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Monodromy and weight filtration for smoothings of isolated singularities

Dedicated to Frans Oort on the occasion of his 60th birthday

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Abstract. We investigate the connection between monodromy and weight filtration for one-parameter smoothings of isolated singularities. We give a formula for the signature of the intersection form in terms of the Hodge numbers of the vanishing cohomology.

Key words: singularity, mixed Hodge structure, monodromy, weight filtration

1. Introduction

Let V be a finite dimensional vectorspace and let N be a nilpotent endomorphism of V. Then for each integer n there exists a unique decreasing filtration $W = W(N, n)$ of V such that $N(W_i) \subset W_{i-2}$ for each i and the induced map $N^i : Gr^W_{n+i} \to Gr^W_{n-i}$ is an isomorphism for all i.

If $F : Z \to \mathbf{C}$ is a flat projective morphism with smooth generic fiber, then associated to the critical value 0 we have a limit mixed Hodge structure $H^n(Z_F)$ whose weight filtration is equal to $W(N, n)$ where N is the logarithm of the unipotent part of the monodromy transformation T around 0.

A similar situation arises in the case of an isolated hypersurface singularity $f\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ and its vanishing cohomology $\tilde{H}^n(X_{f,0})$. Again we have a monodromy operator T, but now the description of the weight filtration is slightly more complicated: write

$$H^n(X_{f,0}) = H^n(X_{f,0})_1 \oplus H^n(X_{f,0})_{\neq 1}, \tag{1}$$

where $H^n(X_{f,0})_1$ (resp. $H^n(X_{f,0})_{\neq 1}$) is the subspace on which T acts with eigenvalue 1 (resp. eigenvalues $\neq$ 1). Then $W = W(N, n+1)$ on $H^n(X_{f,0})_1$ and $W = W(N, n)$ on $H^n(X_{f,0})_{\neq 1}$.

In this note we deal with the case of the weight filtration on the vanishing cohomology of a one-parameter smoothing of an isolated singularity. Part of the results were announced in [9] with a short indication of proof. In this general case the decomposition (1) has to be replaced by a suitable decomposition of $Gr^W H^n(X_{f,0})$.

We also give precise results about the polarizations on these summands and express the index of the intersection form (in the even-dimensional case) in terms of Hodge numbers. This generalizes and simplifies [8] Theorem 4.11 and [9] Theorem 2.23. The main tool in our proof is a strong globalization theorem for one-parameter smoothings of isolated singularities, in the spirit of the Appendix of [4].

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2. Monodromy and weight filtration

Let (X', x) be an isolated singularity of a complex space of pure dimension $n+1$, and $f\colon (X', x) \to (\mathbf{C}, 0)$ a holomorphic function germ. Suppose that $X := f^{-1}(0)$ has an isolated singularity at x. We let $X'_{f,x}$ denote the Milnor fibre of f at x. We first sharpen a globalization theorem due to Looijenga [4]:

THEOREM 1. *Let $f\colon (X', x) \to (\mathbf{C}, 0)$ be a smoothing of an isolated singularity of pure dimension n. Then there exists a flat projective morphism $F: Z \to \mathbf{C}$, a point $z \in Z_0$ and an isomorphism $h\colon (X', x) \to (Z, z)$ such that $F \circ h = f$ and F is smooth along $Z_0 \setminus \{z\}$ and such that the restriction mapping $H^n(Z_F, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. Here Z_F denotes the generic fibre of F.*

