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MARK GREEN
STEFAN MÜLLER-STACH

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Algebraic cycles on a general complete intersection of high multi-degree of a smooth projective variety

MARK GREEN[1] * and STEFAN MÜLLER-STACH[2] **
[1] *U.C.L.A. Los Angeles, Department of Mathematics, CA 90024-7009, Los Angeles, U.S.A. e-mail: mlg@math.ucla.edu*
[2] *University Essen, 45141 Essen, Germany. e-mail: mueller-stach@uni-essen.de*

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Our goal in this paper is to generalize the results of [G], [V] on the Abel–Jacobi map for general 3-folds of degree $d \geqslant 6$ in $\mathbf{P}^4$ to arbitrary general complete intersections of high multi-degrees on any smooth projective variety. Our work on this was highly influenced by conversations with M. Nori, and we have rephrased our original argument to take advantage of Nori's Connectedness Theorem [N]. A feature of our argument is that part of it relies on the geometry of the full family of complete intersections of a given multi-degree in order to force rigidity of the normal function, and part of it relies on using a pencil to construct the cycle we want.

We have phrased our result in terms of rational Deligne cohomology, as this is the most natural framework for it. We will use the notation $H^{2p}_{\mathcal{D}}(X)$ to denote the hypercohomology

$$\mathbf{H}^{2p}(X, \mathbf{Z}(p) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X);$$

this sits naturally in an exact sequence

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X) \to H^{p,p}_{\mathbf{Z}}(X) \to 0.$$

There is a cycle class map

$$\psi_X : CH^p(X) \to H^{2p}_{\mathcal{D}}(X)$$

which projects to the usual cycle class in cohomology, and which gives the Abel–Jacobi map when restricted to cycles homological to zero. For problems relating to existence of cycles and Abel–Jacobi maps, it appears that it is often better to consider the hypercohomology

$$\mathbf{H}^{2p}(X, \mathbf{Q}(p) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X),$$

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which we will denote $H^{2p}_{\mathcal{D}}(X, \mathbf{Q})$. There is an exact sequence

$$0 \to J^p(X)_{\mathbf{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbf{Q}) \to H^{p,p}_{\mathbf{Q}}(X) \to 0,$$

where $J^p(X)_{\mathbf{Q}} = J^p(X)/\text{torsion}$, and a cycle map

$$\psi_{X,\mathbf{Q}}: CH^p(X, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(X, \mathbf{Q}).$$

The reason we favor these rational Deligne cohomology groups is that the Hodge conjecture can only be true if one uses rational cohomology, and the results of Green and Voisin on the Abel–Jacobi map and our generalization of them are only known to be true modulo torsion in the intermediate Jacobian.

If $X \subseteq Y$, then there is a natural map $J^p(Y) \to J^p(X)$ and hence a natural map $J^p(Y) \to H^{2p}_{\mathcal{D}}(X)$ which descends to a map $J^p(Y)_{\mathbf{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbf{Q})$. We will denote by $\bar{\psi}_{X,\mathbf{Q}}$ the composition of the cycle map $\psi_{X,\mathbf{Q}}$ with the canonical projection $H^{2p}_{\mathcal{D}}(X, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(X, \mathbf{Q})/J^p(Y)_{\mathbf{Q}}$.

Deligne cohomology exists for compact varieties with a normal-crossing divisor removed, if we use logarithmic differentials. We make use of this in the proof.

THEOREM. *Let Y be a smooth projective variety of dimension $n + e$ and $L_1, L_2, \ldots, L_e$ sufficiently ample line bundles on Y. If $i: X \hookrightarrow Y$ is a general complete intersection on Y of type $L_1, L_2, \ldots, L_e$ and $p \leqslant n - 1$, then in the diagram*

$$\begin{array}{ccc} CH^p(Y, \mathbf{Q}) & \xrightarrow{\cap X} & CH^p(X, \mathbf{Q}) \\ \downarrow{\scriptstyle \psi_{Y,\mathbf{Q}}} & \searrow^{\alpha} & \downarrow{\scriptstyle \bar{\psi}_{X,\mathbf{Q}}} \\ H^{2p}_{\mathcal{D}}(Y, \mathbf{Q}) & \longrightarrow & H^{2p}_{\mathcal{D}}(X, \mathbf{Q})/J^p(Y)_{\mathbf{Q}} \end{array}$$

we have that $\text{Image}(\alpha) = \text{Image}(\bar{\psi}_{X,\mathbf{Q}})$.

