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The overconvergence of morphisms of etale φ-∇-spaces on a local field

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1. Introduction

Let X be a smooth curve over a perfect field k of positive characteristic p and let $\overline{X}$ be the smooth compactification of X. The category of unit-root F-isocrystals over X is equivalent to the category of p-adic representations of $\pi_1(X, *)$. And it is conjectured that p-adic representations with finite monodromy around $D = \overline{X} - X$, which means that the inertia group of the Galois group around D acts trivially after a finite extension, correspond to overconvergent unit-root F-isocrystals. So the natural functor

$$\begin{pmatrix} \text{overconvergent unit-root} \\ F\text{-isocrystals on } X \text{ around } D \end{pmatrix} \longrightarrow \begin{pmatrix} \text{unit-root} \\ F\text{-isocrystals on } X \end{pmatrix}$$

is expected to be fully faithful. Crew showed finite monodromy theorem in rank one case and he proved full faithfulness for rank one objects. (See [1])

In this paper we will show full faithfulness of the above functor for arbitrary rank. To do this we study the local version of the above problem. In [3] Fontaine defined etale φ-modules on Spec $k((t))$, a spectrum of the field of formal Laurent series over k, whose category is equivalent to the category of p-adic representations of $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$. We will define a subcategory of overconvergent etale φ-∇-modules on Spec $k((t))$. (∇ means a connection) It is conjectured that the category of overconvergent etale φ-∇-modules corresponds to that of p-adic representations of $\mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$ with finite monodromy. We will show that the category of overconvergent etale φ-modules, considering only Frobenius structures and forgetting connections, is a full subcategory of etale φ-modules. As a corollary of this result we obtain the full faithfulness for the local version.

It is quite natural to expect that the natural functor, in general, from the category of overconvergent F-isocrystals, admitting arbitrary slopes, to the category of F-isocrystals or its local version is fully faithful. In this case the connection, however, plays an essential role and our method is useless.

2. Preliminaries

(2.1) Let p be a prime number and let Λ be a discrete valuation ring which is finite over the ring $\mathbb{Z}_p$ of p-adic integers. We put $q = p^f$ to be the number of elements of the residue field of Λ. We denote by Λ_0 the maximum unramified subring of Λ.

Let k be a perfect field of characteristic p. We assume that k contains the field $\mathbb{F}_q$ of q elements. Let $W = W(k)$, R and K be the ring of Witt vectors with k-coefficients, $\Lambda \otimes_{\Lambda_0} W$ and $R[p^{-1}]$ respectively. We define an endomorphism σ on $K = \Lambda[p^{-1}] \otimes_{\Lambda_0} W$, which we call frobenius, by $1 \otimes \mathrm{frob}^f$. Here frob is the usual frobenius on W which is induced by the map $x \mapsto x^p$ on k. We denote by $||$ an absolute value of K.

We consider the ring

$$O_{\mathcal{E}} = \varprojlim((R/p^n R)[[T]][T^{-1}]).$$

Then $O_{\mathcal{E}}$ is a complete discrete valuation ring whose residue field is identified with $k((t))$ by the map $T \mapsto t$. We denote by $\mathcal{E} = O_{\mathcal{E}}[p^{-1}]$ the fraction field of $O_{\mathcal{E}}$. An element of $\mathcal{E}$ is uniquely expressed by formal Laurent series

$$\sum_{n=-\infty}^{n=\infty} a_n T^n, \quad a_n \in K, \quad |a_n| \text{ is bounded and } |a_n| \to 0 \ (n \to -\infty).$$

The absolute value $||_g$ of $\mathcal{E}$, which is called the Gauss norm, is given by

$$\left|\sum a_n T^n\right|_g = \sup\{|a_n| \mid n \in \mathbb{Z}\}.$$

A series $\sum a_n T^n \in \mathcal{E}$ is called overcovergent if and only if there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $|a_n| \leqslant C\eta^{-n}$ for all negative integer n. We denote by $\mathcal{E}^\dagger$ (resp. $O_{\mathcal{E}}^\dagger$) the subfield of $\mathcal{E}$ which consists of overconvergent series (resp. the subring $\mathcal{E}^\dagger \cap O_{\mathcal{E}}$ of $O_{\mathcal{E}}$). $\mathcal{E}^\dagger$ is also a discrete valuation field with the absolute value $|\ |_g$ whose integer ring is $O_{\mathcal{E}}^\dagger$ and whose residue field is also $k((t))$.

We fix a frobenius endomorphism σ on $\mathcal{E}$, which extends the frobenius on K and is also denoted by σ. We assume that the endomorphism σ is a lifting of qth power map on $k((t))$ such that $\mathcal{E}^\dagger$ is stable under σ. It is easy to see that σ is a continuous map on $\mathcal{E}$ and that the condition $\sigma(\mathcal{E}^\dagger) \subset \mathcal{E}^\dagger$ is equivalent to the condition

$$T^\sigma \in \mathcal{E}^\dagger.$$

One can easily see that the fixed part of the frobenius σ in $\mathcal{E}$ is $\Lambda[p^{-1}]$.

