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Dedicated to Frans Oort on the occasion of his 60th birthday

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Abstract. We investigate the connection between monodromy and weight filtration for one-parameter smoothings of isolated singularities. We give a formula for the signature of the intersection form in terms of the Hodge numbers of the vanishing cohomology.

Key words: singularity, mixed Hodge structure, monodromy, weight filtration

1. Introduction

Let \( V \) be a finite dimensional vectorspace and let \( N \) be a nilpotent endomorphism of \( V \). Then for each integer \( n \) there exists a unique decreasing filtration \( W = W(N, n) \) of \( V \) such that \( N(W_i) \subset W_{i-2} \) for each \( i \) and the induced map \( N^i : \text{Gr}_{W^i} \rightarrow \text{Gr}_{W^{i-2}} \) is an isomorphism for all \( i \).

If \( F : Z \rightarrow \mathbb{C} \) is a flat projective morphism with smooth generic fiber, then associated to the critical value 0 we have a limit mixed Hodge structure \( H^n(Z_F) \) whose weight filtration is equal to \( W(N, n) \) where \( N \) is the logarithm of the unipotent part of the monodromy transformation \( T \) around 0.

A similar situation arises in the case of an isolated hypersurface singularity \( f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) and its vanishing cohomology \( H^n(X_f, 0) \). Again we have a monodromy operator \( T \), but now the description of the weight filtration is slightly more complicated: write

\[
H^n(X_f, 0) = H^n(X_f, 0)_1 \oplus H^n(X_f, 0)_{\neq 1}, \tag{1}
\]

where \( H^n(X_f, 0)_1 \) (resp. \( H^n(X_f, 0)_{\neq 1} \)) is the subspace on which \( T \) acts with eigenvalue 1 (resp. eigenvalues \( \neq 1 \)). Then \( W = W(N, n + 1) \) on \( H^n(X_f, 0)_1 \) and \( W = W(N, n) \) on \( H^n(X_f, 0)_{\neq 1} \).

In this note we deal with the case of the weight filtration on the vanishing cohomology of a one-parameter smoothing of an isolated singularity. Part of the results were announced in [9] with a short indication of proof. In this general case the decomposition (1) has to be replaced by a suitable decomposition of \( \text{Gr}_{W} H^n(X_f, 0) \).
We also give precise results about the polarizations on these summands and express the index of the intersection form (in the even-dimensional case) in terms of Hodge numbers. This generalizes and simplifies [8] Theorem 4.11 and [9] Theorem 2.23. The main tool in our proof is a strong globalization theorem for one-parameter smoothings of isolated singularities, in the spirit of the Appendix of [4].

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2. Monodromy and weight filtration

Let \((X', x)\) be an isolated singularity of a complex space of pure dimension \(n + 1\), and \(f: (X', x) \to (\mathbb{C}, 0)\) a holomorphic function germ. Suppose that \(X := f^{-1}(0)\) has an isolated singularity at \(x\). We let \(X'_{f,x}\) denote the Milnor fibre of \(f\) at \(x\). We first sharpen a globalization theorem due to Looijenga [4]:

**THEOREM 1.** Let \(f: (X', x) \to (\mathbb{C}, 0)\) be a smoothing of an isolated singularity of pure dimension \(n\). Then there exists a flat projective morphism \(F: Z \to \mathbb{C}\), a point \(z \in Z_0\) and an isomorphism \(h: (X', x) \to (Z, z)\) such that \(F \circ h = f\) and \(F\) is smooth along \(Z_0 \setminus \{z\}\) and such that the restriction mapping \(H^n(Z_F, \mathbb{C}) \to H^n(X'_{f,x}, \mathbb{C})\) is surjective. Here \(Z_F\) denotes the generic fibre of \(F\).