REMARKS. (1) The one part of the statement which one might hope to dispense with is the fact that we need to mod out by $J^p(Y)_{\mathbf{Q}}$. This is in there because we do not have the Generalized Hodge Conjecture available. Being more careful, one can mod out by the part of $J^p(Y)_{\mathbf{Q}}$ coming from the maximal sub-Hodge structure of weight 1 of $H^{2p-1}(Y)$, or by that part of it for which the Generalized Hodge Conjecture fails.

(2) One cannot do better than the hypothesis $p \leqslant n - 1$, since if $p = n$, one has the example of curves X of high degree in $\mathbf{P}^2$ for which the cycle map $CH^1(X)_0 \to J^1(X)$ is surjective, while this is not true for $CH^1(\mathbf{P}^2)_h \to J^1(X)$. Of course, if $p = n > 1$, then the theorem's conclusion also holds, as an easy consequence of the Lefschetz theorems.

(3) For $Y = \mathbf{P}^{n+e}$, and X a general complete intersection of multi-degree $(d_1, \ldots, d_e)$ with all $d_i >> 0$, the theorem states that the image of $CH^p(X, \mathbf{Q}) \to$

$H^{2p}_{\mathcal{D}}(X, \mathbf{Q})$ is just the image of the hyperplane class on Y. In particular, the image of the Abel–Jacobi map for X is contained in the torsion points of $J^p(X)$. Paranjape [P] obtains a more explicit bound. One probably can use the powerful general theorems of Ein-Lazarsfeld to obtain explicit bounds for how high the multi-degrees have to be for general Y, as in the work of Ravi [R].

Proof. We will use Nori's connectedness theorem [N], and therefore will stick as closely as possible to his notation. Let $S = \prod_{i=1}^{e} \mathbf{P}(H^0(Y, L_i))$, $A = Y \times S$, and $B \subseteq A = \{((s_1, \dots, s_e), y) \mid s_i(y) = 0 \text{ for all } i = 1, \dots, e\}$. For any morphism $p: T \to S$, let $A_T = A \times_S T$ and $B_T = B \times_S T$. If $t \in S$ is a general point of S, and $Z_t \in CH^p(X_t)$, then we may choose a branched cover $p: T \to S$, a $\tilde{t} \in p^{-1}(t)$ and a $\mathcal{Z} \in CH^p(B_T)$ so that $\mathcal{Z} \cdot (X_t \times \{\tilde{t}\}) = Z_t$. If we adopt the notation $Z_s = \mathcal{Z} \cdot (X_{p(s)} \times \{s\})$, the problem is that one cannot control what $Z_{\hat{t}}$ is for other preimages $\hat{t}$ of t. We let T_0 denote a Zariski open subset of T chosen so that the map $p: T_0 \to S$ is smooth.

Nori's connectedness theorem states that the restriction map

$$H^k(A_{T_0}, \mathbf{Q}) \to H^k(B_{T_0}, \mathbf{Q})$$

is injective for $k \leqslant 2n$ and is surjective for $k \leqslant 2n - 1$. This implies that the induced map on Deligne cohomology

$$H^{2p}_{\mathcal{D}}(A_{T_0}, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(B_{T_0}, \mathbf{Q})$$

is an isomorphism if $p \leqslant n - 1$, which we have assumed. If $p: T_0 \to S$ is the projection, then we would like to show that under the natural identification of $H^{2p}_{\mathcal{D}}(X_{\tilde{t}}, \mathbf{Q})$ with $H^{2p}_{\mathcal{D}}(X_t, \mathbf{Q})$ for any $\tilde{t} \in p^{-1}(t)$, we have that $\bar{\psi}_{X_{\tilde{t}}, \mathbf{Q}}(Z_{\tilde{t}})$ represents the same class modulo the image of $J^p(Y)_{\mathbf{Q}}$ for all $\tilde{t} \in p^{-1}(t)$.

