and a subalgebra for the cup-product. In algebraic literature on cohomology, the nc cochains are called normalized. Their inclusion into the total Hochschild complex is known to induce an isomorphism of cohomology [8, Chap. X. Section 2]. Here is a short elementary proof of that fact.

For $A \in \mathcal{M}^a$, define $\mathcal{P}A$ by

$$(\mathcal{P}A)(u_0, \dots, u_a) = \sum_{k=0}^{a} (-1)^k$$

$$\sum_{i_1 < \dots < i_k} u_{i_1} \cdots u_{i_k} A\left(u_0, \dots, \underset{(i_1)}{1}, \dots, \underset{(i_k)}{1}, \dots, u_a\right).$$

LEMMA 3.1. (i) *$\mathcal{P}A$ is nc for each A and $\mathcal{P}A = A$ if A is nc.*
(ii) *$\mathcal{P}(AB) = \mathcal{P}(A)\mathcal{P}(B)$.*
(iii) *$\delta\mathcal{P} = \mathcal{P}\delta$. In particular $\delta\mathcal{M}_{nc} \subset \mathcal{M}_{nc}$ and $\delta \ker \mathcal{P} \subset \ker \mathcal{P}$.*

Proof. (i) and (ii) are obvious. Let $A \in \mathcal{M}^a$ and $B \in \mathcal{M}$ be such that $\delta\mathcal{P}A = \mathcal{P}\delta A$ and $\delta\mathcal{P}B = \mathcal{P}\delta B$. One has

$$\begin{aligned} \delta\mathcal{P}(AB) &= (\delta\mathcal{P}A)\mathcal{P}B + (-1)^{a+1}(\mathcal{P}A)\delta\mathcal{P}B \\ &= (\mathcal{P}\delta A)\mathcal{P}B + (-1)^{a+1}(\mathcal{P}A)\mathcal{P}\delta B = \mathcal{P}\delta(AB). \end{aligned}$$

Since, as easily checked, $\delta\mathcal{P}A = \mathcal{P}\delta A$ for $a \leqslant 0$, it follows by induction on a that $\delta\mathcal{P}A = \mathcal{P}\delta A$ for each A. Indeed, δ and $\mathcal{P}$ are local operators on $\mathcal{M}$ and each A of degree $a > 0$ may be locally written down as a sum of product of elements of degree less than a. □

The elements of $\ker \mathcal{P}$ are generated by

$$\mathrm{id}^{b_0} A_0 \cdots \mathrm{id}^{b_s} A_s, \tag{$*$}$$

where the A_i's are nc, $b_1, \dots, b_s > 0$ and $a_0, \dots, a_{s-1} > -1$.

We define k by

$$k(\mathrm{id}^b A) = \begin{cases} -\mathrm{id}^{b-1} A & \text{for} \quad b > 0 \\ 0 & \text{for} \quad b = 0 \end{cases},$$

and

$$k(AB) = (kA)B + (-1)^{a+1} A k B,$$

provided

$$A(\dots,1)=0 \quad \text{and} \quad B(1,\dots)\neq 0.$$

Recall that

$$\delta\,\mathrm{id}^{2b}=0 \quad \text{and} \quad \delta\,\mathrm{id}^{2b-1}=-\mathrm{id}^{2b}.$$

It is easily seen, by induction on s, that

$$(k\delta+\delta k)(\mathrm{id}^{b_0}A_0\cdots\mathrm{id}^{b_s}A_s)=\begin{cases} s & \mathrm{id}^{b_0}A_0\cdots\mathrm{id}^{b_s}A_s \quad \text{if } b_0=0,\\ (s+1) & \mathrm{id}^{b_0}A_0\cdots\mathrm{id}^{b_s}A_s \quad \text{if } b_0\neq 0.\end{cases}$$

We have thus proved.

PROPOSITION 3.2. *On* $\ker\mathcal{P}$,

$$(k\circ\delta+\delta\circ k)A=n_A A,$$

where n_A *is the number of disjoint blocks* id^b *occurring in* A.