**Proof.** If \(n = 0\) then \(f\) is finite, hence projective. So in the sequel we suppose that \(n \geq 1\). We follow the proof of [4]. Let \(Y\) be an affine variety of dimension \(n + 1\) with a unique singular point \(y\) and \(P\) a regular function on \(Y\) such that the germ \(f: (X', x) \to (\mathbb{C}, 0)\) is biholomorphic to \(P: (Y, y) \to \mathbb{C}\). The existence of \(Y\) such that \((X', x) \cong (Y, y)\) follows from work of Artin [1] and Hironaka [2], and the existence of a polynomial \(P\) with the desired properties follows from finite determinacy for germs with isolated singularities, due to Mather and Looijenga [4]. We assume \(Y\) to be embedded in affine \(N\)-space such that \(y = 0\). Let \(\mathfrak{m}\) denote the ideal of regular functions on \(Y\) vanishing at \(y\). Fix a positive integer \(k\) such that all germs \(P + g\) for \(g \in \mathfrak{m}^k\) are analytically isomorphic to \(P\). Let \(Z'\) denote the projective closure of \(Y\). We may assume that \(Z' \setminus \{y\} \) and \(Z' \setminus Y = Z'_{\infty}\) are smooth.

Choose a sufficiently general (to be made precise below) homogeneous polynomial \(g\) of degree \(d \geq k\) sufficiently big and let \(Q = P + g\). Let \(Z = \{(\xi, t) \in Z' \times \mathbb{C} \mid \xi_i^d Q(\xi_1/\xi_0, \ldots, \xi_N/\xi_0) = t \xi_0^d\}\). We embed \(Y\) in \(Z\) as the graph of \(Q\) and let \(z = (y, 0)\). The projection \(F\) of \(Z\) onto the second factor provides a globalization of \(f\). We will show that we can choose \(g\) in such a way that it has the desired properties. First we require that \(g\) defines a smooth hypersurface in \(\mathbb{P}^{N-1}\) which is transverse to \(Z'_{\infty}\) and that \(z\) is the only critical point of \(F\) on \(F^{-1}(0)\).

We fix a good Stein representative \(f: X' \to \Delta\) for the germ \(f\) in the sense of [3] Chapter 2.B. Write \(\Omega_j = \Omega_{X'/df} \wedge \Omega_{X'}^{-1}\). By [3] Theorem 8.7, the sheaf \(\mathcal{H}^n f_*(\Omega_j)\) is coherent. Let \(\omega_Y = j_* \Omega_Y^{n+1}\), where \(j: Y \setminus \{y\} \to Y\) is the inclusion.
map. Put \( Y_t = Q^{-1}(t) \). First observe that for \( t \neq 0 \) sufficiently small the restriction map \( H^n(Y_t, C) \to H^n(X'_{f,x}, C) \) is surjective. This follows from the specialization sequence

\[
H^n(Y_0, C) \to H^n(Y_t, C) \to H^n(X'_{f,x}, C) \to H^{n+1}(Y_0, C),
\]

(here we use that \( F \) has no critical point at infinity) and the fact that for an affine variety of dimension \( n \) the cohomology groups are zero in degrees > \( n \). Moreover, for such \( t \) there is a natural map \( \rho : H^0(Y, \omega_Y) \to H^0(Y_t, \Omega^n_{Y_t}) \to H^n f_*(\Omega_f)(t) \)

which is the composition of the map \( \eta \mapsto \) the restriction to \( Y_t \) of \( \eta/dP \) and the restriction to \( X'_{f,x} \). Then \( \rho \) is the composition of two surjections, hence surjective.

(The second map is surjective as \( H^n(Y_t, C) \to H^n(X'_{f,x}, C) \) is surjective.) Choose \( \eta_1, \ldots, \eta_r \in H^0(Y, \omega_Y) \) whose images generate \( H^n f_*(\Omega_f)(t) \) for all \( t \neq 0 \) sufficiently small. If \( g \) is a small perturbation of \( f \), they will still generate \( H^ng_*(\Omega_g)(t) \) for all \( t \neq 0 \) sufficiently small, again by Looijenga’s coherence theorem.