To see this, we note that since

$$r: H^{2p}_{\mathcal{D}}(A_{T_0}, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(B_{T_0}, \mathbf{Q})$$

is an isomorphism, we know that $\psi_{B_{T_0}, \mathbf{Q}}(\mathcal{Z}) = r(\phi)$ for some

$$\phi \in H^{2p}_{\mathcal{D}}(A_{T_0}, \mathbf{Q}).$$

If we denote by $q_{\tilde{t}}: H^{2p}_{\mathcal{D}}(A_{T_0}, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(Y, \mathbf{Q})$ the restriction map on Deligne cohomology from A_{T_0} to $Y \times \{\tilde{t}\}$ composed with the natural identification of $Y \times \{\tilde{t}\}$ with Y, and $r_t: H^{2p}_{\mathcal{D}}(Y, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(X_t, \mathbf{Q})$ the restriction map on Deligne cohomology, and $s_{\tilde{t}}: H^{2p}_{\mathcal{D}}(B_{T_0}, \mathbf{Q}) \to H^{2p}_{\mathcal{D}}(X_t, \mathbf{Q})$ the restriction map on Deligne cohomology from B_{T_0} to $X_t \times \{\tilde{t}\}$ composed with the natural identification of $X_t \times \{\tilde{t}\}$ with X_t, then we have a commutative diagram:

$$\begin{array}{ccc} H^{2p}_{\mathcal{D}}(A_{T_0}, \mathbf{Q}) & \xrightarrow{q_{\tilde{t}}} & H^{2p}_{\mathcal{D}}(Y, \mathbf{Q}) \\ \Big\downarrow r & & \Big\downarrow r_t \\ H^{2p}_{\mathcal{D}}(B_{T_0}, \mathbf{Q}) & \xrightarrow{s_{\tilde{t}}} & H^{2p}_{\mathcal{D}}(X_t, \mathbf{Q}). \end{array}$$

Now

$$\begin{aligned} \psi_{X_t,\mathbf{Q}}(Z_{\tilde{t}}) &= s_{\tilde{t}}(\psi_{B_{T_0},\mathbf{Q}}(\mathcal{Z})) \\ &= s_{\tilde{t}}(r(\phi)) \\ &= r_t(q_{\tilde{t}}(\phi)). \end{aligned}$$

Now we note that for any $t_1, t_2 \in T_0$, the difference $q_{t_1}(\phi) - q_{t_2}(\phi) \in J^p(Y)_{\mathbf{Q}}$. Thus if $p(t_1) = p(t_2) = t$, we have that

$$\psi_{X_t,\mathbf{Q}}(Z_{t_1}) - \psi_{X_t,\mathbf{Q}}(Z_{t_2}) \in r_t(J^p(Y)_{\mathbf{Q}})$$

as claimed in the previous paragraph. We will use this in the form: For any $\tilde{t} \in p^{-1}(t)$,

$$\bar{\psi}_{X_t,\mathbf{Q}}\left(\sum_{s \in p^{-1}(t)} Z_s\right) = \deg(p)\bar{\psi}_{X_t,\mathbf{Q}}(Z_{\tilde{t}}).$$

The case of $p = 0$ being clear, we may assume inductively that the theorem is true for all lower values of p than the one we are considering; we will use only finitely many lower cases of the theorem, so that a common lower bound for the multi-degrees can be found.

