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Extensions with restricted ramification and duality for arithmetic schemes

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Let X be a $(d + 1)$ -dimensional smooth arithmetic scheme lying smoothly and properly over an open subset U of $\text{Spec}(\mathcal{O}_K)$ for some algebraic number field K with geometrically connected fibres of dimension d . It is well-known, that there is a duality pairing for the sheaf cohomology of X , which connects the étale cohomology groups of a locally constant constructible sheaf F with the cohomology with compact support of the sheaf $F^D(d)$, where (d) means the d -fold Tate-twist (see [6]).

In this paper we will prove a duality theorem, which is exclusively formulated in terms of the ordinary sheaf cohomology, not using cohomology groups with compact support. One could also view the result as a computation of the cohomology with compact support in terms of ordinary sheaf cohomology using a dualizing sheaf.

With the above notations let $U = \text{Spec}(\mathcal{O}_{K,S})$ where S is a finite set of places of K including the archimedean places. We write S^f for the set of finite places in S and S_∞ for the set of archimedean places of K , thus $S = S^f \cup S_\infty$. As usual we denote the maximal extension of K , unramified outside S by K_S . For an integer n and an abelian group A we denote the kernel of the n -multiplication map by ${}_n A$. Let $C_{S^f}(K_S)$ be the S^f -idele class group of the field K_S (for a definition see below) and we put $I_{n,U} := {}_n C_{S^f}(K_S)$. The $\text{Gal}(K_S/K)$ -module $I_{n,U}$ is the direct limit of its finite submodules and hence defines a sheaf of $\mathbb{Z}/n\mathbb{Z}$ -modules on $U_{\text{ét}}$. Then the following holds.

THEOREM 1. *Let U be an open subscheme of $\text{Spec}(\mathcal{O}_K)$, where K is a finite extension of \mathbb{Q} . Further let*

$$\pi : X \longrightarrow U$$

be a smooth and proper morphisms of schemes with fibres of pure dimension d . Let n be an integer invertible on U and assume K to be totally imaginary if n is even. Then for every locally constant, constructible sheaf F of $\mathbb{Z}/n\mathbb{Z}$ -modules on $X_{\text{ét}}$ the cup-product

$$H_{\text{et}}^i(X, F) \times H_{\text{et}}^{2d+2-i}(X, \text{Hom}(F, \pi^*I_{n,U}(d))) \xrightarrow{\cup} H_{\text{et}}^{2d+2}(X, \pi^*I_{n,U}(d)) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$

defines a perfect pairing of finite abelian groups for all i .

Using generalized Poincaré-duality the heart of the proof of Theorem 1 is the case $d = 0$, which can be reformulated in terms of Galois groups with restricted ramification of algebraic number fields. The proof combines a result of Wingberg on pro- p -extensions with restricted ramification with a theorem of Washington on the behaviour of the prime-to- p part of ideal class groups in \mathbb{Z}_p -extensions of algebraic number fields.

For a profinite group G and a prime number p we denote the maximal pro- p factor group of G by $G(p)$. Using the language of higher etale homotopy groups we prove the following theorem for the case of arithmetic surfaces.

THEOREM 2. *Assume that the fibres of $\pi: X \rightarrow U$ are smooth, projective curves of nonzero genus and let p be a prime number, which is invertible on U . Then one has:*

$$\pi_i^{\text{et}}(X)(p) = 0 \text{ for all } i \geq 2.$$

Combining Theorems 1 and 2 we conclude:

THEOREM 3. *Assume that the fibres of $\pi: X \rightarrow U$ are smooth, projective curves of genus g and let p be a prime number invertible on U . Assume that p is odd or that K is totally imaginary. Then the following holds for the etale fundamental group $G := \pi_1^{\text{et}}(X)$:*

- (i) *If $g \geq 1$ then G is a profinite duality group at p of dimension 4.*
- (ii) *If $g = 0$ then G is a profinite duality group at p of dimension 2.*

REMARK. (a) If $p = 2$ and K is not totally imaginary the theorem remains true, if we replace the word duality group by virtual duality group, i.e. there is an open subgroup being a duality group.

