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Poincaré duality and integral cycles*

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Abstract. We show that the Alexander–Lefschetz duality can be thought of as a homotopy equivalence between a space of integral cycles and a space of maps into integral cycles on a sphere.

Let M be an orientable manifold of dimension n. For every $k \geqslant 0$ and every compact polyhedron $A \subset M - \partial M$ there is the Alexander–Lefschetz duality isomorphism

$$H_k(A;\mathbb{Z}) \cong H^{n-k}(M, M-A;\mathbb{Z}) \tag{1}$$

that specializes to the Poincaré duality isomorphism

$$H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$$

when ∂M is empty and $A = M$ [7].

In 1956 Dold and Thom proved that for every $k \geqslant 0$ and every polyhedron X

$$H_k(X;\mathbb{Z}) \cong \pi_k(AG(X))$$

where $AG(X)$ is a free abelian topological group generated by the points of X [1]. A few years later Almgren generalized this result proving that for every $k \geqslant r \geqslant 0$

$$H_k(X;\mathbb{Z}) \cong \pi_{k-r}(Z_r(X))$$

where $Z_r(X)$ is the group of integral r-dimensional cycles on X [2].

Let $\widetilde{Z}_r(S^n) = Z_r(S^n)$ for $r > 0$ and let $\widetilde{Z}_0(S^n)$ be the connected component of 0 in $Z_0(S^n)$ (that is the space of 0-cycles of degree 0 on S^n). Almgren's isomorphism implies that the space $\widetilde{Z}_r(S^n)$ is a $K(\mathbb{Z}, n-r)$. This observation together with the classical result

$$H^{n-k}(X;\mathbb{Z}) \cong \pi_k \mathrm{Map}(X, K(\mathbb{Z}, n))$$

gives for every $k \geqslant r \geqslant 0$ an isomorphism

$$H^{n-k}(X;\mathbb{Z}) \cong \pi_{k-r}\mathrm{Map}(X, \widetilde{Z}_r(S^n)).$$

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It is easy to see that if $A \subset M$ is a deformation retract of an open set $U \subset M$, then for every $k \geqslant r \geqslant 0$

$$H^{n-k}(M, M-A; \mathbb{Z}) \cong \pi_{k-r}\mathrm{Map}\left((M, M-U), (\widetilde{Z}_r(S^n), 0)\right).$$

Combining the Almgren Theorem with the above isomorphisms we get that for every $k \geqslant r \geqslant 0$

$$\pi_{k-r}(Z_r(A)) \cong \pi_{k-r}\mathrm{Map}\left((M, M-U), (\widetilde{Z}_r(S^n), 0)\right).$$

Thus it is natural to expect that $Z_r(A)$ and $\mathrm{Map}((M, M-U), (\widetilde{Z}_r(S^n), 0))$ are homotopy equivalent. The aim of this note is to construct a homotopy equivalence

$$\Phi\colon\ Z_r(A) \longrightarrow \mathrm{Map}\left((M, M-U), (\widetilde{Z}_r(S^n), 0)\right)$$

that induces the Alexander–Lefschetz duality isomorphism (1).

Actually, we will prove the following result.

THEOREM A. *If M is a smooth orientable manifold of dimension n, then for every $r \geqslant 0$ and every polyhedron $A \subset M - \partial M$ there is a neighborhood U of A in M and a map*

$$\Phi\colon\ Z_r(A) \longrightarrow \mathit{Map}\left((M, M-U), (\widetilde{Z}_r(S^n), 0)\right)$$

so that for every $k \geqslant r \geqslant 0$ the homomorphism

$$\pi_{k-r}(Z_r(A)) \longrightarrow \pi_{k-r}\mathit{Map}\left((M, M-U), (\widetilde{Z}_r(S^n), 0)\right)$$