Proof. If $n = 0$ then f is finite, hence projective. So in the sequel we suppose that $n \geqslant 1$. We follow the proof of [4]. Let Y be an affine variety of dimension $n+1$ with a unique singular point y and P a regular function on Y such that the germ $f: (X', x) \to (\mathbf{C}, 0)$ is biholomorphic to $P: (Y, y) \to \mathbf{C}$. The existence of Y such that $(X', x) \simeq (Y, y)$ follows from work of Artin [1] and Hironaka [2], and the existence of a polynomial P with the desired properties follows from finite determinacy for germs with isolated singularities, due to Mather and Looijenga [4]. We assume Y to be embedded in affine N-space such that $y = 0$. Let $\mathbf{m}$ denote the ideal of regular functions on Y vanishing at y. Fix a positive integer k such that all germs $P + g$ for $g \in \mathbf{m}^k$ are analytically isomorphic to P. Let Z' denote the projective closure of Y. We may assume that $Z' \setminus \{y\}$ and $Z' \setminus Y = Z'_\infty$ are smooth.

Choose a sufficiently general (to be made precise below) homogeneous polynomial g of degree $d \geqslant k$ sufficiently big and let $Q = P + g$. Let $Z = \{(\xi, t) \in Z' \times \mathbf{C} \mid \xi_0^d Q(\xi_1/\xi_0, \ldots, \xi_N/\xi_0) = t\xi_0^d\}$. We embed Y in Z as the graph of Q and let $z = (y, 0)$. The projection F of Z onto the second factor provides a globalization of f. We will show that we can choose g in such a way that it has the desired properties. First we require that g defines a smooth hypersurface in $\mathbf{P}^{N-1}$ which is transverse to Z'_∞ and that z is the only critical point of F on $F^{-1}(0)$.

We fix a good Stein representative $f: X' \to \Delta$ for the germ f in the sense of [3] Chapter 2.B. Write $\Omega^{\cdot}_f = \Omega^{\cdot}_{X'}/df \wedge \Omega^{\cdot-1}_{X'}$. By [3] Theorem 8.7, the sheaf $\mathcal{H}^n f_*(\Omega^{\cdot}_f)$ is coherent. Let $\omega_Y = j_*\Omega^{n+1}_{Y\setminus\{y\}}$ where $j\colon Y\setminus\{y\} \to Y$ is the inclusion map. Put $Y_t = Q^{-1}(t)$. First observe that for $t \neq 0$ sufficiently small the restriction map $H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. This follows from the specialization sequence

$$H^n(Y_0, \mathbf{C}) \to H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C}) \to H^{n+1}(Y_0, \mathbf{C}),$$

(here we use that F has no critical point at infinity) and the fact that for an affine variety of dimension n the cohomology groups are zero in degrees $> n$. Moreover, for such t there is a natural map $\rho\colon H^0(Y, \omega_Y) \to H^0(Y_t, \Omega^n_{Y_t}) \to \mathcal{H}^n f_*(\Omega^{\cdot}_f)(t)$ which is the composition of the map $\eta \mapsto$ the restriction to Y_t of η/dP and the restriction to $X'_{f,x}$. Then ρ is the composition of two surjections, hence surjective. (The second map is surjective as $H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective.) Choose $\eta_1, \dots, \eta_r \in H^0(Y, \omega_Y)$ whose images generate $\mathcal{H}^n f_*(\Omega^{\cdot}_f)(t)$ for all $t \neq 0$ sufficiently small. If g is a small perturbation of f, they will still generate $\mathcal{H}^n g_*(\Omega^{\cdot}_g)(t)$ for all $t \neq 0$ sufficiently small, again by Looijenga's coherence theorem.

There exists $l \in \mathbf{N}$ such that $\eta_1, \dots, \eta_r$ extend to sections of $\omega_{Z'}(lZ_\infty)$. Let $D = Z'_\infty \cap Z'_0 = Z'_\infty \cap Z'_t$. Then $\eta_1/dQ, \dots, \eta_r/dQ$ extend to sections of $\Omega^n_{Z_t}((l-d)D)$. So if $d \geqslant l$ the map $H^0(Z_t, \Omega^n_{Z_t}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. Then a fortiori $H^n(Z_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective.