(b) Under certain restrictions to X and U it was shown in [8], that the maximal pro- p factor group of G has similar duality properties.

1. In this section we prove a duality theorem for Galois groups with restricted ramification of algebraic number fields which will imply Theorem 1 for the case $d = 0$.

Let K be a number field and let S be a finite set of places of K which contains all archimedean places. By K_S we denote the maximal extension of K , which is unramified outside S and we denote $\text{Gal}(K_S/K)$ by G_S .

For an intermediate field $K \subset L \subset K_S$ we use the following notations:

- $S(L)$ the set of places of L lying above S ,
- $E_{L,S}$ the group of $S(L)$ -units of L ,
- J_L the idele group of L ,
- $J_{L,S}$ the S -idele group of L ,
- $C_S(L)$ the S -idele class group of L ,
- $Cl_S(L)$ the S -ideal class group of L .

We further use similar notations for the set of finite primes S^f in S . The definitions are the same word-by-word, however we could not find them in the literature and therefore we give the definitions. Let L be a finite extension of K in K_S :

$$J_{L,S^f} = \{(a_w) \in J_L \mid a_w = 1 \text{ for } w \notin S^f\} \cong \prod_{v \in S^f} L_v^\times,$$

$$E_{L,S^f} = E_{L,S}\text{-the group of } S^f\text{-units,}$$

$$C_{L,S^f} = J_{S^f}(L)/E_{S^f}(L).$$

Further we denote:

$$J_{S^f} = \varinjlim J_{L,S^f}, \quad E_{S^f} = \varinjlim E_{L,S^f}, \quad C_{S^f} = \varinjlim C_{L,S^f},$$

where the limit runs through all finite subextensions L of K in K_S .

REMARK: If K is totally imaginary, every extension of K is automatically unramified at the archimedean places. In this situation the pair (G_S, C_{S^f}) is a P -class formation for the set P of prime numbers l with $l^\infty \mid \#G_S$ having analogous properties like the usual P -class formation (G_S, C_S) (see Proposition 10 below). Each element of $C_S(K)$ which can be represented by an idele with support in the set of archimedean places is a universal norm for (G_S, C_S) . The advantage of (G_S, C_{S^f}) is that it is free of this ‘redundancy at infinity’, i.e. the group of universal norms in $C_{S^f}(K)$ is much smaller.

Let p be a prime number and we assume K to be totally imaginary if $p = 2$. By $S_p(K)$ we denote the set of places of K dividing p . For an abelian group A we will denote the subgroup of elements which are annihilated by a power of p by $\text{Tor}_p(A)$.

THEOREM 4. *If $S \supseteq S_p(K) \cup S_\infty(K)$ then G_S is a duality group at p of dimension 2 with p -dualizing module $I = \text{Tor}_p(C_{S^f}(K_S))$, i.e. for every finite discrete p -primary G_S -module M and all i , the cup-product*

$$H^i(G_S, M) \times H^{2-i}(G_S, \text{Hom}(M, I)) \xrightarrow{\cup} H^2(G_S, I) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

defines a perfect pairing of finite groups.

We will prove Theorem 4 in several steps.

PROPOSITION 5. *Under the above assumptions one has $cd_p(G_S) = 2$. For a finite discrete p -primary G_S -module M the cohomology groups $H^i(G_S, M)$ are finite for all i . The p -dualizing module of G_S is isomorphic to $\text{Tor}_p(C_{Sf}(K_S))$.*

Proof. The statement about the cohomological dimension as well as the finiteness statement on the cohomology can be found at various places in the literature (see [6], [4]). Following the description of the dualizing module of an arbitrary profinite group in [10], [11] it holds for the p -dualizing module I of G_S :

$$I = \varinjlim_{L, \text{cor}^*} H^2(\text{Gal}(K_S/L), \mathbb{Z}/p^n\mathbb{Z})^*,$$

where L runs through the finite subextensions of K in K_S .