induced by Φ is the Alexander–Lefschetz duality isomorphism

$$H_k(A; \mathbb{Z}) \cong H^{n-k}(M, M-A; \mathbb{Z}).$$

In Appendix we included a proof of the classical isomorphism

$$H^n(X; \mathbb{Z}) \cong \pi_0\mathrm{Map}(X, AG_0(S^n))$$

where $H^n(X;\mathbb{Z})$ stands for the nth singular cohomology group of X and $AG_0(X)$ is the connected component of 0 in $AG(X)$. The proof is based on the Dold–Thom Theorem and some basic results of simplicial homotopy theory and is essentially different from the classical proof, that uses obstruction theory and identifies $\pi_0\mathrm{Map}(X, AG_0(S^n))$ with the nth cellular cohomology group of X.

The paper has two sections, which is a reflection of the fact that in the case of zero dimensional cycles there are two constructions of the map Φ from Theorem A. One, allows the extension of Theorem A to the case where M is a topological manifold but has no obvious generalization for higher dimensional cycles. Another, is done in the context of smooth manifolds and easily generalizes to higher dimensional cycles. The rough idea of the proof of Theorem A is to lift $Z_r(M)$ to sections of a $\widetilde{Z}_r(S^n)$-bundle associated with the tangent bundle of M. The orientability of M is shown to be equivalent to the triviality of this bundle. Thereby, identifying sections of this bundle with the maps from M to $\widetilde{Z}_r(S^n)$.

I am grateful to Blaine Lawson for suggesting to me the extension of the smooth case construction to the topological case and many very useful comments.

1. Zero dimensional cycles on a topological manifold

In this section we shall describe a homotopy equivalence map

$$\Phi\colon\ Z_0(A) \longrightarrow \mathrm{Map}\Big((M, M-U), (\widetilde{Z}_0(S^n), 0)\Big)$$

for M being a topological manifold. Actually, we shall replace the space $Z_0(X)$ of 0-dimensional integral cycles (with the flat norm topology) by the free abelian group $AG(X)$ generated by the points of X with the compactly generated topology on it [1]. $Z_0(X)$ and $AG(X)$ have the same set of elements but different topologies. However, they are homotopy equivalent.

THEOREM B. *If M is a topological orientable manifold of dimension n, then for every polyhedron $A \subset M - \partial M$ there is a neighborhood U of A in M and a map*

$$\Phi\colon\ AG(A) \longrightarrow Map\,((M, M-U), (AG_0(S^n), 0))$$

so that for every $k \geqslant 0$ the homomorphism induced by Φ

$$\pi_k(AG(A)) \longrightarrow \pi_k Map\,((M, M-U), (AG_0(S^n), 0))$$

is the Alexander–Lefschetz duality isomorphism

$$H_k(A;\mathbb{Z}) \cong H^{n-k}(M, M-A;\mathbb{Z}).$$

Proof. Let M be a closed orientable topological manifold. First we will construct a map

$$\Phi\colon\ AG(M) \longrightarrow \mathrm{Map}(M, AG_0(S^n)).$$

It will have the property that for every $A \subset M$ the restriction of Φ to $AG(A)$ induces a map

$$\Phi\colon\ AG(A) \longrightarrow \mathrm{Map}\,((M, M-U), (AG_0(S^n), 0))$$

for some neighborhood U of A in M.

Let W be a neighborhood of the diagonal of $M \times M$ so the the diagram

$$M \underset{\Delta}{\longrightarrow} W \underset{\pi_1}{\longrightarrow} M$$

represents the tangent microbundle τM of M where Δ: $M \to W$ is the diagonal map and π_1 is the projection on the first factor. The Thom space $\widehat{\tau M}$ of τM is the quotient $M \times M/(M \times M - W)$. We will denote the quotient map from $M \times M$ onto $\widehat{\tau M}$ by π.

