COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 97, nº 1-2 (1995), p. 157-160

<http://www.numdam.org/item?id=CM_1995__97_1-2_157_0>

Compositio Mathematica **97**: 157–160, 1995.

Examples of liftings of surfaces and a problem in de Rham cohomology

Dedicated to Frans Oort on the occasion of his 60th birthday

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Received 26 August 1994; accepted in final form 7 October 1994

1. Introduction

In 1987, Deligne and Illusie [3] proved that if X is a projective smooth variety over a field k of characteristic $p > 0$ such that X can be lifted to a smooth scheme over the Witt vectors $W(k)$, then the Hodge-de Rham spectral sequence for de Rham cohomology degenerates. Ogus and Illusie have asked whether this remains true if X is liftable over R, a discrete valuation ring ramified over $W(k)$. In this paper, we show that the answer to this question is no. We give an example in each characteristic $p > 0$ of a surface whose Hodge-de Rham spectral sequence does not degenerate, and which lifts over a discrete valuation ring R of characteristic zero with $e = 2$, where e is the ramification index of R over $W(k)$. Thus, the ramification is as small as possible, given the theorem of Deligne–Illusie. This is of interest since some of the pathologies of characteristic p algebraic geometry do not occur for varieties which are liftable over discrete valuation rings satisfying $e < p - 1$ (see, for instance, [9]).

We give the main examples in Section 1. The key is to use a construction due to Raynaud [9] (generalizing work of Godeaux and Serre) of surfaces which are quotients of complete intersections by finite group schemes. Sections 2 and 3 are examples of hyperelliptic surfaces in characteristics two and three which exhibit the same behavior. These are included since the construction is somewhat more explicit than that used in Section 1.

1.1. EXAMPLES IN ALL CHARACTERISTICS

The key to our construction is the following theorem of Raynaud.

THEOREM 1 (Raynaud). *Let S be a local scheme and let G be a finite flat group scheme over S. There exists a projective space P over S with a linear action of G which contains a relative complete intersection surface Y which is stabilized by G and such that: (1) G operates freely on Y; (2) $Y = X/G$ is smooth over*

S. Moreover, if $S = \mathrm{Spec}(R)$, where R is a complete discrete valuation ring of characteristic 0 with algebraically closed residue field of characteristic p, and X_0 is the special fibre of X and G_0 is the special fibre of G, then $\mathrm{Pic}^\tau(X_0) \cong G_0^\vee$, where $G_0^\vee$ is the Cartier dual of G_0.

Proof. See [9], 4.2.3 and 4.2.6.

Apply this result to the case where R is a complete discrete valuation ring of characteristic zero with algebraically closed residue field of characteristic p, $S = \mathrm{Spec}(R)$, and G is a finite flat group scheme over S such that $G_0 \cong \alpha_p$. Thanks to a theorem of Tate and Oort [11], we know that G will exist if and only if we can find elements a and c in the maximal ideal of R such that $ac = p$. Thus, we can choose R so that the ramification index e of R over $W(k)$ is 2.

Since α_p is self-dual, $\mathrm{Pic}^\tau(X_0) \cong \alpha_p$. We now use the following theorem to conclude the Hodge-de Rham spectral sequence of X_0 does not degenerate.

THEOREM 2. *Let Z be a smooth projective variety over an algebraically closed field of characteristic $p > 0$ such that $\mathrm{Pic}^\tau(Z) \cong \alpha_p$. Then the Hodge-de Rham spectral sequence of Z does not degenerate.*

Proof. (Compare [6] and [5], where this is done in the special case of α_2-Enriques surfaces in characteristic 2.) If Z has global 1-forms which are not closed, then certainly the Hodge-de Rham spectral sequence does not degenerate. If all 1-forms are closed, then by a theorem of Oda [8], $H^1_{\mathrm{DR}}(Z) \cong DM(p\ \mathrm{Pic}^\tau(X_0))$. In our case, this gives $h^1_{\mathrm{DR}}(Z) = 1$. But since $H^1(0_Z)$ is the tangent space to the Picard scheme, $h^1(0_Z) = 1$, while $h^0(\Omega^1) \geqslant 1$, since α_p in the Picard scheme leads to a global differential form on Z. Therefore the Hodge-de Rham spectral sequence does not degenerate.

REMARK. One can also prove the non-degeneration of the Hodge-de Rham spectral sequence using Theorem 1 of [7].

To summarize, the surface X_0 constructed above has non-degenerate Hodge-de Rham spectral sequence, and lifts over R, a complete discrete valuation ring of characteristic zero with $e = 2$.

1.2. A HYPERELLIPTIC EXAMPLE IN CHARACTERISTIC TWO

Our example is a hyperelliptic surface X_0 of type(cl) in characteristic two, in the terminology of [2]. This is constructed by taking $X_0 = (C_0 \times D_0)/(\mathbf{Z}/4)$ where C_0 and D_0 are elliptic curves, $j(D_0) = 12^3$, and $\mathbf{Z}/4$ acts by $(z, w) \to (z + a, f(w))$ where a is a point of order 4 on C_0, and $f\colon D_0 \to D_0$ is an automorphism of order 4. By [6], Theorem 4.11, the Hodge-de Rham spectral sequence of X_0 does not degenerate.