We define a derivation D on $\mathcal{E}$ by

$$D\left(\sum a_n T^n\right) = \sum n a_n T^n.$$

The derivation D is a continuous one on $\mathcal{E}$ and one sees easily that $D(x) \in \mathcal{E}^\dagger$ if and only if $x \in \mathcal{E}^\dagger$.

We have the relation $D \circ \sigma = \frac{D(T^\sigma)}{T^\sigma} \sigma \circ D$. If, for example, we choose the frobenius on $\mathcal{E}$ by $T^\sigma = T^q$, then we have $D \circ \sigma = q\sigma \circ D$.

(2.2) We define a system of invariants $\alpha_N : \mathcal{E} \to \mathbb{R}_{\geqslant 0}$ for integers N by

$$\alpha_N \left(\sum a_n T^n\right) = \sup\{|a_n| \; ; \; n \leqslant N\}.$$

It is well-defined since $|a_n| \to 0$ $(n \to -\infty)$. For any element x in $\mathcal{E}$, one has $\alpha_N(x) \leqslant |x|_g$ for all N and equality holds for sufficient large N. By definition, $N \leqslant M$ implies $\alpha_N(x) \leqslant \alpha_M(x)$. From the properties of non-archimedean absolute value we have the following inequalities:

$$\begin{aligned} \alpha_N(x+y) &\leqslant \max\{\alpha_N(x), \alpha_N(y)\} \\ \alpha_N(xy) &\leqslant \sup\{\alpha_L(x)\alpha_M(y) \mid L + M = N\} \end{aligned} \tag{2.2.1}$$

for all $x, y \in \mathcal{E}$. By definition, x is contained in $\mathcal{E}^\dagger$ if and only if there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(x) \leqslant C\eta^{-N}$ for all negative integers N.

Now we show a stability theorem for the frobenius σ which is the key proposition in our arguments.

(2.2.2) PROPOSITION. *Let r be a positive integer and let $a_1, a_2, \ldots, a_r$ be elements in $\mathcal{E}^\dagger$ such that $|a_i|_g \leqslant 1$ for all i. If an element x in $\mathcal{E}$ satisfies the equality:*

$$x^{\sigma^r} + a_1 x^{\sigma^{r-1}} + \cdots + a_{r-1}x^\sigma + a_r x = 0, \tag{2.2.3}$$

then x is contained in $\mathcal{E}^\dagger$.

First we prove (2.2.2) in the case where $T^\sigma = T^q$. This case avoids the confusion arising from σ and helps to understand the case of general σ.

As all a_i's are contained in $\mathcal{E}^\dagger$, there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(a_i) \leqslant C\eta^{-N}$ for negative integer N and for all i. Assume that x is not contained in $\mathcal{E}^\dagger$. We may assume that $|x|_g \leqslant 1$. Then $\alpha_N(x) \leqslant 1$ for all N and there exists a negative integer L such that $\alpha_L(x) > C\eta^{-L}$ and $\alpha_{L-1}(x) < \alpha_L(x)$. Now we calculate $\alpha_{q^r L}$ of each term in the left-hand side of (2.2.3) using (2.2.1). Since $T^\sigma = T^q$, we have

$$\alpha_{q^r L}(x^{\sigma^r}) = \alpha_L(x)$$

and

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) \leqslant \sup\{\alpha_M(a_i)\alpha_N(x);\ M + q^{r-i}N = q^r L\} \quad (1 \leqslant i \leqslant r).$$

If $M \leqslant (q^r - q^{r-i})L$ in the second inequality, then $\alpha_M(a_i)\alpha_N(x) \leqslant C\eta^{-(q^r-q^{r-i})L} \leqslant C\eta^{-L} < \alpha_L(x)$. If $M > (q^r - q^{r-i})L$ in the second inequality, then $N < L$ and $\alpha_M(a_i)\alpha_N(x) < \alpha_L(x)$ since $|a_i|_g \leqslant 1$. So we have

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) < \alpha_L(x)$$

for all $1 \leqslant i \leqslant r$. This contradicts (2.2.3). □

To prove (2.2.2) for general σ, we prepare the following lemma.

(2.2.4) LEMMA. *If x is not contained in $\mathcal{E}^\dagger$, then, for any real number* $0 < \eta < 1$, *there exist infinitely many negative integers L such that*

$$\alpha_L(x)\eta^{L-N} > \alpha_N(x)$$

for all integer $L < N < 0$.