An orientation of M induces a Thom class $U \in H^n(\widehat{\tau M};\mathbb{Z})$ [4]. Since

$$H^n(\widehat{\tau M};\mathbb{Z}) \cong \pi_0 \mathrm{Map}(\widehat{\tau M}, AG_0(S^n))$$

the class U can be represented by a map u: $\widehat{\tau M} \to AG_0(S^n)$ that extends to

$$\begin{aligned} &\widetilde{u}\colon\ AG(\widehat{\tau M}) \longrightarrow AG_0(S^n) \\ &\widetilde{u}\Big(\sum n_i x_i\Big) = \sum n_i u(x_i). \end{aligned}$$

With every cycle $c = \sum n_i x_i \in AG(M)$ we can associate a family of cycles

$$\begin{aligned} &C\colon\ M \longrightarrow AG(M \times M) \\ &C(x) = \sum n_i(x_i, x). \end{aligned}$$

For any $c \in AG(M)$ we define $\Phi(c)$ as the composition of the continuous maps

$$M \underset{C}{\longrightarrow} AG(M \times M) \underset{\pi_*}{\longrightarrow} AG(\widehat{\tau M}) \underset{\widetilde{u}}{\longrightarrow} AG_0(S^n).$$

It is easy to see that for every subspace A of M and every cycle $c \in AG(A)$ the map $\Phi(c)$ is constant zero on a complement of some neighborhood U of A in M (U depends on W). Thus the restriction of Φ to $AG(A)$ induces a map

$$\Phi\colon\ AG(A) \longrightarrow \mathrm{Map}\left((M, M - U), (AG_0(S^n), 0)\right).$$

If M is a manifold, A is a polyhedron contained in M, and U is a small neighborhood of A in M, then U is homotopy equivalent to A and

$$\pi_k \mathrm{Map}\left((M, M - U), (AG_0(S^n), 0)\right) \cong H^{n-k}(M, M - A; \mathbb{Z}).$$

Thus under the above assumptions Φ induces for every $k \geqslant 0$ a homomorphism

$$\Phi_*\colon\ H_k(A;\mathbb{Z}) \longrightarrow H^{n-k}(M, M - A; \mathbb{Z}).$$

□

It is easy to see that if A is a point, Φ_* is an isomorphism. Hence, it is also an isomorphism for A being any simplex of a triangulation of M. In order to prove that Φ_* is an isomorphism for any polyhedron $A \subset M - \partial M$ one uses induction on the number of the top dimensional simplices of A together with the Mayer–Vietoris argument (note that Φ_* is compatible with Mayer–Vietoris sequences). For details see for example the proof of the Alexander–Lefschetz duality in [7]. In order to see that Φ_* coincides with the classical Alexander–Lefschetz duality map note that they coincide for A being a simplex and then use a Mayer–Vietoris argument and the induction on the number of simplices of A.

2. Integral cycles on smooth manifolds

In this section we present a proof of Theorem A. For simplicity we start from the zero dimensional case.

First, note that $\mathrm{Map}(M, \widetilde{Z}_0(S^n))$ can be thought of as a space of sections of the trivial fiber bundle $M \times \widetilde{Z}_0(S^n)$ over M. It turns out that such a bundle is induced by the tangent bundle of M when M is orientable.

Let M be a smooth (not necessary orientable) manifold of dimension n and let P_M be the principal $O(n)$-bundle associated with the tangent bundle TM of M. The *delooped determinant fibration* of M is

$$\mathbf{B}^n \det(TM) = P_M \times_{O(n)} \widetilde{Z}_0(S^n)$$

where the action of $O(n)$ on $\widetilde{Z}_0(S^n)$ is induced from the one-point compactification of the linear action of $O(n)$ on $\mathbb{R}^n$. The name $\mathbf{B}^n \det(TM)$ is justified by the fact that the fibration

$$P_M \times_{O(n)} U(\Omega^n(\widetilde{Z}_0(S^n)))$$

where $U(\Omega^n(\widetilde{Z}_0(S^n)))$ consists of the components of $\Omega^n(\widetilde{Z}_0(S^n))$ that correspond to $\{\pm 1\}$ under the homotopy equivalence $\mathbb{Z} \to \Omega^n(\widetilde{Z}_0(S^n))$ is the Petterson–Stong *determinant fibration* [6]. The fibration is fiberwise homotopy equivalent to the determinant principal $(\mathbb{Z}/2\mathbb{Z})$-bundle of M. If M is an orientable manifold, then the determinant line bundle of M is trivial and $\mathbf{B}^n \det(TM)$ is homotopically trivial. Actually, we prove the following result.