Since $\mu_{2p} \in K_S$ we can take the limit over all L which are totally imaginary. Using Tate’s long exact sequence ([6] I Sect. 4 Thm. 4.10) we obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(\text{Gal}(K_S/L), \mu_{p^n}) \longrightarrow \prod_{v \in S^f(L)} H^0(\text{Gal}(\bar{K}_v/L_v), \mu_{p^n}) \longrightarrow \\ \longrightarrow H^2(\text{Gal}(K_S/L), \mathbb{Z}/p^n\mathbb{Z})^* \longrightarrow H^1(\text{Gal}(K_S/L), \mu_{p^n}). \end{aligned}$$

Going to the limit over L and n we obtain:

$$I \cong \text{coker} \left(\mu_{p^\infty} \xrightarrow{\text{diag}} \prod_{v \in S^f(K)} (\text{Ind}_{G_v}^{G_S} \mu_{p^\infty}) \right),$$

where G_v denotes the decomposition group in G_S of a fixed extension of v to K_S . Using the exact sequence

$$0 \longrightarrow E_{Sf}(K_S) \longrightarrow J_{Sf}(K_S) \longrightarrow C_{Sf}(K_S) \longrightarrow 0$$

and the fact, that $E_{Sf}(K_S)$ is p -divisible we obtain the exact sequence

$$0 \longrightarrow \text{Tor}_p(E_{Sf}(K_S)) \longrightarrow \text{Tor}_p(J_S(K_{Sf})) \longrightarrow \text{Tor}_p(C_S(K_{Sf})) \longrightarrow 0.$$

As $\mu_{p^\infty} \subset E_{Sf}(K_S)$ and by the definition of J_{Sf} we get the exact sequence

$$0 \longrightarrow \mu_{p^\infty} \xrightarrow{\text{diag}} \prod_{v \in S^f(K)} (\text{Ind}_{G_v}^{G_S} \mu_{p^\infty}) \longrightarrow \text{Tor}_p(C_{Sf}(K_S)) \longrightarrow 0,$$

which proves the proposition. □

In order to prove Theorem 4 we have to verify the vanishing of the following limit for $i = 0, 1$ (cf. [10], [11]):

$$D_i(\mathbb{Z}/p\mathbb{Z}) \stackrel{\text{def}}{=} \varinjlim_{\substack{U \subset G_S \\ \text{cor}^*}} H^i(U, \mathbb{Z}/p\mathbb{Z})^*,$$

where the limit runs through the open subgroups of U and the transition maps are the duals of the corestriction maps.

It is easy to see that $D_0(\mathbb{Z}/p\mathbb{Z}) = 0$ since $p^\infty \mid \#G_S$. In order to prove the vanishing of $D_1(\mathbb{Z}/p\mathbb{Z})$ the following theorem is crucial.

THEOREM 6. *If $S \supset S_p \cup S_\infty$ then $C_S(K_S)$ is p -divisible.*

REMARK. It is easily seen, that the p -divisibility of $C_S(K_S)$ is equivalent to the fact, that the local field $(K_S)_v$ is a p -closed local field for all $v \in S(K_S)$. If S omits only finitely many primes of K this is an easy consequence of the theorem of Grunwald-Hasse-Wang (see [7]). However if S is finite this is non-trivial. If S does not contain S_p one even does not know, whether the supernatural order of G_S is divisible by p^∞ .

In [13] Wingberg investigates the similar pro- p situation for Theorem 6. In order to refer to his result we introduce the following notations:

- $K_S(p)$ the maximal pro- p subextension of K in K_S ,
- $G_S(p)$ the Galois group of $K_S(p)/K$,
- K_v the completion of the number field K at the prime v ,
- $K_v(p)$ the maximal pro- p extension of K_v ,
- \mathcal{G}_v the Galois group of $K_v(p)/K_v$,
- \mathcal{T}_v the inertia group of v in \mathcal{G}_v ,
- G_v the decomposition group of v in $G_S(p)$,
- $H_1 \star H_2$ the free pro- p product of the pro- p groups H_1 and H_2 .

