COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 104, nº 1 (1996), p. 27-39
<http://www.numdam.org/item?id=CM_1996__104_1_27_0>

Compositio Mathematica **104:** 27–39, 1996.

Principal parts of line bundles on toric varieties

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Received 6 June 1994; accepted in final form 14 August 1995

Abstract. The main tool for studying the inflections (or Weierstrass points) of a mapping of a smooth projective variety into projective space are the principal parts of line bundles. In recent work by D. Cox, [2], homogeneous coordinates on a toric variety have been introduced, and in subsequent work with V. Batyrev, [1], an Euler sequence is defined. The homogeneous coordinates and the Euler sequence are direct generalizations of the usual notions in the case of projective space. The purpose of this note is to use the Euler sequence to describe the principal parts of line bundles on a toric variety (Theorem 1.2). The essential idea is to compare derivatives with respect to local and global coordinates. Even for the case of projective space, the complete description is apparently not to be found in the literature.

Key words: principal parts, toric varieties, homogeneous coordinates, Euler sequence, Weierstrass points, inflections.

0. Notation

We use standard notation from analysis. If $\alpha = (a_1, \ldots, a_n)$ is a tuple of non-negative integers, let $\alpha! := \Pi_{i=1}^n a_i!$ and $|\alpha| := \Sigma_{i=1}^n a_i$. A monomial of the form $\Pi_{i=1}^n e_i^{a_i}$ will be denoted by e^α. We will often write ∂_{x_i} in place of $\partial/\partial x_i$.

TORIC VARIETIES

As general references for toric varieties, we use [3] and [8]. Let X be an n-dimensional toric variety associated with a fan Δ in an n-dimensional lattice $N \cong \mathbb{Z}^n$. Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice and $\Delta(1)$ be the set of one-dimensional cones of Δ. For each $\rho \in \Delta(1)$, let n_ρ be the generator of $\rho \cap N$ and D_ρ be the associated T-invariant Weil divisor; the set of such D_ρ is a basis for the free abelian group of T-Weil divisors, $\mathbb{Z}^{\Delta(1)}$. To describe the homogeneous coordinate ring of X introduced in [2], recall the exact sequence

$$\begin{aligned} &0 \to M \to \mathbb{Z}^{\Delta(1)} \to A_{n-1}(X) \to 0, \\ &m \mapsto D_m = \sum_\rho \langle m, n_\rho \rangle D_\rho, \end{aligned} \tag{1}$$

where $A_{n-1}(X)$ is the group of Weil divisors modulo rational equivalence and the map $\mathbb{Z}^{\Delta(1)} \to A_{n-1}(X)$ sends a divisor to its class. For each $\rho \in \Delta(1)$, let x_ρ be a variable. There is a 1-1 correspondence between T-Weil divisors and monomials in the x_ρ, namely, $D = \Sigma_\rho a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$ corresponds with $x^D = \Pi_\rho x_\rho^{a_\rho}$. The homogeneous coordinate ring of X is $S = \mathbb{C}[x_\rho \,|\, \rho \in \Delta(1)]$ with grading given by the class group, $A_{n-1}(X)$. This means that two monomials x^D and x^E have the same degree if $[D] = [E]$ in $A_{n-1}(X)$. For each T-Weil divisor D, there is a coherent sheaf, $\mathcal{O}_X(D)$. As explained in [2], it comes from sheafifying the $A_{n-1}(X)$-graded S-module $S(D)$, where $S(D)$ is S with degree shifted by $[D]$, i.e., its $[E]$th graded part is given by $S(D)_{[E]} = S_{[D]+[E]}$. We will always be interested in the case where X is smooth and projective. Hence, each $\mathcal{O}_X(D)$ is a line bundle.

As discussed in [1], for each element $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})$, there corresponds an Euler formula. If $f \in S$ is homogeneous of degree $[D]$, then it is straightforward to check that

$$\sum_{\rho \in \Delta(1)} \phi([D_\rho])\, x_\rho \partial_{x_\rho} f = \phi([D]) f.$$

The case of $X = \mathbb{P}^n$ recovers the usual Euler formula.

Principal parts

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on a smooth n-dimensional variety X over $\mathbb{C}$. Let $P^k(\mathcal{F})$ be the sheaf of kth order principal parts of $\mathcal{F}$. We assume familiarity with principal parts, and recall some basic facts. Some references are [4], [6], [9], [11]. Throughout, we indentify vector bundles over X with locally free sheaves of $\mathcal{O}_X$-modules.