LEMMA. *A smooth manifold M is orientable if and only if there is a continuous map*

$$\tau\colon \mathbf{B}^n \det(TM) \longrightarrow \widetilde{Z}_0(S^n)$$

that is a homotopy equivalence on fibers.

Proof. Suppose, that M is a smooth orientable manifold. The Thom space $\widehat{TM}$ of the tangent bundle of M can be identified with the quotient

$$P_M \times_{O(n)} S^n / M \times_{O(n)} \infty$$

where $M \times_{O(n)} \infty$ denotes the infinity section of $P_M \times_{O(n)} S^n$.

Since M is orientable, the tangent bundle of M has a Thom class that is represented in

$$H^n(\widehat{TM};\mathbb{Z}) \cong \pi_0\mathrm{Map}\left(\widehat{TM}, \widetilde{Z}_0(S^n)\right)$$

by a continuous map

$$u\colon \widehat{TM} \longrightarrow \widetilde{Z}_0(S^n).$$

The composition of u with the quotient map

$$P_M \times_{O(n)} S^n \longrightarrow \widehat{TM}$$

gives the map

$$t\colon P_M \times_{O(n)} S^n \longrightarrow \widetilde{Z}_0(S^n)$$

that can be extended in a natural way to the map

$$\tau\colon \mathbf{B}^n \det(TM) \longrightarrow \widetilde{Z}_0(S^n)$$

whose restriction to every fiber $\widetilde{Z}_0(\widehat{T_xM})$ of $\mathbf{B}^n \det(TM)$ coincides with $\widetilde{Z}_0(t\,|_{\widehat{T_xM}})$. Since t is induced by the Thom class, the restriction of t to every fiber $\widehat{T_xM}$ of $P_M \times_{O(n)} S^n$ is a map $t_x\colon S^n \to \widetilde{Z}_0(S^n)$ representing a generator of $\pi_n(\widetilde{Z}_0(S^n))$. Hence, the induced map

$$\tau_x\colon \widetilde{Z}_0(S^n) \longrightarrow \widetilde{Z}_0(S^n)$$

is a homotopy equivalence*.

Suppose, now that there is a map

$$\tau\colon \mathbf{B}^n \det(TM) \longrightarrow \widetilde{Z}_0(S^n)$$

that is a homotopy equivalence on fibers. Than

$$(\pi,\tau)\colon \mathbf{B}^n \det(TM) \longrightarrow M \times \widetilde{Z}_0(S^n)$$

is a fiber-wise homotopy trivialization of

$$\mathbf{B}^n \det(TM) = P_M \times_{O(n)} \widetilde{Z}_0(S^n).$$

Hence, the determinant bundle of M

$$\det(TM) = P_M \times_{O(n)} U(\Omega^n(\widetilde{Z}_0(S^n)))$$

is trivial as well. Therefore, M is an orientable manifold. □

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* This follows from the isomorphisms

$$[K(G,n), K(G,n)] \cong \mathrm{Hom}(\pi_*(K(G,n)), \pi_*(K(G,n))) \cong \mathrm{Hom}(G,G).$$

Now, we are going to construct a continuous map

$$\varphi\colon\ Z_0(M) \longrightarrow \Gamma(M, \mathbf{B}^n \det(TM))$$

where $\Gamma(M, \mathbf{B}^n \det(TM))$ is the group of continuous sections of $\mathbf{B}^n \det(TM)$.