Using a theorem of Kuz'min, Wingberg proved the following theorem see [13]:

THEOREM 7. (Wingberg [13]). *Assume $\mu_{2p} \subset K$ and $S \supset S_p(K) \cup S_\infty$ and suppose that $C_S(K_S(p))$ is not p -divisible. Then there exists a prime $v \in S_p(K)$ such that the decomposition group G_v of v in the extension $K_S(p)/K$ is the full group $G_S(p)$ and the following holds:*

There exists a finite set T of primes of K containing S such that the homomorphism

$$\phi : \bigstar_{\substack{v' \in S \setminus S_\infty \\ v' \neq v}} \mathcal{G}_{v'} \star \bigstar_{v' \in T \setminus S} (\mathcal{G}_{v'} / \mathcal{T}_{v'}) \xrightarrow{\sim} G_S(p)$$

induced by the maps: $\phi_{v'} : \mathcal{G}_{v'} \hookrightarrow \text{Gal}(K(p)/K) \xrightarrow{\text{can}} G_S(p)$ is an isomorphism.

COROLLARY 8. *If $C_S(K_S(p))$ is not p -divisible then there exists a prime $v \in S_p(K)$ such that the following inequality holds*

$$[K_v : \mathbb{Q}_p] \geq \sum_{\substack{v' \in S_p(K) \\ v' \neq v}} [K_{v'} : \mathbb{Q}_p].$$

Proof of the corollary: As $\mu_p \subset K$ the following (in)equality holds by [9] II Section 5 Thm. 4 for a prime $v' \in S_p(K)$:

$$\text{rank } G_{v'} \leq \text{rank } \mathcal{G}_{v'} = [K_{v'} : \mathbb{Q}_p] + 2.$$

Now assume that $\#S_p(K) > 1$ (otherwise the statement of the corollary is trivial) and assume that $C_S(K_S(p))$ is not p -divisible. By Theorem 7 there exists a prime $v \in S_p(K)$ such that $G_S = G_v$ and the free product decomposition of G_S yields the following inequalities:

$$\begin{aligned} [K_v : \mathbb{Q}_p] &= \text{rank } \mathcal{G}_v - 2 \\ &\geq \text{rank } G_v - 2 = \text{rank } G_S - 2 \\ &\geq \left(\sum_{\substack{v' \in S_p(K) \\ v' \neq v}} \text{rank } \mathcal{G}_{v'} \right) - 2 = \left(\sum_{\substack{v' \in S_p(K) \\ v' \neq v}} ([K_{v'} : \mathbb{Q}_p] + 2) \right) - 2 \\ &\geq \sum_{\substack{v' \in S_p(K) \\ v' \neq v}} [K_{v'} : \mathbb{Q}_p], \end{aligned}$$

which proves the corollary. □

In order to prove Theorem 6 it is obviously sufficient to prove that the group $C_S(L_S(p))$ is p -divisible for a cofinal set of finite extensions L of K in K_S . Since $\mu_{2p} \subset K_S$ we assume without loss of generality that $\mu_{2p} \subset K$, in particular K is totally imaginary, containing the imaginary abelian field $\mathbb{Q}(\zeta_{2p})$. Now what we need is a method of leaving the bad situations described in theorem 7. For this we use a result of Washington, where k_n denotes the unique subextension of degree p^n in the cyclotomic \mathbb{Z}_p -extension of a number field k and h_n denotes the class number of k_n .

THEOREM 9. (Washington [12]). *Let k be an imaginary abelian number field and let*

$$H = \{l \mid l \text{ prime number, } l \text{ divides } h_n \text{ for some } n\}.$$

Then H is infinite.

Now returning to our situation assume that L is a finite subextension of K in K_S and that $C_S(L_S(p))$ is not p -divisible.

CLAIM: There is a finite extension L'/L contained in K_S such that $C_S(L'_S(p))$ is p -divisible.

Proof of the claim: Following Theorem 9, choose a prime number $l > p$ and n such that:

- (i) $l|h(\mathbb{Q}(\zeta_{p^n}))$ and
- (ii) $(l, [L : \mathbb{Q}]) = 1$.