Now let E_i be a line in $\mathbf{P}(H^0(Y, L_i))$, and U the preimage in T of $\prod_{i=1}^{e} E_i$. We let $p_i\colon U \to E_i$ denote the canonical projection. We let Λ_i denote the base locus of the pencil E_i. For any i and t_i we denote by $F^i_{t_i}$ the hypersurface on Y defined by the section of L_i corresponding to t_i. We introduce the notation for any subset $I \subseteq \{1, 2, \ldots, e\}$ and $t = (t_1, \ldots, t_e)$ that $U^I_t = \{s \in U \mid p_i(s) = t_i \text{ for all } i \in I\}$. Then as cycles in $Y \times U$,

$$(X_t \times U) \cdot B_U = \sum_{I \subseteq \{1,2,\ldots,e\}} ((\cap_{i \notin I} \Lambda_i) \cap (\cap_{i \in I} F^i_{t_i})) \times U^I_t.$$

Note that this sum contains, for $I = \{1, 2, \ldots, e\}$, the term $\sum_{s \in p^{-1}(t)} X_t \times \{s\}$. Let $\mathcal{Z}_U = \mathcal{Z} \cdot B_U$. We may, possibly by replacing $\mathcal{Z}_U$ with a rationally equivalent cycle, assume that $(X_t \times U) \cdot \mathcal{Z}_U = \sum_I Z^I_t$ with Z^I_t a cycle of appropriate dimension on

$((\cap_{i\notin I}\Lambda_i)\cap(\cap_{i\in I}F^i_{t_i}))\times U^I_t$, and that $Z_t^{\{1,2,\dots,e\}}=\sum_{s\in p^{-1}(t)} Z_s\times\{s\}$. Now by the projection formula, if $\pi_1\colon Y\times U\to Y$ is the canonical projection,

$$(\pi_{1*}(\mathcal{Z}_U))\cdot X_t=\sum_{s\in p^{-1}(t)} Z_s+\sum_{I\neq\{1,2,\dots,e\}}\pi_{1*}(Z^I_t).$$

For $I\neq\{1,2,\dots,e\}$, $\pi_{1*}(Z^I_t)$ is a codimension $(p-e+|I|)$ cycle on $(\cap_{i\notin I}\Lambda_i)\cap(\cap_{i\in I}F^i_{t_i})$, which if we pick t and the pencils E_i generally is a general complete intersection on Y corresponding to the bundle $(\oplus_{i=1}^e L_i)\oplus(\oplus_{i\notin I}L_i)$. This complete intersection has dimension $n-e+|I|$, and thus by induction on p there is a codimension $(p-e+|I|)$ rational cycle W^I on Y whose rational Deligne class, modulo $J^p(Y)_{\mathbf{Q}}$, restricts to that of Z^I_t. However, this is when viewed as cycles on $(\cap_{i\notin I}\Lambda_i)\cap(\cap_{i\in I}F^i_{t_i})$; if we view them as codimension p cycles on X_t, then their rational Deligne cohomology class is obtained by restricting $\cup_{j\notin I}[L_j]$ cupped with the rational Deligne class of W^I, where $[L_i]$ is the Deligne first Chern class of L_i. On the level of Chow groups, let $\tilde{W}^I=W^I\cdot(\cap_{i\notin I}c_1(L_i))$. We thus know that $\pi_{1*}(\mathcal{Z}_U)-\sum_{I\neq\{1,2,\dots,e\}}\tilde{W}^I$, restricted to X_t, has the same rational Deligne class modulo $J^p(Y)_{\mathbf{Q}}$ as $\deg(p)Z_{\tilde{t}}$ for any $\tilde{t}$ in the preimage of t. Since we are working rationally, we may divide by $\deg(p)$ to obtain the cycle we seek.

REMARK. The Poincaré–Lefschetz–Griffiths program of proving the Hodge Conjecture using normal functions comes down to understanding the image of the Abel–Jacobi map for hypersurfaces of high degree on a smooth projective variety Y (it would be enough to understand the image modulo torsion). What we have shown is that for a general hypersurface X of high degree, the image of the Abel–Jacobi is no larger than what is required if the Hodge Conjecture is going to be true. This is encouraging, in that although one does not have a conjecture about what the image of the Abel–Jacobi map should be in general, one therefore does in this case.

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