By [9] we have the following exact sequences of mixed Hodge structures associated with the Milnor fibre $X'_{f,x}$ of f at 0:

$$0 \to H^{n+1}_{\{x\}}(X') \to H^n(X'_{f,x})_1 \xrightarrow{V} H^n_c(X'_{f,x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0, \quad (2)$$

$$0 \to H^{n-1}(X'_{f,x}) \to H^n_{\{x\}}(X) \to H^n_c(X'_{f,x})$$

$$\xrightarrow{j} H^n(X'_{f,x}) \to H^{n+1}_{\{x\}}(X) \to H^{n+1}_c(X'_{f,x}) \to 0, \quad (3)$$

where the subscript 1 denotes the generalized eigenspace of T for the eigenvalue 1 and $jV = N = \log(T)$ (resp. $Vj = N_c = \log(T_c)$) on $H^n_c(X'_{f,x})_1$ (resp. $H^n(X'_{f,x})_1$). We recall

THEOREM 2.

$$Gr^W_i H^{n+1}_{\{x\}}(X') = 0 \quad \text{for } i \geqslant n+1; \quad (4)$$

$$Gr^W_i H^n_{\{x\}}(X) = 0 \quad \text{for } i \geqslant n; \quad (5)$$

$$Gr^W_i H^{n+2}_{\{x\}}(X') = 0 \quad \text{for } i \leqslant n+1; \quad (6)$$

$$Gr^W_i H^{n+1}_{\{x\}}(X) = 0 \quad \text{for } i \leqslant n. \quad (7)$$

See [9] Corollary 1.12. Both N and N_c map W_i to W_{i-2}.

THEOREM 3. *For all $i \geqslant 0$ the map*

$$N_c^i\colon\ Gr^W_{n+1+i}\,\mathrm{im}(V) \to Gr^W_{n+1-i}\,\mathrm{im}(V)$$

is an isomorphism.

REMARK 4. In the hypersurface case, i.e. when X' is smooth, the map V is an isomorphism and we recover [8] Corollary 4.9.

Proof. We choose a flat projective morphism $F\colon\ Z \to \mathbf{C}$, a point $z \in Z$ and an isomorphism $h\colon\ (X', x) \to (Z, z)$ such that $F \circ h = f$ and F is smooth along $Z_0 \setminus \{z\}$ as in Theorem 1. Let Z_F denote the generic fibre of F. Then one has the exact sequence of mixed Hodge structures

$$\to H^n(Z_0) \to H^n(Z_F) \to H^n(X'_{f,x}) \to 0, \tag{4}$$

where $H^n(Z_F)$ carries the limit mixed Hodge structure. There is a monodromy action T on this sequence, and T acts as the identity on $H^n(Z_0)$. We have the following sequence

$$H^n(Z_F)_1 \xrightarrow{k} H^n(X'_{f,x})_1 \xrightarrow{V} H^n_c(X'_{f,x})_1(-1) \xrightarrow{k^t} H^n(Z_F)_1(-1)$$

and $N = k^t \circ V \circ k$. As k is surjective, its transpose k^t is injective and defines an isomorphism of mixed Hodge structures $\mathrm{im}(V) \to \mathrm{im}(N)$ such that $k^t \circ N_c = N \circ k^t$. As $W = W(N, n)$ on $H^n(Z_F)_1$ we get that $W = W(N, n+1)$ on $\mathrm{im}(N)$. We conclude that $W = W(N_c, n+1)$ on $\mathrm{im}(V)$.