By class field theory there exists a finite cyclic unramified extension $F/\mathbb{Q}(\zeta_{p^n})$ of degree l . As the only prime of $\mathbb{Q}(\zeta_{p^n})$ which divides p is principal it completely splits in F , hence there are l different primes dividing p in F . By condition (ii) we see that F and $L(\zeta_{p^n})$ are linearly disjoint over $\mathbb{Q}(\zeta_{p^n})$. Hence every prime v of $L(\zeta_{p^n})$ dividing p splits into l different primes in LF . Therefore the field LF has the property that for every prime dividing p there are at least $l - 1$ other primes having the same absolute local degree. By Corollary 8 we obtain that the group $C_S((LF)_S(p))$ is p -divisible which proves the claim.

Thus we have proved Theorem 6. □

Now we are able to prove the vanishing of D_1 (see above).

$$\begin{aligned} D_1(\mathbb{Z}/p\mathbb{Z}) &= \varinjlim H^1(\text{Gal}(K_S/L), \mathbb{Z}/p\mathbb{Z})^* \\ &= \varinjlim \text{Gal}(K_S/L)^{ab}/p \\ &= \varinjlim C_S(L)/p \\ &= C_S(K_S)/p = 0. \end{aligned}$$

Thus the proof of Theorem 4 is complete. □

In order to get a better understanding of Theorem 4 we give the following proposition for the case of totally imaginary K . (Compare [6] I, Sect. 4, 4.2, 4.5, 4.6.)

PROPOSITION 10. *Let K be a totally imaginary number field and let $S \supset S_\infty(K)$ be a finite set of primes. Then the following holds.*

- (i) *The pair (G_S, C_{Sf}) is a P -class formation for the set P of primes l with $l^\infty \mid \#G_S$.*
- (ii) *The reciprocity map: $\text{rec}: C_{Sf}(K) \xrightarrow{\text{rec}} G_S^{ab}$ is surjective with divisible kernel.*
- (iii) *If p is a prime number such that all primes dividing p are in S and if M is a finite p -primary G_S -module then the cup-product defines isomorphisms:*

$$\alpha^r : \text{Ext}_{G_S}^r(M, C_{Sf}) \xrightarrow{\sim} H^{2-r}(G_S, M)^*$$

for all r .

Proof. Using the definition of C_S and C_{Sf} we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & & \\
 & & & \downarrow & & \downarrow & & & & \\
 & & & \text{Ind}^{G_S} \mathbb{C}^\times & = & \text{Ind}^{G_S} \mathbb{C}^\times & & & & \\
 & & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & E_S & \longrightarrow & J_S & \longrightarrow & C_S & \longrightarrow & 0 & \\
 & & \parallel & & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & E_{Sf} & \longrightarrow & J_{Sf} & \longrightarrow & C_{Sf} & \longrightarrow & 0 & \\
 & & & & \downarrow & & \downarrow & & & \\
 & & & & 0 & & 0 & & &
 \end{array}$$

Hence we obtain for all $i \geq 1$ isomorphisms: $H^i(G_S, C_S) \xrightarrow{\sim} H^i(G_S, C_{Sf})$. Since (G_S, C_S) is a P -class formation the same follows for (G_S, C_{Sf}) . We call the reciprocity maps associated to these class formations by rec and rec_f and we denote the kernel of rec (rec_f) by $D_S(K)$ ($D_{Sf}(K)$). It is well-known ([6] I, Sect. 4, 4.5), that rec is surjective and that $D_S(K)$ is divisible. Hence we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & & \\
 & & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & \prod_{v \in S_\infty(K)} \mathbb{C}^\times & \longrightarrow & D_S(K) & \longrightarrow & D_{Sf}(K) & \longrightarrow & 0 & \\
 & & \parallel & & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & \prod_{v \in S_\infty(K)} \mathbb{C}^\times & \longrightarrow & C_S(K) & \longrightarrow & C_{Sf}(K) & \longrightarrow & 0 & \\
 & & & & \downarrow \text{rec} & & \downarrow \text{rec}_f & & & \\
 & & & & G_S^{ab} & = & G_S^{ab} & & & \\
 & & & & \downarrow & & \downarrow & & & \\
 & & & & 0 & & 0 & & &
 \end{array}$$

This proves (ii). By Theorem 1.13 of [6] I Sect. 1 and by (ii) we obtain that α^r is an isomorphism for $r \geq 1$ (cf. [6] I Sect. 4 Thm. 4.6.(a)). The statement for $r = 0$ follows from Proposition 5. For $r < 0$ both groups are zero. \square