Proof. If there exists no integer $L \leqslant -2$ which satisfies this property, then, for any negative integer N, there exists an integer $N < M < 0$ such that $\alpha_N(x)\eta^{N-M} \leqslant \alpha_M(x)$. Applying this finitely many time, we have $\alpha_N(x) \leqslant \alpha_{-1}(x)\eta^{-N-1}$ for all negative integers N. We have a contradiction. If there exists only finitely many such negative integers L, then we choose η close enough to 1 and we get a contradiction similar to the first part. □

Proof of (2.2.2). We define $u \in \mathcal{E}^\dagger$ by $T^\sigma = T^q u$. Then $|u-1|_g < 1$. As all a_i's are contained in $\mathcal{E}^\dagger$, there exist real numbers $C > 0$ and $0 < \eta < 1$ such that $\alpha_N(u) \leqslant \eta^{-N}$ and $\alpha_N(a_i) \leqslant C\eta^{-N}$ for all negative integers N for all i. We may assume that $|x|_g \leqslant 1$, equivalently that $\alpha_N(x) \leqslant 1$ for all integers N. Assume that x is not contained in $\mathcal{E}^\dagger$. Then there exists a negative integer L such that $\alpha_L(x) > \max\{C, 1\}\eta^{-L}$, $\alpha_{L-1}(x) < \alpha_L(x)$ and $\alpha_L(x)\eta^{L-N} > \alpha_N(x)$ for all $L < N < 0$ by (2.2.4). First we will show

$$\begin{cases} \alpha_{q^j L}(x^{\sigma^j}) = \alpha_L(x) > \max\{C,1\}\eta^{-L}; & \\ \alpha_N(x^{\sigma^j}) < \alpha_{q^j L}(x^{\sigma^j}) & \text{if } N < q^j L; \\ \alpha_{q^j L}(x^{\sigma^j})\eta^{q^j L - N} > \alpha_N(x^{\sigma^j}) & \text{if } q^j L < N < 0 \end{cases} \tag{2.2.5}$$

for all $0 \leqslant j \leqslant r$ inductively. If $j = 0$, we have nothing to prove by the assumption on L. We put $y = \sum y_n T^n = x^{\sigma^{j-1}}$. We observe that $|y|_g = |x|_g \leqslant 1$. For any integer N we have

$$\alpha_N(y_n^\sigma T^{qn} u^n) = \alpha_{N-qn}(y_n u^n) \begin{cases} < \alpha_n(y) & \text{if } N > qn; \\ \leqslant \alpha_n(y) & \text{if } N = qn; \\ \leqslant \alpha_n(y)\eta^{-N+qn} & \text{if } N < qn \end{cases} \tag{2.2.6}$$

by the assumption on u and (2.2.1) for each n. In the case $N = q^j L$ we get

$$\alpha_{q^j L}(y^{\sigma}_{q^{j-1}L} T^{q^j L} u^{q^{j-1}L}) = \alpha_0(y_{q^{j-1}L} u^{q^{j-1}L}) = \alpha_{q^{j-1}L}(y)$$

and

$$\alpha_{q^j L}(y^{\sigma}_n T^{qn} u^n) < \begin{cases} \alpha_{q^{j-1}L}(y) & \text{if } n < q^{j-1}L; \\ \alpha_{q^{j-1}L}(y)\eta^{(q^{j-1}L-n)-(q^j L - qn)} & \text{if } q^{j-1}L < n < 0; \\ \eta^{-q^j L} & \text{if } n > 0 \end{cases}$$

by (2.2.6) and the assumption on the induction. Thus we obtain

$$\alpha_{q^j L}(y^{\sigma}) = \alpha_{q^{j-1}L}(y) = \alpha_L(x).$$

from (2.2.1). In the case $N < q^j L$ we get

$$\alpha_N(y^{\sigma}_n T^{qn} u^n) < \begin{cases} \alpha_{q^{j-1}L}(y) & \text{if } qn \leqslant N; \\ \alpha_{q^{j-1}L}(y) & \text{if } N < qn \leqslant q^j L; \\ \alpha_{q^{j-1}L}(y)\eta^{(q^{j-1}L-n)-(N-qn)} & \text{if } q^j L < qn < 0; \\ \eta^{-N} < \eta^{-q^j L} & \text{if } qn \geqslant 0 \end{cases}$$

by (2.2.6) and the assumption on the induction. Since

$$(q^{j-1}L - n) - (N - qn) > (q-1)(n - q^{j-1}L) > 0$$

for $q^j L < qn < 0$, we have

$$\alpha_N(y^{\sigma}) < \alpha_{q^j L}(y^{\sigma})$$

from (2.2.1). In the case $q^j L < N < 0$ we get

$$\alpha_N(y^{\sigma}_n T^{qn} u^n) < \begin{cases} \alpha_{q^{j-1}L}(y) & \text{if } qn \leqslant q^j L; \\ \alpha_{q^{j-1}L}(y)\eta^{q^{j-1}L-n} & \text{if } q^j L < qn \leqslant N; \\ \alpha_{q^{j-1}L}(y)\eta^{q^{j-1}L-n-(N-qn)} & \text{if } N < qn < 0; \\ \eta^{(q^j-1)L-N} & \text{if } qn \geqslant 0. \end{cases}$$

by (2.2.6) and the assumption on the induction. Since $q^j L - N \leqslant q^j L - qn < q^{j-1}L - n < 0$ for $q^j L < qn \leqslant N$ and $q^j - N < (q^{j-1}L - n) - (N - qn)$ for $N < qn < 0$, we have

$$\alpha_{q^j L}(y^{\sigma})\eta^{q^j L - N} > \alpha_N(y^{\sigma})$$

by (2.2.1).