It follows that $Gr^W(\mathrm{im}(V))$ is completely determined by the kernel of N_c on $\mathrm{im}(V)$. In order to determine this kernel, observe that (4) implies that $\ker(V)$ has weights $\leqslant n$ and that (7) implies that $\mathrm{coker}(j)$ has weights $\geqslant n+1$. Hence $\ker(V) \subset \mathrm{im}(j)$. So we have the exact sequence

$$0 \to \ker(j) \to \ker(N_c) \xrightarrow{j} \ker(V) \to 0 \tag{5}$$

and hence $\ker(N_c)$ has weights $\leqslant n$. By considering the action of N_c on the exact sequence

$$0 \to \mathrm{im}(V) \to H^n(X'_{f,x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0$$

we obtain the exact sequence

$$0 \to \ker(N_c;\ \mathrm{im}(V)) \to \ker(N_c)(-1) \to W_{n+2}H^{n+2}_{\{x\}}(X') \to 0$$

and hence $\ker(N_c;\ \mathrm{im}(V)) = W_{n+1}(\ker(N_c)(-1))$. So from (5) we obtain

LEMMA 5. *We have the exact sequence of mixed Hodge structures*

$$0 \to \ker(j)(-1) \to \ker(N_c;\ \operatorname{im}(V)) \xrightarrow{j} W_{n+1}(\ker(V)(-1)) \to 0$$

THEOREM 6. *Regarding the map* $H^n_c(X'_{f,x}) \xrightarrow{j} H^n(X'_{f,x})$ *we have that*

$$N^i : Gr^W_{n+i} \operatorname{im}(j) \to Gr^W_{n-i} \operatorname{im}(j)$$

is an isomorphism for all $i \geqslant 0$, *i.e.* $W = W(N, n)$ *on* $\operatorname{im}(j)$.

Proof. Choose a globalization $F\colon\ Z \to \mathbf{C}$ of f as in the proof of Theorem 2. Then j is factorized as

$$H^n_c(X'_{f,x}) \xrightarrow{k^t} H^n(Z_F) \xrightarrow{k} H^n(X'_{f,x}).$$

Let $P^n(Z_F) = \ker(L\colon\ H^n(Z_F) \to H^{n+2}(Z_F))$ denote the primitive cohomology. Here L is the cup product with the hyperplane class. As a general hyperplane does not pass through the point x, the image of k^t is contained in $P^n(Z_F)$.

We have the nondegenerate pairing S on $P^n(Z_F)$, given by

$$S(x, y) = (-1)^{n(n-1)/2} \int_{Z_F} x \wedge y.$$

It is $(-1)^n$-symmetric, $W_\alpha = (W_{2n-1-\alpha})^\perp$ and $S(Nx, y) + S(x, Ny) = 0$. Moreover $N^\alpha : Gr^W_{n+\alpha} P^n(Z_F) \to Gr^W_{n-\alpha} P^n(Z_F)$ is an isomorphism for all $\alpha \geqslant 0$. If $P_{n+\alpha} := \ker(N^{\alpha+1} : Gr^W_{n+\alpha} P^n(Z_F) \to Gr^W_{n-\alpha-2} P^n(Z_F))$, the form $(x, y) \mapsto S(Cx, N^\alpha \overline{y})$ is hermitian positive definite on $P_{n+\alpha}$ by [7], Lemma 6.25.

Let $Q_\alpha = Gr^W_{n-\alpha} \ker(k) \subset Gr^W_{n-\alpha} P^n(Z_F)$. Then $Gr^W_{n+\alpha} \operatorname{im}(j) \simeq (Q_\alpha)^\perp$ as $Gr^W_{n+\alpha} \ker(j) = 0$. Therefore,

$$Gr^W_{n-\alpha} \operatorname{im}(j) \simeq N^\alpha (Q_\alpha)^\perp / Q_\alpha \cap N^\alpha (Q_\alpha)^\perp$$

so we have to show that

$$Q_\alpha \cap N^\alpha (Q_\alpha)^\perp = (0).$$

Clearly, $Q_\alpha \subset N^\alpha P_{n+\alpha}$ as $N = 0$ on $\ker(k)$. So let $x \in N^\alpha(Q_\alpha)^\perp \cap Q_\alpha$. Write $x = N^\alpha x'$ with $x' \in P_{n+\alpha} \cap (Q_\alpha)^\perp$. Then $S(Cx', N^\alpha \overline{x'}) = 0$ hence $x = 0$.