REMARK. Now we can conclude Theorem 4 from Proposition 10 (iii) and from Theorem 6: C_{Sf} is a quotient of C_S , hence p -divisible. Therefore the spectral sequence

$$E_2^{pq} = H^r(G_S, \text{Ext}_{\mathbb{Z}}^q(M, C_{Sf})) \implies \text{Ext}_{G_S}^{p+q}(M, C_{Sf})$$

degenerates to a sequence of isomorphisms: $H^p(G_S, \text{Hom}(M, C_{Sf})) \xrightarrow{\sim} \text{Ext}_{G_S}^p(M, C_{Sf})$, which by Proposition 10(iii) implies Theorem 4.

2. In this section we prove Theorem 1.

Let K be a number field, $U \subset \text{Spec}(\mathcal{O}_K)$ an open subscheme and let S be the finite set of places of K , containing the archimedean places and such that $U = \text{Spec}(\mathcal{O}_{K,S})$. As it is well known every finite discrete G_S -module M defines a locally constant constructible sheaf on U_{et} . If M is a discrete G_S module, annihilated by some integer n , then M is the direct limit of its finite submodules and therefore defines a sheaf of $\mathbb{Z}/n\mathbb{Z}$ -modules on U_{et} , which is the direct limit of locally constant constructible sheaves. The following proposition is well-known.

PROPOSITION 11. *Assume that n is invertible on U . Then for all i :*

$$H_{\text{et}}^i(U, M) \cong H^i(G_S, M).$$

Proof. If M is finite this is [6] II, Sect. 2, 2.9. The general case follows since étale cohomology as well as Galois cohomology commute with direct limits. \square

Therefore Theorem 1 for $d = 0$ is equivalent to Theorem 4 which is already proved. Further one knows that the p -dualizing module of a duality group is divisible (or else $\text{Tor}_p(C_{Sf}(K_S))$ is p -divisible as $C_S(K_S)$ is p -divisible). For further use we note:

COROLLARY 12. *The stalks of the sheaf $I_{n,U}$ are injective $\mathbb{Z}/n\mathbb{Z}$ -modules.*

Now let X, π, U, n be as in the introduction. By $D(X, \mathbb{Z}/n\mathbb{Z})$ resp. $D(U, \mathbb{Z}/n\mathbb{Z})$ we denote the derived category of $\mathbb{Z}/n\mathbb{Z}$ -module sheaves on X_{et} resp. U_{et} . As π and n fulfill the conditions of generalized Poincaré-duality we get the following isomorphism of objects in $D(U, \mathbb{Z}/n\mathbb{Z})$ (see Theorem 3.2.5. in [1] XVIII):

$$R\pi_* \mathcal{H}om_{D(X, \mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d)[2d]) \cong \mathcal{H}om_{D(U, \mathbb{Z}/n\mathbb{Z})}(R\pi_* F, I_{n,U}). \quad (*)$$

Here (d) is the d -fold Tate-twist and $[2d]$ is the shift by $2d$.

By Corollary 12 the stalks of $I_{n,U}$ (resp. $\pi^* I_{n,U}$) are injective $\mathbb{Z}/n\mathbb{Z}$ -modules. Therefore we have an isomorphism

$$\mathcal{H}om_{D(X, \mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d)[2d]) \cong \mathcal{H}om_{D(X, \mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d))[2d].$$

Applying $R\Gamma(U, -)$ on the left hand side of $(*)$ we get a complex of abelian groups whose r^{th} cohomology group is isomorphic to

$$H_{\text{et}}^{r+2d}(X, \mathcal{H}om(F, \pi^* I_{n,U}(d))).$$

By the proper-smooth base change theorem [5] VI, Sect. 4, 4.2) the sheaves $R^i \pi_* F$ are locally constant and constructible on U . Applying $R\Gamma(U, -)$ on the right hand side of (*) we get a complex of abelian groups whose r^{th} cohomology group by theorem 1 for $d = 0$ is isomorphic to