Now we calculate $\alpha_{q^r L}$ of each term in the left-hand side of (2.2.3) using (2.2.5). We have

$$\alpha_{q^r L}(x^{\sigma^r}) = \alpha_L(x)$$

and

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) \leqslant \sup\{\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) \mid M + N = q^r L\} \quad \text{for } 1 \leqslant i \leqslant r.$$

If $M \leqslant (q^r - q^{r-i})L$ in the last inequality, then $\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) \leqslant C\eta^{-(q^r - q^{r-i})L} \leqslant C\eta^{-L} < \alpha_L(x)$ for $|x^{\sigma^{r-i}}|_g = |x|_g \leqslant 1$. If $M > (q^r - q^{r-i})L$ in the last inequality, then $N < q^{r-i}L$ and $\alpha_M(a_i)\alpha_N(x^{\sigma^{r-i}}) < \alpha_L(x)$ for $|a_i|_g \leqslant 1$. So we obtain

$$\alpha_{q^r L}(a_i x^{\sigma^{r-i}}) < \alpha_L(x)$$

for all $1 \leqslant i \leqslant r$. This contradicts (2.2.3). □

(2.2.7) REMARK. The assertion of (2.2.2) does not always hold if $|a_i|_g > 1$ for some i. For example, we consider the case that $T^\sigma = T^q$. Consider the element

$$x = \sum_{i=0}^{\infty} \pi^i T^{-q^i} \notin \mathcal{E}^\dagger.$$

Then x satisfies the relation

$$\pi x^{\sigma^2} - (1 + \pi T^{1-q})x^\sigma + T^{1-q}x = 0.$$

3. Overconvergent etale φ-∇-spaces on Spec $k((t))$

We keep the notation as in Section 2. In this section we define overconvergent etale Λ-φ-∇-spaces on Spec $k((t))$ and show some properties of them which we will use.

(3.1) We fix a lifting $O_\mathcal{E}$ of $k((t))$ in characteristic 0 and a lifting σ on $\mathcal{E}$ of qth power map on $k((t))$ as in the previous section. Now we define some notions.

DEFINITION (1) A Λ-φ-module (resp. an overconvergent Λ-φ-module) on Spec $k((t))$ is a free $O_\mathcal{E}$ (resp. $O_\mathcal{E}^\dagger$)-module M of finite rank with a σ-linear endomorphism $\varphi : M \to M$, which we call Frobenius, satisfying the following condition;

$$\begin{aligned}&\varphi(M) \text{ spans } \mathcal{E} \otimes_{O_\mathcal{E}} M \text{ (resp. } \mathcal{E}^\dagger \otimes_{O_\mathcal{E}^\dagger} M) \text{ as an}\\ &\mathcal{E} \text{ (resp. an } \mathcal{E}^\dagger)\text{-vector space.}\end{aligned} \qquad (3.1.1)$$

(2) A Λ-φ-∇-module (resp. an overconvergent Λ-φ-∇-module) on Spec $k((t))$ is a Λ-φ-module (M, φ) (resp. an overconvergent Λ-φ-module (M, φ)) with an additive endomorphism $\nabla : M \to M$, which we call a connection, satisfying the following conditions;

$$\nabla(ax) = D(a)x + a\nabla(x) \text{for all } a \in O_{\mathcal{E}} (\text{resp. } a \in O_{\mathcal{E}}^{\dagger}) \text{and } x \in M; \quad (3.1.2)$$

$$\nabla \circ \varphi = \frac{D(T^{\sigma})}{T^{\sigma}} \varphi \circ \nabla. \quad (3.1.3)$$

(3) A Λ-φ-space (resp. an overconvergent Λ-φ-space, a Λ-φ-∇-space, an overconvergent Λ-φ-∇-space) on Spec $k((t))$ is an $\mathcal{E}$ (resp. $\mathcal{E}^{\dagger}$, $\mathcal{E}$, $\mathcal{E}^{\dagger}$)-vector space E of finite dimension with φ (resp. φ, φ and ∇, φ and ∇) which satisfy the above conditions (3.1.1)–(3.1.3) when we replace M and $O_{\mathcal{E}}$ with E and $\mathcal{E}$ (resp. $\mathcal{E}^{\dagger}$, $\mathcal{E}$, $\mathcal{E}^{\dagger}$), respectively.