Choose a Riemannian metric g on M and $\varepsilon > 0$ so that ε is less than the injectivity radius of M. The geodesic ε-disc centered at $x \in M$ will be denoted by $D_x(\varepsilon)$.

For every $x \in M$ the surjective map

$$f_x\colon\ D_x(\varepsilon) \longrightarrow T_xM, \qquad f_x(y) = \frac{\exp_x^{-1}(y)}{\varepsilon - \exp_x^{-1}(y)}$$

extends to the continuous map

$$F_x\colon\ M \longrightarrow \widehat{T_xM}$$

that coincides with t_x on $D_x(\varepsilon)$ and sends the complement of this disc to the point at infinity ∞ of $\widehat{T_xM}$. Let

$$\varphi_x\colon\ Z_0(M) \longrightarrow \widetilde{Z}_0(\widehat{T_xM})$$

be defined by the formula

$$\varphi_x\Big(\sum n_i x_i\Big) = \sum n_i F_x(x_i) - \Big(\sum n_i\Big) \cdot \infty.$$

Since $\widetilde{Z}_0(\widehat{T_xM})$ is a topological group and F_x depends in a continuous way on x the same is true for φ_x. Thus we get a continuous map

$$\varphi\colon\ Z_0(M) \longrightarrow \Gamma(M, \mathbf{B}^n \det(TM)).$$

Note that τ induces the map

$$\tau_*\colon\ \Gamma(M, \mathbf{B}^n \det(TM)) \longrightarrow \mathrm{Map}(M, \widetilde{Z}_0(S^n))$$
$$\tau_*(s) = \tau \circ s.$$

The composition of φ with τ_* gives a map

$$\Phi\colon\ Z_0(M) \longrightarrow \mathrm{Map}(M, \widetilde{Z}_0(S^n)).$$

Note that for any polyhedron $A \subset M - \partial M$ the restriction of Φ to $Z_0(A)$ induces a map

$$\Phi\colon\ Z_0(A) \longrightarrow \mathrm{Map}\Big((M, M - U_\varepsilon(A)), (\widetilde{Z}_0(S^n), 0)\Big)$$

where $U_\varepsilon(A)$ is the ε-neighborhood of A in M. It is easy to see that for ε sufficiently small $U_\varepsilon(A)$ is a deformational retract of A and for every $k \geqslant 0$

$$\pi_k \mathrm{Map}\Big((M, M - U_\varepsilon(A)), (\widetilde{Z}_0(S^n), 0)\Big) \cong H^{n-k}(M, M - A; \mathbb{Z}).$$

Using the same arguments as in the topological case one proves that Φ is a homotopy equivalence.

In order to generalize the construction of $\Phi = \tau_* \circ \varphi$ to higher dimensional cycles we replace φ by the map

$$\varphi^k\colon\ Z_k(M) \longrightarrow \Gamma(M, \mathbf{B}^{n-k}\mathrm{det}(TM))$$
$$\varphi^k(x)(c) = (F_x)_\sharp(c)$$

where for $k > 0$

$$\mathbf{B}^{n-k}\mathrm{det}(TM) = P_M \times_{O(n)} Z_k(S^n)$$

and replace τ by a map

$$\tau^k\colon\ \mathbf{B}^{n-k}\mathrm{det}(TM) \longrightarrow Z_k(S^n)$$

defined as follows.

The trivialization map

$$\tau\colon\ P_M \times_{O(n)} \widetilde{Z}_0(S^n) \longrightarrow \widetilde{Z}_0(S^n)$$

induces the map

$$\Omega^k\tau\colon\ P_M \times_{O(n)} \Omega^k\widetilde{Z}_0(S^n) \longrightarrow \Omega^k\widetilde{Z}_0(S^n).$$

Since both $\Omega^k\widetilde{Z}_0(S^n)$ and $Z_k(S^n)$ are $K(\mathbb{Z}, n-k)$s, they are homotopy equivalent and any homotopy equivalence map

$$h^k\colon\ Z_k(S^n) \longrightarrow \Omega^k\widetilde{Z}_0(S^n)$$

induces the following fiberwise homotopy equivalence of fibrations

$$H^k\colon\ P_M \times_{O(n)} Z_k(S^n) \longrightarrow P_M \times_{O(n)} \Omega^k\widetilde{Z}_0(S^n).$$

We define τ^k as the composition $h^k \circ \Omega^k\tau \circ H^k$.