$$\text{Hom}(H_{\text{et}}^{2-r}(X, F), \mathbb{Z}/n\mathbb{Z}).$$

This proves Theorem 1. □

3. In this section we prove Theorems 2 and 3. For a scheme X we denote by X_{et} its etale homotopy type, i.e. a pro-simplicial set. The etale homotopy groups of X are by definition the homotopy groups of X_{et} and it is well-known that these pro-groups are pro-finite, whenever the scheme X is noetherian, connected and geometrically unibranch ([2] Theorem 11.1). By \tilde{X}_{et} we denote the universal covering of X_{et} . If p is a prime number and Y is a pro-simplicial set, we denote the pro- p completion of Y by $Y.^{\wedge p}$. The maximal pro- p factor group of a pro-group G is denoted by $G(p)$. For the following we need

PROPOSITION 13. *Assume that Y is simply connected (i.e. $\pi_1(Y) = 0$) and that $\pi_i(Y)$ is pro-finite for all $i \geq 2$. Then we have isomorphisms for all i :*

$$\pi_i(Y)(p) \rightarrow \pi_i(Y.^{\wedge p}).$$

Proof. If G is an abelian pro-finite group, the canonical surjection: $G \rightarrow G(p)$ has a kernel with trivial p -Sylow subgroup, i.e. the supernatural order of the kernel is prime to p . Therefore G is a p -good pro-group in the sense of [2], i.e. for every finite p -primary $G(p)$ -module M the canonical homomorphism: $H^i(G(p), M) \xrightarrow{\text{infl}} H^i(G, M)$ is an isomorphism for all i . Hence Proposition 13 follows by induction on i from Theorem 6.7. of [2]. □

Now let K be a number field, let $U \subset \text{Spec}(\mathcal{O}_K)$ be an open subscheme and let p be a prime number invertible on U . Then the following holds:

PROPOSITION 14. *The higher etale homotopy groups of U have no p -part, i.e.*

$$\pi_i^{\text{et}}(U)(p) = 0 \quad \text{for } i \geq 2.$$

Further the canonical morphism:

$$U_{\text{et}}.^{\wedge p} \rightarrow K(\pi_1^{\text{et}}(U)(p), 1)$$

with $\pi_1^{\text{et}}(U)(p)$ the maximal pro- p factor group of the etale fundamental group of U is a weak homotopy equivalence.

Proof. Let S be the finite subset of places of K , containing the archimedean places and such that $U = \text{Spec}(\mathcal{O}_{K,S})$. Then $G_S(K) \cong \pi_1^{\text{et}}(U)$ and Proposition 11 implies that the universal covering \tilde{U}_{et} has no cohomology with values in p -primary coefficient groups. Hence the pro- p completion of \tilde{U}_{et} is contractible and therefore the first statement of the proposition follows from Proposition 13. By a theorem of O. Neumann (see [4] Prop. 22) for every finite p -primary $G_S(p)$ -torsion module M the canonical homomorphism $H^i(G_S(p), M) \xrightarrow{\text{infl}} H^i(G_S, M)$ is an isomorphism for all i . The same arguments as for the first statement then also show the second. \square

Similar arguments apply in the geometric situation. Here we denote the genus of a curve C by $g(C)$.

PROPOSITION 15. *Let k be a field and let C be a connected, smooth curve over k . Assume either that C is incomplete or $g(C) > 0$. Then*

$$\pi_i^{\text{et}}(C) = 0 \quad \text{for } i \geq 2,$$

i.e. the canonical morphism

$$C_{\text{et}} \longrightarrow K(\pi_1^{\text{et}}(C), 1)$$

is a weak homotopy equivalence. If k is separably closed and p is an arbitrary prime number then also the canonical morphism

$$C_{\text{et}}^{\wedge p} \longrightarrow K(\pi_1^{\text{et}}(C)(p), 1)$$

is a weak homotopy equivalence.