THEOREM 7. (i) *For all* $i > 0$ *the map*

$$V \circ N^{i-1}\colon\ Gr^W_{n+i} H^n(X'_{f,x})_1 \to Gr^W_{n-i} H^n_c(X'_{f,x})_1$$

is an isomorphism;

(ii) *for all* $i \geqslant 0$ *the map*

$$N^i \circ j\colon\ Gr^W_{n+i} H^n_c(X'_{f,x}) \to Gr^W_{n-i} H^n(X'_{f,x})$$

is an isomorphism.

Proof. For $i > 0$ we have $Gr^W_{n+i}\ker(V) = 0$ so

$$Gr^W_{n+i}H^n(X'_{f,x})_1 \simeq Gr^W_{n+i}\operatorname{im}(V).$$

This space is mapped isomorphically to $Gr^W_{n-i+2}\operatorname{im}(V)$ by N^{i-1} according to Theorem 3. As $\operatorname{coker}(V)$ has weights $\geqslant n+2$, we have

$$Gr^W_{n-i+2}\operatorname{im}(V) \simeq Gr^W_{n-i}H^n_c(X'_{f,x})_1.$$

This proves (i). Ones proves (ii) similarly using Theorem 6 instead of Theorem 3.

3. Primitive decomposition

Let V be a finite dimensional vector space and N a nilpotent endomorphism of V, n an integer and $W = W(N,n)$. Then we have the following decomposition of $Gr^W(V)$. Recall that $N^i\colon\ Gr^W_{n+i}(V) \to Gr^W_{n-i}(V)$ is an isomorphism for all $i \geqslant 0$. Put

$$P_{n+i} = \ker(N^{i+1}\colon\ Gr^W_{n+i}(V) \to Gr^W_{n-i-2}(V))$$

for $i \geqslant 0$ and 0 else. Then we have the primitive decomposition

$$Gr^W_\alpha(V) \simeq \bigoplus_{i\geqslant 0} N^i P_{\alpha+2i}.$$

We will give an analogous but more subtle decomposition of $Gr^W H^n(X'_{f,x})_1$ and $Gr^W H^n_c(X'_{f,x})_1$ (we use the same notation as in the preceding section). This was first mentioned in [6] and proved by Saito in a letter to the author. Define

$$B_{n+i} = \ker(N_c^{i+1}\colon\ Gr^W_{n+i}H^n_c(X'_{f,x})_1 \to Gr^W_{n-i-2}H^n_c(X'_{f,x})_1)$$

for $i \geqslant 0$ and 0 else, and

$$A_{n+i} = \ker(N^i\colon\ Gr^W_{n+i}H^n(X'_{f,x})_1 \to Gr^W_{n-i}H^n(X'_{f,x})_1)$$

for $i > 0$ and 0 else. By Theorem 7 B_{n+i} is mapped isomorphically to $Gr^W_{n-i}\ker(V)$ by $N^i \circ j$ and A_{n+i} is mapped isomorphically to $Gr^W_{n-i}\ker(j)$ by $V \circ N^{i-1}$.

THEOREM 8. *We have*

$$Gr^W_\alpha H^n_c(X'_{f,x})_1 = \bigoplus_{i\geqslant 0} N^i B_{\alpha+2i} \oplus \bigoplus_{i\geqslant 0} VN^i A_{\alpha+2+2i}$$

and

$$Gr^W_\alpha H^n(X'_{f,x})_1 = \bigoplus_{i\geqslant 0} N^i j B_{\alpha+2i} \oplus \bigoplus_{i\geqslant 0} N^i A_{\alpha+2i}.$$

Proof. Define a graded vectorspace C by $C_{2\alpha} = 0$ and

$$C_{2\alpha+1} = Gr^W_{\alpha+1} H^n(X'_{f,x})_1 \oplus Gr^W_\alpha H^n_c(X'_{f,x})_1.$$