There exists \( l \in \mathbb{N} \) such that \( \eta_1, \ldots, \eta_r \) extend to sections of \( \omega Z'_{l lZ_{\infty}} \). Let \( D = Z_{lZ_{\infty}} \cap Z_{0} = Z_{lZ_{\infty}} \cap Z_{lZ_{\infty}} \). Then \( \eta_1/dQ, \ldots, \eta_r/dQ \) extend to sections of \( \Omega^2_{Z_t}((l - d)D) \).

So if \( d \geq l \) the map \( H^0(Z_t, \Omega^n_{Z_t}) \to H^n(X'_{f,x}, C) \) is surjective. Then a fortiori \( H^n(Z_t, C) \to H^n(X'_{f,x}, C) \) is surjective.

By [9] we have the following exact sequences of mixed Hodge structures associated with the Milnor fibre \( X'_{f,x} \) of \( f \) at 0:

\[
0 \to H^{n+1}_{\{x\}}(X') \to H^n(X'_{f,x})_1 \overset{V}{\to} H^n_c(X'_{f,x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0, \quad (2)
\]

\[
0 \to H^{n-1}(X'_{f,x}) \to H^n_{\{x\}}(X) \to H^n_c(X'_{f,x})
\]

\[
\overset{j}{\to} H^n(X'_{f,x}) \to H^{n+1}_{\{x\}}(X) \to H^{n+1}_c(X'_{f,x}) \to 0, \quad (3)
\]

where the subscript 1 denotes the generalized eigenspace of \( T \) for the eigenvalue 1 and \( jV = N = \log(T) \) (resp. \( Vj = N_c = \log(T_c) \)) on \( H^n_c(X'_{f,x})_1 \) (resp. \( H^n(X'_{f,x})_1 \)). We recall

**THEOREM 2.**

\[
Gr^W_i H^{n+1}_{\{x\}}(X') = 0 \quad \text{for } i \geq n + 1; \quad (4)
\]

\[
Gr^W_i H^n_{\{x\}}(X) = 0 \quad \text{for } i \geq n; \quad (5)
\]

\[
Gr^W_i H^{n+2}(X') = 0 \quad \text{for } i \leq n + 1; \quad (6)
\]

\[
Gr^W_i H^{n+1}_{\{x\}}(X) = 0 \quad \text{for } i \leq n. \quad (7)
\]

See [9] Corollary 1.12. Both \( N \) and \( N_c \) map \( W_i \) to \( W_{i-2} \).
THEOREM 3. For all $i \geq 0$ the map
\[ N_c^i : Gr_{n+1+i}^W \text{im}(V) \to Gr_{n+1-i}^W \text{im}(V) \]

is an isomorphism.

REMARK 4. In the hypersurface case, i.e. when $X'$ is smooth, the map $V$ is an isomorphism and we recover [8] Corollary 4.9.

Proof. We choose a flat projective morphism $F : Z \to C$, a point $z \in Z$ and an isomorphism $h : (X', x) \to (Z, z)$ such that $F \circ h = f$ and $F$ is smooth along $Z_0 \setminus \{z\}$ as in Theorem 1. Let $Z_F$ denote the generic fibre of $F$. Then one has the exact sequence of mixed Hodge structures
\[ \to H^n(Z_0) \to H^n(Z_F) \to H^n(X'_{f, x}) \to 0, \tag{4} \]

where $H^n(Z_F)$ carries the limit mixed Hodge structure. There is a monodromy action $T$ on this sequence, and $T$ acts as the identity on $H^n(Z_0)$. We have the following sequence
\[ H^n(Z_F)_1 \to H^n(X'_{f, x})_1 \to H^n(X'_{f, x})_1(-1) \xrightarrow{k^t} H^n(Z_F)_1(-1) \]

and $N = k^t \circ V \circ k$. As $k$ is surjective, its transpose $k^t$ is injective and defines an isomorphism of mixed Hodge structures $\text{im}(V) \to \text{im}(N)$ such that $k^t \circ N_c = N \circ k^t$. As $W = W(N, n)$ on $H^n(Z_F)_1$ we get that $W = W(N, n+1)$ on $\text{im}(N)$.