(4) A morphism of Λ-φ-modules (resp. $\cdots$) is an $O_{\mathcal{E}}$ (resp. $\cdots$)-linear homomorphism which commutes with all additional structures.

In the rest of this paper we use the terminology φ-module (resp. $\cdots$) instead of Λ-φ-module (resp. $\cdots$) for simplicity.

REMARK. (1) In [3] Fontaine defined φ-modules and our φ-modules (resp. overconvergent φ-modules) are $O_{\mathcal{E}}$-φ-modules (resp. $O_{\mathcal{E}}^{\dagger}$-$\varphi$-modules) of his definition. In the etale case Fontaine's φ-module has a natural connection and the category of etale φ-modules on Spec $k((t))$ which is defined by Fontaine coincides with our category of etale φ-∇-modules. (See (3.2) and (3.3))

(2) We omit the conditions of convergence for a connection in the definition of φ-modules (resp. $\cdots$). In the etale case the connection satisfies the condition of topological quasi-nilpotence from (3.1.3), so the category of unit-root F-isocrystals on Spec $k((t))$ (see [5] and [1]) is equivalent to the category of etale φ-∇-spaces.

We define tensor products and duals of φ-modules (resp. $\cdots$) as follows.

Let $(E_1, \varphi_1, \nabla_1)$ and $(E_2, \varphi_2, \nabla_2)$ be φ-∇-spaces. We define their tensor product (E, φ, ∇) by $E = E_1 \otimes_{\mathcal{E}} E_2, \varphi = \varphi_1 \otimes \varphi_2$ and $\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2$.

For a φ-∇-space (E, φ, ∇), we define the dual $E^{\vee}$ of E by $E^{\vee} = \mathrm{Hom}_{\mathcal{E}}(E, \mathcal{E})$, $\varphi^{\vee}(f) = (1 \otimes \sigma) \circ \sigma^* f \circ (1 \otimes \varphi)^{-1}$ and $\nabla^{\vee}(f)(x) = D(f(x)) - f(\nabla(x))$ for $f \in E^{\vee}$ and $x \in E$, where $1 \otimes \varphi \colon \sigma^* E \to E$ is the isomorphism which is induced by φ. Here $\sigma^* E$ is the scalar extension of E by $\sigma \colon \mathcal{E} \to \mathcal{E}$.

The other cases are same as in the above definition.

There exists a natural functor ν^* from the category of overconvergent φ-modules (resp. $\cdots$) to that of φ-modules (resp. $\cdots$) which is defined by the scalar extension $O_{\mathcal{E}}^{\dagger} \to O_{\mathcal{E}}$ (resp. $\cdots$). We can easily see that the functor ν^* commutes with taking a tensor product and dual.

We show that there exists a cyclic vector in a φ-space (resp. in an overconvergent φ-space). (c.f. [7](3.3))

(3.1.4) LEMMA. *Let E be an $\mathcal{E}$ (*resp. *an $\mathcal{E}^\dagger$)-vector space of finite dimension with a σ-linear endomorphism $\varphi : E \to E$ such that $\varphi(E)$ spans E as an $\mathcal{E}$ (*resp. *an $\mathcal{E}^\dagger$)-vector space. Then there exists a cyclic vector $e \in E$, that is, there exists an element $e \in E$ such that $e, \varphi(e), \dots, \varphi^{r-1}(e)$ is a basis of E where $r = \dim E$.*

Proof. We will show the case of $\mathcal{E}$-vector spaces. Set

$$s = \max\{s(x) \mid x, \varphi(x), \dots, \varphi^{s(x)-1}(x) \text{ are linearly independent over } \mathcal{E} \text{ in E}\}.$$

If $s = r$, then there is nothing to prove. Assume $s < r$ and $x, \varphi(x), \dots, \varphi^{s-1}(x)$ are linearly independent over $\mathcal{E}$ in E. As $\varphi(E)$ spans E over $\mathcal{E}$, there exists an element $y \in E$ such that $x, \varphi(x), \dots, \varphi^{s-1}(x)$ and $\varphi^s(y)$ are linearly independent over $\mathcal{E}$. By the assumption on s, we have

$$(x + ay) \wedge \varphi(x + ay) \wedge \cdots \wedge \varphi^s(x + ay) = 0 \text{ in } \bigwedge^{s+1} E$$

for all $a \in \mathcal{E}$. Since there exist sufficiently many elements in $\mathcal{E}$ (for example we may choose $1, T, \dots,$ for a), we have

$$x \wedge \varphi(x) \wedge \cdots \wedge \varphi^{s-1}(x) \wedge \varphi^s(y) = 0.$$

This contradicts the choice of y. The case of $\mathcal{E}^\dagger$-vector spaces is similar. □