4. The complex $(\mathcal{M}_{nc},\delta)$

We restrict us to a domain of chart $(U,(x^1,\dots,x^m))$ of M, where U is relatively compact. On U, each $A\in\mathcal{M}^a$ reads

$$A(u_0,\dots,u_a)=\sum_{\alpha_0,\dots,\alpha_a}A_{\alpha_0,\dots,\alpha_a}D^{\alpha_0}u_0\cdots D^{\alpha_a}u_a,$$

where

$$D^\alpha u=D_1^{\alpha^1}\cdots D_m^{\alpha^m}u$$

if $\alpha=(\alpha^1,\dots,\alpha^m)\in\mathbb{N}^m$ and D_i is the partial derivative with respect to the ith local coordinate. We set $|\alpha|=\alpha^1+\cdots+\alpha^m$ and call $|\alpha_0|+\cdots+|\alpha_a|$ the order of $D^{\alpha_0}u_0\cdots D^{\alpha_a}u_a$.

Define the following maps on $\mathcal{M}_{nc}$:

$$(\Phi A)(u_0,\dots,u_{a-1})=\sum_{t=1}^m \sum_{i<j<a}(-1)^i A(u_0,\dots,u_{i-1},x^t,\dots,D_t u_j,\dots,u_{a-1}),$$

and

$$\Psi A=\sum_{t=1}^m(x^t\bigtriangleup A)D_t=(-1)^{a+1}\sum_{t=1}^m(i_{x^t}A)D_t,$$

where x^t and D_t are considered as elements of $\mathcal{M}^{-1}$ and $\mathcal{M}^0$ respectively.

PROPOSITION 4.1. *If* $A \in \mathcal{M}_{nc}$ *is of order* r, *then*

$$(\Phi \circ \delta + \delta \circ \Phi)A = -(r + \Psi)A.$$

Proof. We prove the formula by induction on the degree a of A. For $a = 0$, it reduces to the identity

$$\sum_{t=1}^{m} (x^t D^\alpha D_t u - D^\alpha(x^t D_t u)) = -|\alpha| D^\alpha u.$$

Next

$$\Psi(AB) = (\Psi A)B + A\Psi(B) + (-1)^a \sum_{t=1}^{m} (i_t A)(D_t B + (-1)^b B D_t),$$

where i_t stands for i_{x^t}. Moreover

$$\begin{aligned}(\Phi \circ \delta + \delta \circ \Phi)(AB) &= ((\Phi \circ \delta + \delta \circ \Phi)A)B + A(\Phi \circ \delta + \delta \circ \Phi)B \\ &+ \sum_{t=1}^{m} ((i_t \circ \delta + \delta \circ i_t)A) i_{D_t} B \\ &+ \sum_{t=1}^{m} (-1)^a (i_t A)(\delta \circ i_{D_t} - i_{D_t} \circ \delta)B.\end{aligned}$$

As graded commutators of derivations of the cup-product, $i_t \circ \delta + \delta \circ i_t$ and $\delta \circ i_{D_t} - i_{D_t} \circ \delta$ are derivations too. The first obviously vanishes on $\mathcal{M}_{nc}^{-1} \oplus \mathcal{M}_{nc}^0$. It thus vanishes on $\mathcal{M}_{nc}$. In addition

$$A \to -(D_t A + (-1)^a A D_t)$$

is also a derivation. As it coincides with $\delta \circ i_{D_t} - i_{D_t}\delta$ on $\mathcal{M}_{nc}^{-1} \oplus \mathcal{M}_{nc}^0$, it coincides with it on the whole $\mathcal{M}_{nc}$. Proposition 4.1 follows then immediately. □

Define now Ψ_s by

$$\Psi_s A = \sum_{t_1,\ldots,t_s=1}^{m} (ad_\triangle x^{t_1} \circ \cdots \circ ad_\triangle x^{t_s}(A)) D_{t_1} \cdots D_{t_s},$$

where $ad_\triangle X$ is the adjoint action $A \to X \triangle A$. When $s = a + 1$, then $\Psi_s A$ is a differential cochain of order 1 in each argument. It is skew-symmetric because

$$ad_\triangle x \circ ad_\triangle y + ad_\triangle y \circ ad_\triangle x = ad_\triangle(x \triangle y)$$

vanishes when the degrees of x and y are -1. We can thus write

$$\Psi_s A = \sum_{t_1<\cdots<t_s} (ad_\triangle x^{t_1} \circ \cdots \circ ad_\triangle x^{t_s}(A)) D_{t_1} \wedge \cdots \wedge D_{t_s}.$$

PROPOSITION 4.2. (i) $\Psi_{s+1} = \Psi \circ \Psi_s + s\Psi_s$,

(ii) $((-1)^s/s!)\Psi_s \colon \mathcal{M}_{nc}^{s-1} \to \mathcal{M}_{nc}^{s-1}$ *is the projector onto the skew-symmetric cochains of order* 1 *in each argument, vanishing on the cochains of order* > 1 *in some argument or symmetric with respect to at least two of their arguments,*

(iii) $\Psi_s \circ \delta = \delta \circ \Psi_s$.