It follows that $Gr^W(\text{im}(V))$ is completely determined by the kernel of $N_c$ on $\text{im}(V)$. In order to determine this kernel, observe that (4) implies that $\ker(V)$ has weights $\leq n$ and that (7) implies that $\text{coker}(j)$ has weights $\geq n+1$. Hence $\ker(V) \subset \text{im}(j)$. So we have the exact sequence
\[ 0 \to \ker(j) \to \ker(N_c) \xrightarrow{j} \ker(V) \to 0 \tag{5} \]

and hence $\ker(N_c)$ has weights $\leq n$. By considering the action of $N_c$ on the exact sequence
\[ 0 \to \text{im}(V) \to H^n(X'_{f, x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0 \]

we obtain the exact sequence
\[ 0 \to \ker(N_c; \text{im}(V)) \to \ker(N_c)(-1) \xrightarrow{W_{n+2}} H^{n+2}_{\{x\}}(X') \to 0 \]

and hence $\ker(N_c; \text{im}(V)) = W_{n+1}(\ker(N_c)(-1))$. So from (5) we obtain

LEMMA 5. We have the exact sequence of mixed Hodge structures
THEOREM 6. Regarding the map \( H^n_c(X'_f, x) \rightarrow H^n(X'_f, x) \) we have that

\[ N^i : \text{Gr}_{n+i} W \rightarrow \text{Gr}_{n-i} W \]

is an isomorphism for all \( i \geq 0 \), i.e. \( W = W(N, n) \) on \( \text{im}(j) \).

**Proof.** Choose a globalization \( F : Z \rightarrow \mathbb{C} \) of \( f \) as in the proof of Theorem 2. Then \( j \) is factorized as

\[ H^n_c(X'_f, x) \xrightarrow{k^t} H^n(Z_F) \xrightarrow{k} H^n(X'_f, x). \]

Let \( P^n(Z_F) = \ker(L : H^n(Z_F) \rightarrow H^{n+2}(Z_F)) \) denote the primitive cohomology. Here \( L \) is the cup product with the hyperplane class. As a general hyperplane does not pass through the point \( x \), the image of \( k^t \) is contained in \( P^n(Z_F) \).

We have the nondegenerate pairing \( S \) on \( P^n(Z_F) \), given by

\[ S(x, y) = (-1)^{n(n-1)/2} \int_{Z_F} x \wedge y. \]

It is \((-1)^n\)-symmetric, \( W_\alpha = (W_{2n-1-\alpha})^\perp \) and \( S(Nx, y) + S(x, Ny) = 0 \). Moreover \( N^\alpha : \text{Gr}^W_{n+\alpha} P^n(Z_F) \rightarrow \text{Gr}^W_{n-\alpha} P^n(Z_F) \) is an isomorphism for all \( \alpha \geq 0 \). If \( P_{n+\alpha} := \ker(N^{\alpha+1} : \text{Gr}^W_{n+\alpha} P^n(Z_F) \rightarrow \text{Gr}^W_{n-\alpha-2} P^n(Z_F)) \), the form \( (x, y) \mapsto S(Cx, N^\alpha y) \) is hermitian positive definite on \( P_{n+\alpha} \) by [7], Lemma 6.25.

Let \( Q_\alpha = \text{Gr}^W_{n-\alpha} \ker(k) \subset \text{Gr}^W_{n-\alpha} P^n(Z_F) \). Then \( \text{Gr}^W_{n+\alpha} \text{im}(j) \simeq (Q_\alpha)^\perp \) as \( \text{Gr}^W_{n+\alpha} \ker(j) = 0 \). Therefore,

\[ \text{Gr}^W_{n-\alpha} \text{im}(j) \simeq N^\alpha(Q_\alpha)^\perp / Q_\alpha \cap N^\alpha(Q_\alpha)^\perp \]

so we have to show that

\[ Q_\alpha \cap N^\alpha(Q_\alpha)^\perp = (0). \]

Clearly, \( Q_\alpha \subset N^\alpha P_{n+\alpha} \) as \( N = 0 \) on \( \ker(k) \). So let \( x \in N^\alpha(Q_\alpha)^\perp \cap Q_\alpha \). Write \( x = N^\alpha x' \) with \( x' \in P_{n+\alpha} \cap (Q_\alpha)^\perp \). Then \( S(Cx', N^\alpha x') = 0 \) hence \( x = 0 \).