(3.2) Let $k((t))^{\mathrm{alg}}$ be an algebraic closure of $k((t))$ and let $W(k((t))^{\mathrm{alg}})$ be the ring of Witt vectors with $k((t))^{\mathrm{alg}}$-coefficients. We can embed $O_{\mathcal{E}}$ into $\Lambda \otimes_{\Lambda_0} W(k((t))^{\mathrm{alg}})$ such that the frobenius σ on $O_{\mathcal{E}}$ commutes with the endomorphism $1 \otimes \mathrm{frob}^f$ on $\Lambda \otimes_{\Lambda_0} W(k((t))^{\mathrm{alg}})$, where frob is the usual frobenius on $W(k((t))^{\mathrm{alg}})$. We denote by $\widetilde{\mathcal{E}}$ the fraction field of $\Lambda \otimes_{\Lambda_0} W(k((t))^{\mathrm{alg}})$ and we regard $\mathcal{E}$ as a subfield of $\widetilde{\mathcal{E}}$.

Let E be a φ-space (resp. an overconvergent φ-space) on Spec $k((t))$. Then $\widetilde{\mathcal{E}} \otimes E$ is naturally an F-space on Spec $k((t))^{\mathrm{alg}}$. By the classification theorem of F-spaces on algebraically closed field, the F-space $\widetilde{\mathcal{E}} \otimes_{\mathcal{E}} E$ is determined by its slopes. (See [2] Chapter IV, for example, and we can generalize the classification theorem for our Λ-F-spaces on algebraically closed field) We define slopes of a φ-space E (resp. an overconvergent φ-space) on Spec $k((t))$ by the slopes of $\widetilde{\mathcal{E}} \otimes E$ as an F-space on Spec $k((t))^{\mathrm{alg}}$.

Now we define an etale φ-space (resp. an overconvergent φ-space). A φ-space (resp. an overconvergent φ-space) on Spec $k((t))$ is etale if and only if all its slopes are 0. A φ-module (resp. an overconvergent φ-module) is etale if and only if all its slopes are 0 after inverting p. Equivalently, a φ-module M (resp. an overconvergent

φ-module M) is etale if and only if the morphism $1 \otimes \varphi \colon \sigma^* M \to M$, which is induced by φ, is an isomorphism. By the theory of slopes, tensor products and dual of etale objects are also etale.

According to the classification theorem of F-spaces on algebraically closed field, we characterize etale φ-spaces (resp. overconvergent etale φ-spaces) using a cyclic vector;

(3.2.1) PROPOSITION. *Let E be a φ-space* (resp. *an overconvergent φ-space) on* Spec $k((t))$ *and let e be a cyclic vector of E which satisfies the relation*

$$\varphi^r(e) + a_1\varphi^{r-1}(e) + \cdots + a_r e = 0 \quad a_i \in \mathcal{E} \text{ (resp. } \mathcal{E}^\dagger). \tag{3.2.2}$$

Then E is etale if and only if $|a_i|_g \leqslant 1$ for all $1 \leqslant i \leqslant r-1$ and $|a_r|_g = 1$.

(3.2.3) REMARK. The Newton polygone of (3.2.2) gives slopes of φ-spaces E. Therefore all slopes of E are greater than or equal to 0 if and only if $|a_i|_g \leqslant 1$ for all i.

(3.3) Now we discuss the relation between the Frobenius and the connection.

In the case of an etale φ-∇-space E (resp. an overconvergent etale φ-∇-space E) over Spec $k((t))$ the Frobenius φ determines the connection ∇ and there exists a φ-∇-lattice of E, that is, an $O_{\mathcal{E}}$ (resp. $O_{\mathcal{E}}^\dagger$)-submodule which spans E over $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) and which is stable under the Frobenius φ and the connection ∇. By (3.1.4) and (3.2.1) there exists a basis $e_1, e_2, \ldots, e_r$ ($r = \dim E$) and a matrix $A \in \mathrm{GL}_r(O_{\mathcal{E}})$ (resp. a matrix $A \in \mathrm{GL}_r(O_{\mathcal{E}}^\dagger)$) such that

$$\varphi(e_1, \ldots, e_r) = (e_1, \ldots, e_r)A.$$

We set a matrix C of degree r with $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$)-coefficients by

$$\nabla(e_1, \ldots, e_r) = (e_1, \ldots, e_r)C.$$

From the condition (3.1.3) we have the following relation;

$$A^D + CA = \frac{D(T^\sigma)}{T^\sigma} AC^\sigma,$$

where $A^D = (D(a_{ij}))$ and $C^\sigma = (c_{ij}^\sigma)$ for $A = (a_{ij})$ and $C = (c_{ij})$. For a matrix $X \in \mathrm{M}_r(\mathcal{E})$ we define $\psi(X) = \frac{D(T^\sigma)}{T^\sigma} AX^\sigma A^{-1}$. As $|\frac{D(T^\sigma)}{T^\sigma}|_g < 1$, ψ is a contraction operator on $\mathrm{M}_r(\mathcal{E})$ for p-adic topology and we have

$$C = -(1-\psi)^{-1}(A^D A^{-1}). \tag{3.3.1}$$

So C is uniquely determined by A and all the coefficients of C are contained in $O_{\mathcal{E}}$ (resp. $O_{\mathcal{E}}^\dagger$).