In the case where $\mathcal{F} = F$ is locally free, the principal parts sheaves are locally free, and there are exact sequences of vector bundles

$$0 \to S^k\Omega_X \otimes_{\mathcal{O}_X} F \to P^k(F) \xrightarrow{\pi_k} P^{k-1}(F) \to 0,$$

where $S^k\Omega_X$ denotes the kth symmetric power of the cotangent bundle of X. We call these the *fundamental exact sequences* for principal parts bundles. One uses these sequences to get a local description of the principal parts bundles, which we recall for the case where $F = L$ is a line bundle. First suppose that X is affine, with coordinate ring $A = \mathbb{C}[x_1, \ldots, x_n]$, and define $B = A[dx_1, \ldots, dx_n]$ where the dx_i's are indeterminates. Then X and L can be identified with A, and the bundle $S^k\Omega_X$, (resp., $P^k(L)$), can be indentified with elements of B which are homogeneous of degree k, (resp., $\leqslant k$), in the dx_i's. For arbitrary X, the local picture is similar to the affine case just described: one takes local coordinates $x_1, \ldots, x_n$ at a point x and replaces A by the completion of the local ring of X at x (isomorphic to $\mathbb{C}[[x_1, \ldots, x_n]]$).

In general, for each k, there is a canonical map of sheaves of abelian groups

$$d_k \colon \mathcal{F} \to P^k(\mathcal{F}).$$

These maps commute with the projections to lower-order principal parts bundles: $\pi_k \circ d_k = d_{k-1}$. For $\mathcal{F} = L$, a line bundle, using local coordinates as above, d_k sends a section of L to its truncated Taylor series

$$f = f(x_1, \dots, x_n) \mapsto \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}.$$

Composing with π_k amounts to truncating the Taylor series one degree earlier.

Grothendieck, [4], defines differential operators so that they are represented by principal parts bundles. A map $D: \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups is called a *differential operator of order* $\leqslant k$ if it factors

$$\begin{tikzcd} & P^k\mathcal{F} \arrow[d, "u"] \\ \mathcal{F} \arrow[ur, "d_k"] \arrow[r, "\delta_k"] & \mathcal{G} \end{tikzcd},$$

where u is $\mathcal{O}_X$-linear. In the case where $\mathcal{F} = L$ is a line bundle, this definition is equivalent to saying that locally, D is given by $\mathcal{O}_X$-linear combinations of partial derivative operators in the local variables.

To see the connection between principal parts bundles and inflections, let L be a line bundle, W an $(n+1)$-dimensional vector space over $\mathbb{C}$, and $W \to \Gamma(X, L)$ a map of vector spaces. For each integer $k \geqslant 0$, we define an $\mathcal{O}_X$-linear *truncated Taylor series* map by evaluating global sections and taking principal parts

$$\phi_k: W \otimes_{\mathbb{C}} \mathcal{O}_X \to \Gamma(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{ev} L \xrightarrow{d_k} P^k(L).$$

These maps are compatible with the projections: $\pi_k \circ \phi_k = \phi_{k-1}$.

Assuming ϕ_0 is surjective, there is a corresponding map

$$f: X \to \mathbb{P}(W) \cong \mathbb{P}^n.$$

The study of inflections of f is equivalent to studying the degeneracy locii of the ϕ_k. Let r_k be the generic rank of ϕ_k. A point $x \in X$ such that the rank of $\phi_k(x)$ drops below r_k is called a kth order *inflection* or *Weierstrass point* for f. Let $U = \{x \in X | \mathrm{rk}\phi_k = r_k\}$. Restricting ϕ_k to U determines a surjection onto a subbundle of $P^k(L)|_U$ which corresponds to a rational map

$$f_k: X \dashrightarrow G_{r_k - 1}\mathbb{P}^n$$

to the Grassmannian of $(r_k - 1)$-planes in $\mathbb{P}^n$. This defines the kth order *associated map* of f sending a point to its kth order *osculating space*. Using local coordinates on X to parametrize the mapping, the point $x \in X$ is sent to the span of the derivatives of the mapping up to order k.

1. Principal parts on toric varieties

Let X be a smooth toric variety over $\mathbb{C}$ with s one-dimensional cones $\rho_1, \dots, \rho_s$ having associated invariant Weil divisors $D_i = D_{\rho_i}$ and homogenous coordinates $x_i = x_{\rho_i}$ for $i = 1, \dots, s$. We want to describe the principal parts of line bundles on X. Let

$$\Upsilon := \bigoplus_{i=1}^{s} \mathcal{O}_X(-D_i) = \sum_{i=1}^{s} \mathcal{O}_X(-D_i)e_i,$$

where the e_i are indeterminates. Take symmetric powers to define

$$S^k\Upsilon := \bigoplus_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant s} \mathcal{O}_X(-D_{i_1} - \cdots - D_{i_k}) = \left(\sum_{i=1}^{s} \mathcal{O}_X(-D_i)e_i\right)^k,$$

with basis consisting of monomials in the e_i's of degree k.