Appendix

The aim of this appendix is to prove the isomorphism

$$H^n(X;\mathbb{Z}) \cong \pi_0\mathrm{Map}(X, AG_0(S^n))$$

where $H^n(X;\mathbb{Z})$ stands for the n-th singular cohomology group of X.

The following proof is based on the Dold–Thom theorem and the equivalence of the homotopy categories [**Top**], [$\mathbf{\Delta}$(Sets)], [$\mathbf{\Delta}$(Gr)], $\mathbf{D}(\mathbf{K}_+)$ of topological spaces, simplicial sets, simplicial groups, and chain complexes bounded from below respectively [5, 3]. For C being one of the above categories $\mathrm{Hom}_C(X, Y)$ stands for the set of morphisms of C between the objects X, Y of C. Thus $\mathrm{Map} = \mathrm{Hom}_{\mathrm{Top}}$.

Let us start by recalling some basic definitions.

Let (A_*, d_*) be a chain complex

$$\cdots \leftarrow A_{n-1} \stackrel{d_n}{\leftarrow} A_n \stackrel{d_{n+1}}{\leftarrow} A_{n+1} \leftarrow \cdots$$

The shifted complex $(A_*[p], d_*[p])$ is defined as follows

$$\begin{aligned}(A_*[p])_n &= A_{n-p},\\ (d_*[p])_n &= (-1)^p d_{n-p}.\end{aligned}$$

Any two chain complexes $(A_*, d_*^A), (B_*, d_*^B)$ induce a cochain complex $(\mathrm{Hom}^*(A_*, B_*), \delta^*)$ so that

$$\mathrm{Hom}^n(A_*, B_*) = \mathrm{Hom}_{K_+}(A_*, B_*[n])$$

and the coboundary operator

$$\delta^n\colon\ \mathrm{Hom}^n(A_*, B_*) \to \mathrm{Hom}^{n+1}(A_*, B_*)$$

is defined by the formula

$$(\delta^n f_*)_p = d^B[n]_{p+1} \circ f_{p+1} - f_p \circ d^A_{p+1}.$$

Two chain maps $f_*, g_*\colon A_* \to B_*$ are homotopic if there is a chain map

$$s_*\colon\ A_* \to B_*[-1]$$

so that

$$f_* - g_* = d^B_{*+1} \circ s_* + s_{*-1} \circ d^A_*.$$

Homotopy is an equivalence relation and the group of homotopy classes of chain maps from A_* to B_* is denoted by $\pi_0\mathrm{Hom}_{K_+}(A_*, B_*)$. A straightforward consequence of the above definitions is the following isomorphism

$$H^n\left(\mathrm{Hom}^*(A_*, B_*), \delta^*\right) \cong \pi_0\mathrm{Hom}_{K_+}(A_*, B_*). \tag{2}$$

To every simplicial abelian group $G = (G_*, \partial_*, s_*)$ one can assign a chain complex $C_*(G)$ so that $C_n(G) = G_n$ and $d_n\colon C_n(G) \to C_{n-1}(G)$ is given by the formula

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i.$$

The classical result of Dold and Kan says that $C_*\colon \mathbf{\Delta}(\mathrm{Gr}) \to \mathbf{K}_+$ induces an equivalence of the appropriate homotopy categories

$$[\mathbf{\Delta}(\mathrm{Gr})] \longrightarrow \mathbf{D}(\mathbf{K}_+).$$

Let $AGS(X)$ be the free abelian simplicial group of the singular complex $S(X)$ of X. The chain complex $C_*(AGS(X))$ is nothing but the singular chain complex of X. In the sequel we will denote it by $C_*^{\text{sing}}(X)$.