(3.3.2) THEOREM. *The category of etale φ-∇-modules (*resp. *overconvergent etale φ-∇-modules, etale φ-∇-spaces, overconvergent etale φ-∇-spaces) is naturally a full subcategory of that of etale φ-modules (*resp. *overconvergent etale φ-modules, etale φ-spaces, overconvergent etale φ-spaces) by the forgetful functor. Moreover the category of etale φ-∇-modules (*resp. *etale φ-∇-spaces) on* Spec $k((t))$ *is equivalent to that of etale φ-modules (*resp. *etale φ-spaces).*

In the overconvergent case, on the contrary, the category of overconvergent φ-spaces is not equivalent to that of overconvergent φ-∇-spaces. Because all coefficients of C are not always contained in $\mathcal{E}^\dagger$ when we determine the connection by the relation (3.3.1).

EXAMPLE. Fix a frobenius σ by $T^\sigma = T^p$. Let $a = 1 - \frac{\pi}{T} \in O_{\mathcal{E}}^\dagger$. Then we have $c = -(1-p\sigma)^{-1}(a^{-1}D(a)) \notin O_{\mathcal{E}}^\dagger$. Define a φ-∇-space E on Spec $k((t))$ of rank one by $\varphi(e) = ae$ and $\nabla(e) = ce$, where e is a basis of E. So the coefficients of the frobenius structure of E is an overconvergent, but the coefficients of the connection are not contained in $\mathcal{E}^\dagger$. Moreover the p-adic representation of Gal$(k((t))^{\mathrm{sep}}/k((t)))$ which corresponds to E in the sense of Fontaine [3] is not of finite monodromy. [1]

REMARK. In general, the connection of φ-∇-spaces is not determined by its Frobenius structures. Because the operator ψ is not a contraction in this case.

4. Full faithfulness

We keep the same notation as in the previous section. In this section we will show the local version that the natural functors

$$\begin{pmatrix}\text{overconvergent etale} \\ \varphi\text{-spaces on Spec } k((t))\end{pmatrix} \xrightarrow{\nu^*} \begin{pmatrix}\text{etale} \\ \varphi\text{-spaces on Spec } k((t))\end{pmatrix}$$

$$\begin{pmatrix}\text{overconvergent etale} \\ \varphi\text{-}\nabla\text{-spaces on Spec } k((t))\end{pmatrix} \xrightarrow{\nu^*} \begin{pmatrix}\text{etale} \\ \varphi\text{-}\nabla\text{-spaces on Spec } k((t))\end{pmatrix}$$

defined in (3.1) are fully faithful.

(4.1) First we consider only frobenius structures.

Let r be a positive integer and let A be an $r \times r$ invertible matrix with $\widetilde{\mathcal{E}}$-coefficients. We define a Λ-φ-space E_A on Spec $k((t))^{\mathrm{alg}}$ associated to A by its frobenius linear morphism

$$\varphi((e_1, e_2, \ldots, e_r)) = (e_1, e_2, \ldots, e_r)A,$$

where $e_1, e_2, \ldots, e_r$ is a basis of E_A on $\widetilde{\mathcal{E}}$. We say that A is etale if and only if all slopes of E_A are 0.

(4.1.1) PROPOSITION. *Let A be an invertible matrix of degree r with $\mathcal{E}^\dagger$-coefficients such that A is etale. Assume that $\boldsymbol{x} = (x_1, x_2, \ldots, x_r) \in \mathcal{E}^r$ satisfies the relation*

$$A^t\boldsymbol{x}^\sigma = {}^t\boldsymbol{x}.$$

Then $\boldsymbol{x}$ is contained in $(\mathcal{E}^\dagger)^r$.

Proof. Let L be an $\mathcal{E}^\dagger$-subvector space of $\mathcal{E}$ which is generated by $x_1, x_2, \ldots, x_r$. Then L is stable under the frobenius σ on $\mathcal{E}$. So we can regard L as an overconvergent φ-space. By the construction, there is a natural surjection from $E_A^\vee$ to L as overconvergent φ-spaces. So all slopes of L are 0 by the slope theory of φ-spaces. From (3.1.4) and (3.2.1), there exists a cyclic vector $y \in L$ which satisfies the reration

$$y^{\sigma^s} + a_1 y^{\sigma^{s-1}} + \cdots + a_{s-1}y^\sigma + a_s y = 0$$

such that $|a_i|_g \leqslant 1$ for all $1 \leqslant i \leqslant s-1$ and $|a_s|_g = 1$. Here $s = \dim_{\mathcal{E}^\dagger} L$. So y contains in $\mathcal{E}^\dagger$ and $s = 1$ from (2.2.2). Therefore $x_i's$ are included in $\mathcal{E}^\dagger$. □

(4.1.2) REMARK. The assertion of (4.1.1) is also true if all slopes of E_A are less than or equal to 0 by the remark (3.2.3).