DEFINITION 1.1. If D is a T-Weil divisor, the bundle of kth order homogeneous principal parts of $\mathcal{O}_X(D)$ is

$$S^k\Upsilon(D) := S^k\Upsilon \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) = \bigoplus_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant s} \mathcal{O}_X(D - D_{i_1} - \cdots - D_{i_k}).$$

As an analogue of the projection of principal parts, π_k, define the $\mathcal{O}_X$-linear map

$$\begin{aligned} \sigma = \sigma_k \colon S^k\Upsilon(D) &\to A_{n-1}(X) \otimes_{\mathbb{Z}} S^{k-1}\Upsilon(D), \\ f &\mapsto \sum_{i=1}^{s} [D_i] \otimes x_i \partial_{e_i} f. \end{aligned}$$

The map σ is not generally surjective, but we will see that like the standard projection, its kernel is $S^k\Omega_X \otimes \mathcal{O}_X(D)$. It is surjective, however, in the case where $k = 1$ and $D = 0$. The resulting exact sequence has been called by [1] 'the generalized Euler exact sequence:'

$$0 \to \Omega_X \to \bigoplus_{i=1}^{s} \mathcal{O}_X(-D_i) \xrightarrow{\sigma_1} A_{n-1}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0.$$

For example, if $X = \mathbb{P}^n$ with homogeneous coordinates $x_0, \dots, x_n$ and corresponding divisors $D_0, \dots, D_n$, we can can identify $A_{n-1}(X)$ with $\mathbb{Z}$ so that σ_1 is determined by sheafifying the map

$$\bigoplus_{i=0}^{n} S(-D_i) \xrightarrow{[x_0 \cdots x_n]} S.$$

This gives the standard Euler sequence on projective space. For a general X, the map σ_1 will be determined by a matrix, as above, but with many rows: one for each Euler formula on the toric variety.

In the case of $X = \mathbb{P}^n$, the Euler sequence is exactly the fundamental exact sequence for principal parts with $k = 1$ and $D = 0$, [5]. A complete description of $P^k(\mathcal{O}_X(d))$ and the fundamental exact sequence for a range of values of k may also be described using the Euler sequence (cf. Sect. 2). The main purpose of this note is to generalize these ideas to the case of toric varieties.

To begin to compare homogenous principal parts on a general toric variety with the standard principal parts, consider the map

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}] \to S^k\mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}] = \left(\sum_{i=1}^{s} \mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]e_i\right)^k,$$

$$f \mapsto \frac{1}{k!}\left(\sum_{i=1}^{s} e_i \partial_{x_i}\right)^k f = \sum_{|\alpha|=k} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial x^\alpha} e^\alpha .$$

For each divisor D, the sheaf $\mathcal{O}_X(D)$ is naturally a subsheaf of the constant sheaf on $\mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$ (see the definition of $\mathcal{O}_X(D)$ in [2]). Hence, for each D, the map just described induces a differential operator of order k

$$\delta_k \colon \mathcal{O}_X(D) \to S^k \Upsilon(D).$$

To verify that δ_k is a differential operator, note that on a standard affine open set, the operators ∂_{x_i} which do not come directly from the coordinates can be written as combinations of those that do by using Euler formulas (cf. Sect. 0). In detail, let U be a standard affine open set and assume that $x_1, \dots, x_n$ are coordinates. The sheaf $\mathcal{O}_X(D)$ is the set of elements of degree $[D]$ in $\mathbb{C}[x_1, \dots, x_n, x_{n+1}^{\pm 1}, \dots, x_s^{\pm 1}]$, (cf. [2]). Take $[D_{n+1}], \dots [D_s]$ as a basis for $A_{n-1}(X)$, and take the dual basis, $\phi_{n+1}, \dots, \phi_s$, for $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})$. For $f \in \mathcal{O}_X(D)(U)$, the Euler formulas of Section 0 can be written (solving for ∂_{x_j})

$$\partial_{x_j} f = \frac{1}{x_j}\left(\phi_j([D]) - \sum_{i=0}^{n} \phi_j([D_i]) x_i \partial_{x_i}\right) f, \quad j = n+1, \dots, s.$$

(The reader may find the exact description of $S^k\Upsilon(D)(U)$, given in the proof of Theorem 1.2, useful in understanding the remarks just made.) Hence, using the above formulas, δ_k can be described using only partial derivatives with respect to the coordinates.