THEOREM 7. (i) For all \( i > 0 \) the map

\[ V \circ N^{i-1} : \text{Gr}^W_{n+i} H^n(X'_f, x) \rightarrow \text{Gr}^W_{n-i} H^n(X'_f, x) \]

is an isomorphism;

(ii) for all \( i \geq 0 \) the map

\[ N^i \circ j : \text{Gr}^W_{n+i} H^n(X'_f, x) \rightarrow \text{Gr}^W_{n-i} H^n(X'_f, x) \]

is an isomorphism.
Proof. For $i > 0$ we have $Gr^n_{W,n+i} \ker(V) = 0$ so

$$Gr^n_{W,n+i} H^n(X'_{f,x})_1 \simeq Gr^n_{W,n+i} \text{im}(V).$$

This space is mapped isomorphically to $Gr^n_{W,n-i+2} \text{im}(V)$ by $N^{-1}$ according to Theorem 3. As $\text{coker}(V)$ has weights $\geq n + 2$, we have

$$Gr^n_{W,n-i+2} \text{im}(V) \simeq Gr^n_{W,n-i} H^n_c(X'_{f,x})_1.$$

This proves (i). Ones proves (ii) similarly using Theorem 6 instead of Theorem 3.

3. Primitive decomposition

Let $V$ be a finite dimensional vector space and $N$ a nilpotent endomorphism of $V$, $n$ an integer and $W = W(N,n)$. Then we have the following decomposition of $Gr^n_{W}(V)$. Recall that $N^i : Gr^n_{W,n+i}(V) \to Gr^n_{W,n-i}(V)$ is an isomorphism for all $i \geq 0$. Put

$$P_{n+i} = \ker(N^{i+1}: Gr^n_{W,n+i}(V) \to Gr^n_{W,n-i-2}(V))$$

for $i \geq 0$ and 0 else. Then we have the primitive decomposition

$$Gr^n_{\alpha}(V) \simeq \bigoplus_{i \geq 0} N^i P_{\alpha+2i}.$$

We will give an analogous but more subtle decomposition of $Gr^n_{W,H^c}(X'_{f,x})_1$ and $Gr^n_{W,H^n_c}(X'_{f,x})_1$ (we use the same notation as in the preceding section). This was first mentioned in [6] and proved by Saito in a letter to the author. Define

$$B_{n+i} = \ker(N^{i+1}_c: Gr^n_{W,n+i} H^n_c(X'_{f,x})_1 \to Gr^n_{W,n-i-2} H^n_c(X'_{f,x})_1)$$

for $i \geq 0$ and 0 else, and

$$A_{n+i} = \ker(N^i: Gr^n_{W,n+i} H^n(X'_{f,x})_1 \to Gr^n_{W,n-i} H^n(X'_{f,x})_1)$$

for $i > 0$ and 0 else. By Theorem 7 $B_{n+i}$ is mapped isomorphically to $Gr^n_{W,n-i} \ker(V)$ by $N^i \circ j$ and $A_{n+i}$ is mapped isomorphically to $Gr^n_{W,n-i} \ker(j)$ by $V \circ N^{i-1}$.

THEOREM 8. We have

$$Gr^n_{W,H^n_c}(X'_{f,x})_1 = \bigoplus_{i \geq 0} N^i B_{\alpha+2i} \oplus \bigoplus_{i \geq 0} V N^i A_{\alpha+2+2i}$$

and

$$Gr^n_{W,H^n (X'_{f,x})_1} = \bigoplus_{i \geq 0} N^i j B_{\alpha+2i} \oplus \bigoplus_{i \geq 0} N^i A_{\alpha+2i}.$$
Proof. Define a graded vectorspace $C$ by $C_{2\alpha} = 0$ and 

$$C_{2\alpha+1} = Gr^W_{\alpha+1} H^n(X'_{f,x})_1 \oplus \text{Gr}^W_{\alpha} H^n_c(X'_{f,x})_1.$$ 