Define an endomorphism λ of degree -2 of C as $\lambda(x, y) = (j(y), V(x))$. From Theorem 7 we obtain that for all $i \geqslant 0$ the map $\lambda^i : C_{2n+i} \to C_{2n-i}$ is an isomorphism. Hence, if $D_{2n+i} = \ker(\lambda^{i+1} \colon C_{2n+i} \to C_{2n-i-2})$ for $i \geqslant 0$ and 0 else, then we have that the map $\lambda^\alpha \colon D_{2n+i} \to C_{2n+i-\alpha}$ is injective for $\alpha \leqslant 2i$ and else the zero map. We obtain the primitive decomposition

$$C_\alpha = \bigoplus_{i \geqslant 0} \lambda^i D_{\alpha+2i}.$$

Finally observe that $D_{2n+2i+1} = A_{n+i+1} \oplus B_{n+i}$.

REMARK 9. The previous theorem leads to the decomposition

$$Gr^W H^n(X'_{f,x})_1 = A \oplus B$$

with $B = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i j B_{\alpha+2i}$ and $A = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i A_{\alpha+2i}$. We have $W = W(N, n)$ on B and $W = W(N, n+1)$ on A. Similarly we have

$$Gr^W H^n_c(X'_{f,x})_1 = A' \oplus B'$$

with $B' = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i B_{\alpha+2i}$ and $A' = \bigoplus_\alpha \bigoplus_{i \geqslant 0} V N^i A_{\alpha+2+2i}$. These are decompositions as graded mixed Hodge structures. We have $W = W(N_c, n)$ on B' and $W = W(N_c, n-1)$ on A'. The maps $V : A \to A'(-1)$ and $j : B' \to B$ are isomorphisms. Observe that $A = 0$ if and only if (X, x) is a rational homology manifold and that $B = 0$ if and only if (X', x) is a rational homology manifold.

See also [5] for the case of isolated complete intersection singularities.

We finally want to indicate how one can polarize the mixed Hodge structures $Gr^W H^n(X'_{f,x})$ and $Gr^W H^n_c(X'_{f,x})$. For the part of these on which the monodromy acts with eigenvalues $\neq 1$, we can use the global case, and these mixed Hodge structures are polarized by N. So let us consider the eigenvalue 1 part.

By Remark 9 it suffices to define polarizations on the Hodge structures A_i and B_i, i.e. on the graded quotients of the local cohomology groups.

Define the pairing

$$\langle \cdot, \cdot \rangle \colon H^n(X'_{f,x}) \otimes H^n_c(X'_{f,x}) \to \mathbf{C}$$

by

$$\langle \omega, \eta \rangle := (-1)^{n(n-1)/2} \int_{X'_{f,x}} \omega \wedge \eta.$$

THEOREM 10. *The form* $(x,y)\mapsto\langle j(x),N^iy\rangle$ *polarizes* B_{n+i} *for all* $i\geqslant 0$. *The form* $(x,y)\mapsto\langle x,VN^{i-1}y\rangle$ *polarizes* A_{n+i} *for all* $i\geqslant 1$.

Proof. Fix a globalization $F\colon Z\to\mathbf{C}$ as in Theorem 1. We have the inclusion $k^t\colon Gr^W_{n+i}H^n_c(X'_{f,x})_1\to Gr^W_{n+i}P^n(Z_F)_1$; observe that $\langle k(z),\eta\rangle=S(z,k^t(\eta))$ for $\eta\in H^n_c(X'_{f,x})$ and $z\in H^n(Z_F)_1$.

Let $i\geqslant 0$. For $0\neq\xi\in B_{n+i}$ we have $N_c^{i+1}\xi=0$ hence $k^t(\xi)\in P_{n+i}$. This implies that $\langle Cj(\xi),N^i(\overline{\xi})\rangle=S(Ck^t\xi,N^i(\overline{k^t\xi}))>0$.