Proof. (i) Since, for x and B of degree -1 and b respectively,

$$x \triangle (AB) = (-1)^{b+1}(x \triangle A)B + A(x \triangle B),$$

one has

$$\Psi \circ \Psi_s A = \sum_{t,t_1,\ldots,t_s} (-1)^s (ad_\triangle x^t \circ ad_\triangle x^{t_1} \circ \cdots \circ ad_\triangle x^{t_s} A) D_{t_1} \cdots D_{t_s} D_t$$
$$+ \sum_{t,t_1,\ldots,t_s} (ad_\triangle x^{t_1} \circ \cdots \circ ad_\triangle x^{t_s} A) x^t \triangle (D_{t_1} \cdots D_{t_s} D_t).$$

The first term is $\Psi_{s+1} A$. As

$$x^t \triangle (D_{t_1} \cdots D_{t_s} D_t) = \sum_{i=1}^{s} (-1)^{s+i-1} \delta_{t_i,t} D_{t_1} \cdots \hat{t_i} \cdots D_{t_s} D_{t_i},$$

the second term is $-s\Psi_s A$.

(ii) Let $A \in \mathcal{M}_{nc}^{s-1}$ be given. It is clear that $\Psi_s A = 0$ if A is of order $\neq 1$ in some of its arguments or if it is symmetric with respect to at least two of its arguments.

Moreover, it is clear that

$$\Psi_s(D_{t_1} \wedge \cdots \wedge D_{t_s}) = (-1)^s s! D_{t_1} \wedge \cdots \wedge D_{t_s}.$$

(iii) It follows from a straightforward computation that $\Psi = \Psi_1$ and δ commute. By (i) above, Ψ_s and δ commute too. □

PROPOSITION 4.3. *Let* Φ_s, $s \leqslant p$, *be defined by induction by*

$$\Phi_1 = -\frac{1}{r}\Phi, \; \Phi_{s+1} = -\frac{1}{r-s}((\Psi + s)\Phi_s + \Phi)$$

on the space of cochains of order r *of* $\mathcal{M}_{nc}^{p-1}$. *Then*

$$\mathrm{id} - \alpha = K \circ \delta + \delta \circ K$$

on $\mathcal{M}_{nc}$, where αA is the skew-symmetrization of the terms of order 1 *in each argument of A and where $K = \Phi_p$ on $\mathcal{M}^p_{nc}$.*

Proof. The formula

$$\mathrm{id} - \alpha_s \Psi_s = \Phi_s \circ \delta + \delta \circ \Phi_s$$

reduces to that of Proposition 4.1 for $s = 1$, $\alpha_1 = -r^{-1}$ and $\Phi_1 = \alpha_1 \Phi$. If it is true for s, applying Ψ to both members and using (ii) of Proposition 4.2, we see that it is true for $s + 1$ if

$$\alpha_{s+1} = -(r - s)^{-1} \alpha_s,$$

and

$$\Phi_{s+1} = -(r - s)^{-1}((\Psi + s)\Phi_s + \Phi).$$

Hence, it is true for $s \leqslant p$ on $\mathcal{M}^{p-1}_{nc}$ since the order r of $A \in \mathcal{M}^{p-1}_{nc}$ is at least p, with $\alpha_s = (-1)^s (r - s)!/r!$

Let $A \in \mathcal{M}^{p-1}_{nc}$. If it is of order $> p$, then $\Psi_p A = 0$. If it is of order p, it is of order 1 in each argument and, by Proposition 4.2 again, $\Psi_p A = (-1)^p p! \alpha A$. Hence the result. □

5. Application to the dual of a Lie algebra

Let L^* be the dual of the Lie algebra L, endowed with the Poisson structure

$$P(u, v)|_\xi = \langle \xi, [du, dv] \rangle.$$

We want to use the homotopy formula above to construct a $*$-product on L^*. Recall that a $*$-product is an associative formal deformation of the usual product

$$m_\lambda(u, v) = \sum_{k=0}^{\infty} \lambda^k C_k(u, v), \quad u, v \in N,$$

with

(i) $C_0 = m$, $C_1 = P$,

(ii) $C_k (k > 0)$ are bidifferential maps vanishing on the constants,

(iii) $C_k(u, v) = (-1)^k C_k(v, u)$, $\forall k, u, v$.