(4.1.3) THEOREM. *The natural functor ν^* from the category of overconvergent etale φ-spaces on* Spec $k((t))$ *to that of etale φ-spaces on* Spec $k((t))$ *is fully faithful.*

Proof. Let $\mathcal{C}$ be either the category of overconvergent etale φ-spaces or the category of etale φ-spaces. Define

$$\mathrm{H}^0_{\mathcal{C}}(E) = \{y \in E \mid \varphi(y) = y\}$$

for any object E of $\mathcal{C}$. Then one can easily see that

$$\mathrm{Hom}_{\mathcal{C}}(E_1, E_2) = \mathrm{H}^0_{\mathcal{C}}(E_1^\vee \bigotimes E_2).$$

So we have to show the natural map

$$\mathrm{H}^0_{(\mathrm{over}.\varphi-sps)}(E) \to \mathrm{H}^0_{(\varphi-sps)}(\nu^* E)$$

is an isomorphism for any overconvergent etale φ-space E. The injectivity is trivial. We show the surjectivity. Let $e_1, e_2, \ldots, e_r$ be a basis of E on $\mathcal{E}^\dagger$ and define $A \in \mathrm{GL}_r(\mathcal{E}^\dagger)$ by $\varphi(\boldsymbol{e}) = \boldsymbol{e}A$ where $\boldsymbol{e} = (e_1, e_2, \ldots, e_r)$. If $y = \boldsymbol{e}^t\boldsymbol{x} = \boldsymbol{e}^t(x_1, x_2, \ldots, x_r) \in \mathrm{H}^0_{(\varphi-sps)}(\nu^* E)$, then $\boldsymbol{x}$ satisfies the relation

$$A^t\boldsymbol{x}^\sigma = {}^t\boldsymbol{x}.$$

From (4.1.1) $\boldsymbol{x}$ is contained in $(\mathcal{E}^\dagger)^r$. Hence $y \in \mathrm{H}^0_{(\mathrm{over}.\varphi-sps)}(E)$. □

REMARK. In general case, not restricting to etale case, the full faithfulness of the natural functor from the category of overconvergent φ-spaces to that of φ-spaces does not hold by (2.2.7).

(4.2) Since the category of etale φ-∇-spaces (resp. overconvergent etale φ-∇-spaces) on Spec $k((t))$ is a full subcategory of etale φ-spaces (resp. overconvergent etale φ-spaces) on Spec $k((t))$ (3.3.2), the theorem below follows from (4.1.3).

(4.2.1) THEOREM. *The natural functor* ν^* *from the category of overconvergent etale* φ-∇*-spaces on* Spec $k((t))$ *to that of etale* φ-∇*-spaces on* Spec $k((t))$ *is fully faithful.*

(4.3) Now we consider lattices. As $M = (\mathcal{E}^\dagger \otimes_{O_\mathcal{E}^\dagger} M) \bigcap (O_\mathcal{E} \otimes_{O_\mathcal{E}^\dagger} M)$ for a free $O_\mathcal{E}^\dagger$-module M, we have

(4.3.1) THEOREM. *The natural functors*

$$\begin{pmatrix} \textit{overconvergent etale} \\ \varphi\textit{-modules on } \operatorname{Spec} k((t)) \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{etale} \\ \varphi\textit{-modules on } \operatorname{Spec} k((t)) \end{pmatrix}$$

$$\begin{pmatrix} \textit{overconvergent etale} \\ \varphi\text{-}\nabla\textit{-modules on } \operatorname{Spec} k((t)) \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{etale} \\ \varphi\text{-}\nabla\textit{-modules on } \operatorname{Spec} k((t)) \end{pmatrix}$$

are fully faithful.

5. Application to the case of curves

(5.1) Let X be a smooth curve over k, where k is a perfect field of positive characteristic. Let $\overline{X}$ and D be the smooth compactification of X and $\overline{X} - X$, respectively. We apply (4.2.1) and (4.3.1) to the unit-root F-isocrystals on X. (See [1])

(5.1.1) THEOREM. *The natural functors*

$$\begin{pmatrix} \textit{overconvergent unit-root} \\ \Lambda\text{-}F\textit{-isocrystals on } X \textit{ around } D \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{unit-root} \\ \Lambda\text{-}F\textit{-isocrystals on } X \end{pmatrix}$$

$$\begin{pmatrix} \textit{overconvergent unit-root} \\ \Lambda\text{-}F\textit{-crystals on } X \textit{ around } D \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{unit-root} \\ \Lambda\text{-}F\textit{-crystals on } X \end{pmatrix}$$

are fully faithful.

Proof. By the argument of Crew in [1](4.6)–(4.10) we can reduce (5.1.1) to the local case. Therefore (5.1.1) follows from (4.2.1) and (4.3.1). □