Let $i\geqslant 1$; then the map $k\colon Gr^W_{n+i}P^n(Z_F)_1\to Gr^W_{n+i}H^n(X'_{f,x})_1$ is an isomorphism, as k is surjective and $\ker k=\operatorname{im}(H^n(Z_0)\to H^n(Z_F))$ is of weight $\leqslant n$. Let $\eta\in A_{n+i}$ and $z\in P_{n+i}$ such that $\eta=k(z)$, then $N^i\eta=0$ implies that $N^iz\in\ker(k)\subset\ker(N)$ so $N^{i+1}z=0$. Hence again $z\in P_{n+i}$. So if $z\neq 0$ we have $\langle C\eta,VN^{i-1}\overline{\eta}\rangle=\langle Ck(z),VN^{i-1}\overline{k(z)}\rangle=S(Cz,N^i\overline{z})>0$.

As an application we consider the intersection form h on $H^n_c(X'_{f,x},\mathbf{R})$ given by $h(\omega,\eta)=\int_{X'_{f,x}}\omega\wedge\eta=(-1)^{n(n-1)/2}\langle j(\xi),\eta\rangle$. Clearly its null space is equal to $\ker(j)$. In the case that n is even, h is a symmetric bilinear form, and we will compute its index in terms of the Hodge numbers

$$h^{pq}=\dim Gr^p_FGr^W_{p+q}H^n(X'_{f,x},\mathbf{C}).$$

Note that if $h^{pq}_c=\dim Gr^p_FGr^W_{p+q}H^n_c(X'_{f,x},\mathbf{C})$ then $h^{pq}_c=h^{n-p,n-q}$.

THEOREM 11. *Let n be even. Then the index* $\sigma(h)$ *of h is given by*

$$\sigma(h)=\sum_{p+q=n}(-1)^p\left(h^{pq}+2\sum_{i\geqslant 1}(-1)^ih^{p+i,q+i}\right).$$

Proof. First note that $W_{n-1}H^n_c(X'_{f,x})$ is an isotropic subspace of h which contains its null space. Moreover the orthogonal complement of $W_{n-1}H^n_c(X'_{f,x})$ with respect to h is equal to $W_nH^n_c(X'_{f,x})$. Therefore h induces a symmetric bilinear form h' on $Gr^W_nH^n_c(X'_{f,x})$ such that $\sigma(h')=\sigma(h)$. We extend h' to a hermitian form on $Gr^W_nH^n_c(X'_{f,x},\mathbf{C})$. Let

$$\tilde{B}_{n+i}=\ker(N_c^{i+1}\colon Gr^W_{n+i}H^n_c(X'_{f,x})\to Gr^W_{n-i-2}H^n_c(X'_{f,x})).$$

Then we have the decomposition

$$Gr^W_nH^n_c(X'_{f,x},\mathbf{C})=\bigoplus_{i\geqslant 0}\bigoplus_{p+q=n}N^i\tilde{B}^{p+i,q+i}_{n+2i}\oplus\bigoplus_{i\geqslant 1}\bigoplus_{p+q=n}VN^{i-1}A^{p+i,q+i}_{n+2i}$$

which is orthogonal with respect to h'. It follows from Theorem 10 that h' is definite on each of these summands, and its sign on $N^i\tilde{B}^{p+i,q+i}_{n+2i}$ and $VN^{i-1}A^{p+i,q+i}_{n+2i}$ is equal to $(-1)^{p+i}$ (note that $C = (-1)^{p+n/2}$ on these summands). Finally observe that

$$\dim \tilde{B}^{p+i,q+i}_{n+2i} = h_c^{p+i,q+i} - h_c^{p-i-1,q-i-1} = h^{p-i,q-i} - h^{p+i+1,q+i+1}$$

and

$$\dim A^{p+i,q+i}_{n+2i} = h^{p+i,q+i} - h^{p-i,q-i}.$$

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