The associativity of m_λ is expressed by

$$J_k = \sum_{\substack{i+j=k \\ i,j>0}} C_i \bigtriangleup C_j = 2\delta C_k, \tag{1}$$

for each k. If (1) is true for $k < t$, it is known that $\delta J_t = 0$. So, to construct m_λ, we must manage to obtain, at each step, a J_t which is a coboundary, i.e. such that $\alpha J_t = 0$.

A way to choose C_t is then to apply our homotopy formula. However, the symmetry condition on C_t must be satisfied.

LEMMA 5.1. *Consider the permutations*

$$\tau : (0, 1) \to (1, 0) \quad \textit{and} \quad \nu : (0, 1, 2) \to (2, 1, 0).$$

Then

$$-\nu \circ \delta = \delta \circ \tau,$$

hence

$$\left(\frac{1 \pm \nu}{2}\right) \circ \delta = \delta \circ \left(\frac{1 \mp \tau}{2}\right).$$

Proof. Straightforward. □

It is easy to see that if the C_k's, $k < t$, have the required symmetry, then

$$J_t = \tfrac{1}{2}(1 - (-1)^t \nu) J_t.$$

Thus, if $J_t = 2\delta C_t$

$$J_t = \delta(1 + (-1)^t \tau) C_t,$$

and $(1 + (-1)^t \tau) C_t$ has the required symmetry.

Since $(K - \tau \circ K \circ \nu) J_t$ (see Proposition 4.3) is symmetric or skew-symmetric when t is even or odd, defining

$$C_t = (K - \tau \circ K \circ \nu) J_t$$

gives a way to construct m_λ provided that one proves that J_t is a coboundary for each t.

Let $\xi_1, \ldots, \xi_m$ be coordinates on L^* and let

$$E = \sum_i \xi_i D_{\xi_i}$$

be the Euler vector field of L^*. It is clear that

$$L_E \xi^\alpha = |\alpha| \xi^\alpha \quad \text{and} \quad L_E D^\alpha = -|\alpha| D^\alpha.$$

Moreover it is a derivation both for the composition of maps and for $\triangle$. A cochain C will be said *homogeneous of weight k* if $L_E C = kC$.

THEOREM 5.2. *The space L^* has a unique $*$-product m_λ such that its terms C_k are homogeneous of weight $-k$ and that*

$$u^{*_\lambda p} = u^p,$$

for all $u \in L$. It is given by induction by

$$C_t = (K - \tau \circ K \circ \nu)J_t.$$

Proof. We show by induction that the above formula defines a $*$-product m_λ such that the C_i's $(i \geqslant 1)$ have polynomial coefficients of order > 1 and are homogeneous of weight $-i$. If it is true for $i < t$, then $L_E J_t = -tJ_t$. For $t \geqslant 3$, it follows that J_t has no terms of order 1 in each arguments. Hence $\alpha J_t = 0$ and J_t is a coboundary. Since K preserves the coefficients and the order of the cochain on which it is applied, $C_t = (K - \tau \circ K \circ \tau)J_t$ has the required properties. As easily seen, $\alpha J_2 = \alpha(P \triangle P) = 0$. Hence the existence of m_λ.