Proof. First, if necessary, we replace k by a suitable extension, such that C is geometrically connected. We denote the base change of C to an separable closure k^s of k by C_{k^s} . Since the morphism $C_{k^s} \rightarrow C$ is pro-étale the canonical morphism:

$$\tilde{C}_{k^s, \text{et}} \xrightarrow{\text{can}} \tilde{C}_{\text{et}}$$

from the universal covering of $C_{k^s, \text{et}}$ to the universal covering of C_{et} is a weak homotopy equivalence. Therefore we can assume k to be separably closed also in the first statement. In order to show the first statement we show that the higher homotopy groups of C_{et} (which are abelian profinite groups) have no p -part for an arbitrary prime number p . If either C is incomplete or $p = \text{char}(k)$ then $H_{\text{et}}^i(C, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 2$. Therefore $H^i(\tilde{C}_{\text{et}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 1$, which implies $\pi_i^{\text{et}}(C)(p) = 0$ for $i \geq 2$. If C is complete and $p \neq \text{char}(k)$ then $H_{\text{et}}^i(C, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 2$ and Poincaré-duality [5] V, Sect. 2, Thm. 2.1) implies an isomorphism:

$H_{\text{et}}^2(C, \mathbb{Z}/p\mathbb{Z}) \cong H_{\text{et}}^0(C, \mathbb{Z}/p\mathbb{Z})^* = \mathbb{Z}/p\mathbb{Z}$. If C' is a finite etale cover of C of degree n then the following diagram is commutative:

$$\begin{array}{ccc} H_{\text{et}}^2(C, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{Z}/p\mathbb{Z} \\ \downarrow \text{can} & & \downarrow \cdot n \\ H_{\text{et}}^2(C', \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{Z}/p\mathbb{Z} . \end{array}$$

Therefore $H^2(\tilde{C}_{\text{et}}, \mathbb{Z}/p\mathbb{Z}) = 0$ if and only if there are etale covers of C of degree divisible by arbitrary high powers of p . This is obviously the case if and only if $g(C) > 0$. This proves the first statement. The same arguments also prove the second statement. \square

Now let $\pi: X \rightarrow U$ be smooth, proper with fibres which are curves of nonzero genus. We denote by $U(i) = \tilde{U}$ the pro-scheme representing the universal covering of U . By $X(i)$ we denote the pro-scheme $X \times_U U(i)$. Fixing a (geometric) base point u of U we denote $X \times_U u$ by X_u . Further let p be a prime number invertible on U . Then by Theorem 11.5. of [3] and Propositions 14, 13 we get isomorphisms for all $r \geq 1$:

$$\pi_r((X_u)_{\text{et}}^{\wedge p}) \xrightarrow{\sim} \pi_r(X(i)_{\text{et}}^{\wedge p}).$$

By Proposition 15 we obtain $\pi_r(X(i)_{\text{et}}^{\wedge p}) = 0$ for $r \geq 2$, in particular we have $H_{\text{et}}^r(X(i), \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \geq 2$. If Y is a finite etale cover of X , the same argument applies for $Y(i)$ (possibly Y is defined over an etale cover of U). Going to the limit we obtain for the universal cover \tilde{X}_{et} of X_{et} : $H^r(\tilde{X}_{\text{et}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \geq 1$. Since the groups $\pi_r^{\text{et}}(X)$ are profinite they are p -good for $r \geq 2$ and we obtain $0 = \pi_r(\tilde{X}_{\text{et}})(p) = \pi_r^{\text{et}}(X)(p)$ for $r \geq 2$. This proves Theorem 2. \square

Now Theorem 3 part (i) is an easy consequence of Theorem 2 and Theorem 1: For every finite p -primary $\pi_1^{\text{et}}(X)$ -module M the canonical homomorphism

$$H^r(\pi_1^{\text{et}}(X), M) \longrightarrow H_{\text{et}}^r(X, M)$$

is an isomorphism for all r by Theorem 2. Hence Theorem 1 implies the duality statement (i). Part (ii) is even easier. As the fibres are simply connected we have an isomorphism $\pi_1^{\text{et}}(X) \xrightarrow{\sim} \pi_1^{\text{et}}(U)$ which proves the assertion in view of Theorem 1 for U and proposition 14. \square

References

1. Artin, M., Grothendieck, A. and Verdier, J. L.: *Theorie des Topos et Cohomologie Etale des Schemas*(SGA4) LNM 269, 270, 305.