Define an endomorphism $\lambda$ of degree $-2$ of $C$ as $\lambda(x, y) = (j(y), V(x))$. From Theorem 7 we obtain that for all $i \geq 0$ the map $\lambda^i : C_{2n+i} \to C_{2n-i}$ is an isomorphism. Hence, if $D_{2n+i} = \ker(\lambda^{i+1} : C_{2n+i} \to C_{2n-i-2})$ for $i \geq 0$ and $0$ else, then we have that the map $\lambda^\alpha : D_{2n+i} \to C_{2n+i-\alpha}$ is injective for $\alpha \leq 2i$ and else the zero map. We obtain the primitive decomposition

$$C_\alpha = \bigoplus_{i \geq 0} \lambda^i D_{\alpha+2i}.$$ 

Finally observe that $D_{2n+2i+1} = A_{n+i+1} \oplus B_{n+i}$.

REMARK 9. The previous theorem leads to the decomposition

$$Gr^W H^n(X'_{f,x})_1 = A \oplus B$$

with $B = \bigoplus_{\alpha} \bigoplus_{i \geq 0} N^i j B_{\alpha+2i}$ and $A = \bigoplus_{\alpha} \bigoplus_{i \geq 0} N^i A_{\alpha+2i}$. We have $W = W(N, n)$ on $B$ and $W = W(N, n+1)$ on $A$. Similarly we have

$$Gr^W H^n_c(X'_{f,x})_1 = A' \oplus B'$$

with $B' = \bigoplus_{\alpha} \bigoplus_{i \geq 0} N^i B_{\alpha+2i}$ and $A' = \bigoplus_{\alpha} \bigoplus_{i \geq 0} V N^i A_{\alpha+2+2i}$. These are decompositions as graded mixed Hodge structures. We have $W = W(N_c, n)$ on $B'$ and $W = W(N_c, n-1)$ on $A'$. The maps $V : A \to A'(-1)$ and $j : B' \to B$ are isomorphisms. Observe that $A = 0$ if and only if $(X, x)$ is a rational homology manifold and that $B = 0$ if and only if $(X', x)$ is a rational homology manifold.

See also [5] for the case of isolated complete intersection singularities.

We finally want to indicate how one can polarize the mixed Hodge structures $Gr^W H^n(X'_{f,x})$ and $Gr^W H^n_c(X'_{f,x})$. For the part of these on which the monodromy acts with eigenvalues $\neq 1$, we can use the global case, and these mixed Hodge structures are polarized by $N$. So let us consider the eigenvalue 1 part.

By Remark 9 it suffices to define polarizations on the Hodge structures $A_i$ and $B_i$, i.e. on the graded quotients of the local cohomology groups.

Define the pairing

$$\langle \cdot, \cdot \rangle : H^n(X'_{f,x}) \otimes H^n_c(X'_{f,x}) \to \mathbb{C}$$

by

$$\langle \omega, \eta \rangle := (-1)^{n(n-1)/2} \int_{X'_{f,x}} \omega \wedge \eta.$$
THEOREM 10. The form \((x, y) \mapsto \langle j(x), N^i y \rangle\) polarizes \(B_{n+i}\) for all \(i \geq 0\). The form \((x, y) \mapsto \langle x, VN^{i-1} y \rangle\) polarizes \(A_{n+i}\) for all \(i \geq 1\).

Proof. Fix a globalization \(F: Z \to C\) as in Theorem 1. We have the inclusion \(k^*: Gr^W_{n+i} H^m_c(X_{f,x}) \to Gr^W_{n+i} P^n(Z_F)\); observe that \(\langle k(z), \eta \rangle = S(z, k^i(\eta))\) for \(\eta \in H^m_c(X'_{f,x})\) and \(z \in H^n(Z_F)\).

Let \(i \geq 0\). For \(0 \neq \xi \in B_{n+i}\) we have \(N^{i+1} \xi = 0\) hence \(k^i(\xi) \in P_{n+i}\). This implies that \(\langle C j(\xi), N^i(\xi) \rangle = S(C k^i(\xi), N^i(k^i(\xi))) > 0\).