It is easy to see, by induction, that if

$$C_1(u, v) = \sum_{i,j} P_{ij} D_i u D_j v,$$

then C_k is a linear combination of terms

$$P_{i_1 j_1} D^{\alpha_1} P_{i_2 j_2} \cdots D^{\alpha_{k-1}} P_{i_k j_k} D^\beta u D^\gamma v, \tag{2}$$

where

$$\alpha_1 + \cdots + \alpha_{k-1} + \beta + \gamma = \sum_{s=1}^{k} (e_{i_s} + e_{j_s})$$

(e_ℓ denotes the ℓth unit vector in $\mathbb{N}^m$). We prove by induction that $u^{*_\lambda p} = u^p$ when u is linear. It is true for $p = 1$. If it is true for p, setting

$$u(\xi) = \sum_i t_i \xi_i,$$

we show that the coefficients $C_k(u, u^p)$ of

$$m_\lambda(u, u^{*_\lambda p}) = u^{*_\lambda (p+1)}$$

are vanishing for $k \geqslant 1$. In fact, $C_k(u, u^p)$ is a linear combination of terms like (2) where β is of length 1. Since C_k is homogeneous of weight $-k$, one has

$|\beta + \gamma| \geqslant k + 1$. Thus $\beta + \gamma \geqslant e_{i_\ell} + e_{j_\ell}$ for some ℓ. In particular (2) contains a factor

$$D^{\alpha_\ell} P_{i_\ell j_\ell} t_{i_\ell} t_{j_\ell}$$

and therefore vanishes as C_1 is skewsymmetric.

The uniqueness of m_λ follows from the next property. □

PROPOSITION 5.3. *Let*

$$A = \bigoplus_{p \geqslant 0} A_p$$

be an associative commutative graded algebra with unit $e \in A_0$ *and generated by* A_1. *An associative deformation*

$$u * v = uv + \sum_{i>0} C_i(u, v)$$

of the product of A *such that*

(i) $u \in A_p,\ v \in A_q \Rightarrow C_i(u, v) \in A_{p+q-i}$,

(ii) $u^{*p} = u^p$ *for each* $u \in A_1$,

is uniquely determined by its values on A_1.

In particular, if C_2 *vanishes on* $A_1 \times A_1$, $*$ *is uniquely determined by* C_1 *on* $A_1 \times A_1$.

Proof. If the deformed product m is known on $(\oplus_{i \leqslant p} A_i) \times A$, it is known on $A_{p+1} \times A$. Indeed, A_{p+1} is generated by $A_1 A_p$. If $x = x_1 x_p (x_i \in A_i)$, we have $x = x_1 * x_p + x'$ with $x' \in \oplus_{i \leqslant p} A_i$. Thus

$$x * y = x_1 * (x_p * y) + x' * y$$

is known for each $y \in A$.

If m is known on $A_1 \times (\oplus_{i<p} A_p)$, it is known on $A_1 \times A_p$. Given $x, y \in A_1$, equating the terms of order 1 in x in the equality $(x + y)^{*p+1} = (x + y)^{p+1}$ gives

$$\sum_{i=0}^{p} y^{*i} * x * y^{*(p-i)} = (p+1) x y^p.$$

The left-hand side reads

$$\sum_{j=0}^{p} C_{p+1}^{j+1} (ad_*(y)^j x) * y^{*(p-j)}.$$

Thus

$$(p+1)x * y^p = (p+1)xy^p - \sum_{j=1}^{p} C_{p+1}^{j+1}(ad_*(y)^j x) * y^{*(p-j)}$$

is known. Since A is commutative, A_p is generated by y^p, $y \in A_1$.

Thus m is determined by the values on $A_1 \times A_1$. Observe that $C_i = 0$ on $A_1 \times A_1$ for $i > 2$, hence the particular case. □

COROLLARY 5.4. *Let $S(L)$ be the algebra of polynomials over L^*. Denote by m the product induced on $S(L)$ by the $*$-product m_λ of Proposition 5.2 evaluated at $\lambda = \frac{1}{2}$. Then $(S(L), m)$ is a model of the universal enveloping algebra of L.*

Proof. There exists indeed a product $\circ$ on $S(L)$ such that $(S(L), \circ)$ is a model of the universal enveloping algebra of L. It reads

$$u \circ v = uv + \sum_{k>0} C_k(u, v),$$

where C_k maps $S_p(L) \times S_q(L)$ into $S_{p+q-k}(L)$. The term C_1 is

$$(u, v) \to \frac{1}{2}\langle \xi, [du, dv] \rangle$$

and, as well known, $u^{\circ p} = u^p$ and $C_k(u, v) = 0$ if u, v are linear and $k > 1$. Hence the result by Proposition 5.3. □

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