Let $\mathbb{Z}_*$ denote a chain (simplicial) complex so that

$$\mathbb{Z}_n = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

and all boundary (face and degeneracy) operators being the zero maps. It is easy to see that the cochain complex

$$\operatorname{Hom}^*(C_*^{\text{sing}}(X), \mathbb{Z}_*)$$

is isomorphic to the singular cochain complex $C^*_{\text{sing}}(X)$ of X. From the isomorphism (2) it follows that

$$\begin{aligned} H^n(X;\mathbb{Z}) &\overset{\text{def}}{=} H^n\left(\operatorname{Hom}^*(C_*^{\text{sing}}(X), \mathbb{Z}_*)\right) \\ &\cong \pi_0 \operatorname{Hom}_{K_+}\left(C_*^{\text{sing}}(X), \mathbb{Z}_*[n]\right). \end{aligned}$$

Since the chain complexes $\mathbb{Z}_*[n]$ and $C_*(AG_0S(S^n))$ are quasi-isomorphic

$$\pi_0 \operatorname{Hom}_{K_+}\left(C_*^{\text{sing}}(X), \mathbb{Z}_*[n]\right) \cong \pi_0 \operatorname{Hom}_{K_+}\left(C_*^{\text{sing}}(X), C_*(AG_0S(S^n))\right).$$

From the fact that $C_*\colon [\boldsymbol{\Delta}(\mathrm{Gr})] \longrightarrow \mathbf{D}(\mathbf{K}_+)$ is an equivalence of categories it follows that

$$\begin{aligned} &\pi_0 \operatorname{Hom}_{K_+}\left(C_*^{\text{sing}}(X), C_*(AG_0S(S^n))\right) \\ &\quad\cong \pi_0 \operatorname{Hom}_{\Delta(\mathrm{Gr})}\left(AGS(X), AG_0S(S^n)\right). \end{aligned}$$

Every map $AGS(X) \to AG_0S(S^n)$ is determined by its restriction to $S(X)$. Thus

$$\begin{aligned} &\pi_0 \operatorname{Hom}_{\Delta(\mathrm{Gr})}\left(AGS(X), AG_0S(S^n)\right) \\ &\quad\cong \pi_0 \operatorname{Hom}_{\Delta(\mathrm{Sets})}\left(S(X), AG_0S(S^n)\right). \end{aligned}$$

From the Dold–Thom isomorphism $AG_0S(S^n)$ is homotopy equivalent to $S(AG_0(S^n))$. Hence

$$\begin{aligned} &\pi_0 \operatorname{Hom}_{\Delta(\mathrm{Sets})}\left(S(X), AG_0S(S^n)\right) \\ &\quad\cong \pi_0 \operatorname{Hom}_{\Delta(\mathrm{Sets})}\left(S(X), S(AG_0(S^n))\right). \end{aligned}$$

Since for every simplicial set T and a topological space X

$$\pi_0 \operatorname{Hom}_{\Delta(\mathrm{Sets})}(T, S(X)) \cong \pi_0 \operatorname{Hom}_{\mathrm{Top}}(|T|, X) = \pi_0 \operatorname{Map}(|T|, X)$$

we have

$$\pi_0 \operatorname{Hom}_{\Delta(\mathrm{Sets})}\left(S(X), S(AG_0(S^n))\right) \cong \pi_0 \operatorname{Map}\left(|S(X)|, AG_0(S^n)\right).$$

Finally, because for every X the spaces X and $|S(X)|$ are homotopy equivalent, we get the required isomorphism

$$H^n(X;\mathbb{Z}) \cong \pi_0\mathrm{Map}\,(X, AG_0(S^n))\,.$$

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