The universal property of principal parts bundles says there is an $\mathcal{O}_X$-linear map u_k factorizing δ_k through the canonical differential operator d_k

$$\begin{tikzcd}
 & P^k(\mathcal{O}_X(D)) \arrow[d, "u_k"] \\
\mathcal{O}_X(D) \arrow[ur, "d_k"] \arrow[r, "\delta_k"] & S^k\Upsilon(D)
\end{tikzcd}$$

Finally, to make the projections σ_k and π_k compatible with the maps u_k, define

$$\begin{aligned}
\tau = \tau_k\colon S^k\Upsilon(D) &\to A_{n-1}(X) \otimes_{\mathbb{Z}} S^k\Upsilon(D),\\
f &\mapsto \sum_{i=1}^{s} [D_i] \otimes x_i\partial_{x_i} f.
\end{aligned}$$

Although τ involves partial derivatives with respect to the x_i, it is $\mathcal{O}_X$-linear. If f is homogeneous of degree $[E] = \Sigma_{i=1}^{s} a_i[D_i]$, then

$$\tau(f) = \sum_{i=1}^{s} a_i[D_i] \otimes f = [E] \otimes f.$$

Hence, τ is just multiplication by degree.

We can now state the main theorem:

THEOREM 1.2. *For each $k \geqslant 0$, there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & S^k\Omega_X \otimes \mathcal{O}_X(D) \arrow[r] & S_k\Upsilon(D) \arrow[r, "\sigma_k"] & A_{n-1}(X) \otimes_{\mathbb{Z}} S^{k-1}\Upsilon(D) & \\
 & & & S^{k-1}\Upsilon(D) \arrow[u, "\tau_{k-1}"] & \\
0 \arrow[r] & S^k\Omega_X \otimes \mathcal{O}_X(D) \arrow[r] \arrow[uu, "="] & P^k(\mathcal{O}_X(D)) \arrow[r, "\pi_k"] \arrow[uu, "u_k"] & P^{k-1}(\mathcal{O}_X(D)) \arrow[r] \arrow[u, "u_{k-1}"] & 0
\end{tikzcd} .$$

The bottom row is the fundamental sequence for principal parts bundles. Since π_k is surjective, $P^k(\mathcal{O}_X(D))$ is the pullback of σ_k and $\tau_{k-1} \circ u_{k-1}$. (See Section 2 for remarks about cases where u_k is injective.)

Proof. We will first show that the right-hand square of the diagram commutes. Since $P^k(\mathcal{O}_X(D))$ is generated by the image of d_k and d_k is compatible with the projection, π_k, it suffices to show that $\sigma_k \circ \delta_k = \tau_{k-1} \circ \delta_{k-1}$

$$\begin{aligned}
\sigma_k(\delta_k(f)) &= \sum_i [D_i] \otimes x_i \partial_{e_i} \left(\frac{1}{k!} \left(\sum_j e_j \partial_{x_j} \right)^k f \right) \\
&= \sum_i [D_i] \otimes x_i \partial_{x_i} \left(\frac{1}{(k-1)!} \left(\sum_j e_j \partial_{x_j} \right)^{k-1} f \right) \\
&= \sum_i [D_i] \otimes x_i \partial_{x_i} (\delta_{k-1}(f)) = \tau_{k-1}(\delta_{k-1}(f)).
\end{aligned}$$

We now take local coordinates to check that the induced map between $\ker \sigma_k$ and $\ker \pi_k$ is an isomorphism. Let U be a standard maximal affine open set of X. We may assume that $x_1, \ldots, x_n$ correspond to one-dimensional cones spanning a maximal cone in the fan for X and that U is the corresponding affine subset. Take the primitive lattice points on these one-dimensional cones as a basis for N, the dual basis for M, and $[D_{n+1}], \ldots, [D_s]$ as a basis for $A_{n-1}(X)$. The exact sequence, (1), for $A_{n-1}(X)$ becomes

$$0 \to \mathbb{Z}^n \xrightarrow{\begin{bmatrix} I_n \\ -C \end{bmatrix}} \mathbb{Z}^s \xrightarrow{[C \ I_{s-n}]} \mathbb{Z}^{s-n} \to 0,$$

for some matrix $C = [c_{n+i,j}]_{1 \leqslant i \leqslant s-n,\, 1 \leqslant j \leqslant n}$.