For definiteness, we take D_0 to be the elliptic curve with Weierstrass equation $y^2 + y = x^3 + x^2$, and f to be the automorphism defined by $f(x, y) = (x + 1, y + x + 1)$. Note that the pair (D_0, f) is defined over F_2. We lift this to a pair (D, F) over the Gaussian integers $\mathbf{Z}[i]$ by letting D be the elliptic curve with Weierstrass equation $y^2 + (1 - i)xy - iy = x^3 - ix^2$, and letting F be the automorphism defined by $F(x, y) = (-x + i, -iy - ix - 1)$. (it is amusing to note the elliptic curve D has bad reduction only at primes dividing 5.) The ideal $p = (1 + i)$ is prime in $\mathbf{Z}[i]$. If we localize $\mathbf{Z}[i]$ at this prime and then complete, we get a complete discrete valuation ring R with maximal ideal m such that R is totally ramified over $W(F_2)$ of degree 2 and such that $R/m \cong F_2$. The pair (D, F) can be considered as a lifting of (D_0, f) over $S = \operatorname{Spec}(R)$.

Now take (C_0, a) to be an elliptic curve defined over F_2 together with an F_2-rational point of order 4. There is only one possibility for C_0, namely the curve with Weierstrass equation $y^2 + xy = x^3 + 1$ (see [4], Sect. 3.6). There are two possibilities for a, $a = (1, 1)$ or $(1, 0)$. We take $a = (1, 1)$. Lift the pair (C_0, a) to a pair (C, A) over R, where A is a point of order 4 on C. For instance, we can take for C the curve with Weierstrass equation $y^2 + xy = x^3 - 2x + 3$ and $A = (1, 1)$. (This example was found with the help of the tables in [1].)

Let $X = (C \times D)/(\mathbf{Z}/4)$, where a generator of $\mathbf{Z}/4$ acts by $(z, w) \to (z + A, F(w))$. Then X is a smooth projective S-scheme lifting X_0.

1.3. A HYPERELLIPTIC EXAMPLE IN CHARACTERISTIC THREE

Our example is a hyperelliptic surface X_0 of type (b1) in characteristic three, in the terminology of [2]. This is constructed by taking $X_0 = (C_0 \times D_0)/(\mathbf{Z}/3)$, where C_0 and D_0 are elliptic curves, $j(D_0) = 0$, and $\mathbf{Z}/3$ acts by $(z, w) \to (z+a, f(w))$, where a is a point of order 3 on C_0, and $f: D_0 \to D_0$ is an automorphism of order 3. It is known that the Hodge-de Rham spectral sequence for X_0 does not degenerate ([6], Theorems 4.9 and 4.10).

For definiteness, we take D_0 to be the elliptic curve over F_3 defined by the Weierstrass equation $y^2 = x^3 - x$, and we let f be the automorphism defined by $f(x, y) = (x - 1, y)$.

We seek to lift the pair (D_0, f) to a pair (D, F) over $\mathbf{Z}[\omega]$, where ω is the primitive cube root of unity $\omega = (-1 + \sqrt{-3})/2$. With the help of the "Formulaire" for elliptic curves [1], we find that we can take D to be the elliptic curve with Weierstrass equation $y^2 = x^3 + \sqrt{-3x^2} - x$ and F to be the automorphism defined by $F(x, y) = (\omega^2 x - \omega, y)$.

The ideal $p = (1 - \omega)$ is prime in $\mathbf{Z}[\omega]$. If we localize $\mathbf{Z}[\omega]$ at this prime and then complete, we get a complete discrete valuation ring R with maximal ideal m such that R is totally ramified of degree 2 over $W(F_3)$ and such that $R/m \cong F_3$. The pair (D, F) can be considered as a lifting of the pair (D_0, f) over $S = \operatorname{Spec}(R)$.

Now take an elliptic curve C_0 defined over F_3 together with an F_3-rational point of order 3, and lift this pair to a pair (C, A) defined over R, where A is a point of order 3 on C. This can be done in many ways. One possibility is to take C to be the curve with Weierstrass equation $y^2 + xy + y = x^3$, $A = (0, 0)$ (see [10], Appendix A, Prop. 1.3).

Let $X = (C \times D)/(\mathbf{Z}/3)$, where a generator of $\mathbf{Z}/3$ acts by $(z, w) \to (z + A, F(w))$. Then X is a smooth projective S-scheme lifting X_0.

Acknowledgements

It is a pleasure for me to dedicate this paper to Frans Oort. His work has had a large direct and indirect influence on this project. In addition, he was very helpful to me in the early stages of my career, for which I thank him now.

Work on this paper was begun during a sabbatical year at the Mathematical Sciences Research Institute (MSRI) in Berkeley. I would like to thank MSRI for their hospitality, and MSRI and Brigham Young University for partial financial support.

It is a pleasure to thank A. Ogus for informing me of this problem and for many stimulating and helpful discussions. I would also like to thank L. Illusie and W. Messing for helpful conversations.

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