Let \(i \geq 1\); then the map \(k: Gr^W_{n+i} P^n(Z_F) \to Gr^W_{n+i} H^n(X'_{f,x})\) is an isomorphism, as \(k\) is surjective and \(\ker k = \text{im}(H^n(Z_0) \to H^n(Z_F))\) is of weight \(\leq n\). Let \(\eta \in A_{n+i}\) and \(z \in P_{n+i}\) such that \(\eta = k(z)\), then \(N^i \eta = 0\) implies that \(N^i z \in \ker(k) \subset \ker(N)\) so \(N^{i+1} z = 0\). Hence again \(z \in P_{n+i}\). So if \(z \neq 0\) we have \(\langle C \eta, VN^{i-1}\eta \rangle = \langle C k(z), VN^{i-1}k(z) \rangle = S(c z, N^i z) > 0\).

As an application we consider the intersection form \(h\) on \(H^m_c(X'_{f,x}, R)\) given by \(h(\omega, \eta) = \int_{X'_{f,x}} \omega \wedge \eta = (-1)^{n(n-1)/2} \langle j(\xi), \eta \rangle\). Clearly its null space is equal to \(\ker(j)\). In the case that \(n\) is even, \(h\) is a symmetric bilinear form, and we will compute its index in terms of the Hodge numbers

\[ h^{pq} = \dim Gr^PG^W_{p+q} H^m(X'_{f,x}, C). \]

Note that if \(h^{pq}_c = \dim Gr^PG^W_{p+q} H^m_c(X'_{f,x}, C)\) then \(h^{pq} = h^{n-p,n-q}\).

THEOREM 11. Let \(n\) be even. Then the index \(\sigma(h)\) of \(h\) is given by

\[ \sigma(h) = \sum_{p+q=n} (-1)^p \left( h^{pq} + 2 \sum_{i \geq 1} (-1)^i h^{p+i,q+i} \right). \]

Proof. First note that \(W_{n-1} H^m_c(X'_{f,x})\) is an isotropic subspace of \(h\) which contains its null space. Moreover the orthogonal complement of \(W_{n-1} H^m_c(X'_{f,x})\) with respect to \(h\) is equal to \(W_n H^m_c(X'_{f,x})\). Therefore \(h\) induces a symmetric bilinear form \(h'\) on \(Gr^W_n H^m_c(X'_{f,x}, C)\) such that \(\sigma(h') = \sigma(h)\). We extend \(h'\) to a hermitian form on \(Gr^W_n H^m_c(X'_{f,x}, C)\). Let

\[ \tilde B_{n+i} = \ker(N^{i+1}_c: Gr^W_{n+i} H^m_c(X'_{f,x}) \to Gr^W_{n+i-2} H^m_c(X'_{f,x})). \]

Then we have the decomposition

\[ Gr^W_n H^m_c(X'_{f,x}, C) = \bigoplus_{i \geq 0} \bigoplus_{p+q=n} N^i \tilde B_{n+2i}^{p+i,q+i} \oplus \bigoplus_{i \geq 1} \bigoplus_{p+q=n} VN^{i-1} A_{n+2i}^{p+i,q+i} \]

which is orthogonal with respect to \(h'\). It follows from Theorem 10 that \(h'\) is definite on each of these summands, and its sign on \(N^i \tilde B_{n+2i}^{p+i,q+i}\) and \(VN^{i-1} A_{n+2i}^{p+i,q+i}\) is
equal to \((-1)^{p+i}\) (note that \(C = (-1)^{p+n/2}\) on these summands). Finally observe that

\[
\dim \tilde{D}_{n+2i}^{p+i,q+i} = h_c^{p+i,q+i} - h_c^{p-i-1,q-i-1} = h^{p-i,q-i} - h^{p+i+1,q+i+1}
\]

and

\[
\dim A_{n+2i}^{p+i,q+i} = h^{p+i,q+i} - h^{p-i,q-i}.
\]

References