According to [2], if E is a T-Weil divisor, then $\Gamma(U, \mathcal{O}_X(E))$ is the set of elements of degree $[E]$ in the localized ring $\mathbb{C}[x_1, \ldots, x_s]_{x_{n+1} \cdots x_s}$. Since linearly equivalent divisors give rise to isomorphic line bundles, we may assume that $D = a_{n+1} D_{n+1} + \cdots + a_s D_s$ for some integers a_i. Hence, the affine coordinate ring on U is

$$\Gamma(U, \mathcal{O}_X) = B = \mathbb{C}[w_1, \ldots, w_n], \qquad w_i := \frac{x_i}{\Pi_{j=n+1}^s x_j^{c_{j,i}}}.$$

Also, letting $x^D = \Pi_{i=n+1}^s x_i^{a_i}$, we have

$$\Gamma(U, \mathcal{O}_X(D)) = x^D B$$

and

$$\begin{aligned}
\Gamma(U, S^k \Upsilon(D)) = x^D \Bigg(& B \frac{e_1}{\Pi_{j=n+1}^s x_j^{c_{j,1}}} + \cdots \\
& + B \frac{e_n}{\Pi_{j=n+1}^s x_j^{c_{j,n}}} + B \frac{e_{n+1}}{x_{n+1}} + \cdots + B \frac{e_s}{x_s} \Bigg)^k.
\end{aligned}$$

To make the kernel of σ_k apparent, it is convenient to change e-variables, letting

$$z_i = \begin{cases} \frac{e_i}{\Pi_{j=n+1}^s x_j^{c_{j,i}}} - w_i \sum_{\ell=n+1}^s c_{\ell,i} \frac{e_\ell}{x_\ell}, & \text{for } 1 \leqslant i \leqslant n, \\ \frac{e_i}{x_i}, \quad \text{for } n+1 \leqslant i \leqslant s. \end{cases}$$

It follows that $\Gamma(U, S^k\Upsilon(D)) = x^D(\Sigma_{i=1}^s B z_i)^k$, and since $[D_i] = \Sigma_{j=n+1}^s c_{j,i}[D_j]$ for $i = 1, \dots, n$, we get that

$$\sigma_k = [D_{n+1}] \otimes \partial_{z_{n+1}} + \cdots + [D_s] \otimes \partial_{z_s}.$$

Hence, the kernel of σ_k consists of polynomials of z-degree k in $x^D B[z_1, \dots, z_s]$ which do not involve any of the z_i with $i > n$.

We identify $P^k(\mathcal{O}_X(D))|_U$ with the elements of degree $\leqslant k$ in $B[\mathrm{d}w_1, \dots, \mathrm{d}w_n]$, thinking of the $\mathrm{d}w_i$'s as indeterminates. For $x^D f(w_1, \dots, w_n) \in \Gamma(U, \mathcal{O}_X(D)) = x^D B$, the map d_k gives the truncated Taylor series expansion of f,

$$\begin{aligned} d_k(x^D f) &= \frac{1}{k!}\left(1 + \sum_{i=1}^n \mathrm{d}w_i\, \partial_{w_i}\right)^k f \\ &= \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial w^\alpha}\, \mathrm{d}w^\alpha. \end{aligned}$$

The kernel of π_k is the set of polynomials of degree exactly k in the $\mathrm{d}w_i$'s. We will be finished if we show that for each $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ with $|\alpha| = k$, the map u_k sends the monomial $\mathrm{d}w^\alpha$ to the corresponding monomial in the kernel of σ_k, namely, $x^D z^\alpha$. A priori, we know that u_k maps the kernel of π_k to the kernel of σ_k, so $u_k(\mathrm{d}w^\alpha)$ has no terms involving z_i with $i > n$. Thus, we just need to check the coefficients of terms only involving z_i with $i \leqslant n$.

Using a standard identity from analysis,

$$\begin{aligned} u_k(\mathrm{d}w^\alpha) &= u_k\left(\sum_{\beta \leqslant \alpha} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} w^{\alpha-\beta} d_k(x^D w^\beta)\right) \\ &= \sum_{\beta \leqslant \alpha} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} w^{\alpha-\beta} \delta_k(x^D w^\beta). \end{aligned}$$

For each $\gamma = (\gamma_1, \dots, \gamma_n, 0, \dots, 0) \in \mathbb{Z}^s_{\geqslant 0}$ with $|\gamma| = k$, we need to find the z^γ-term in the above expression. In $\delta_k(x^D w^\beta)$, this term is

$$\delta_k(x^D w^\beta)_{z^\gamma} = \left[\frac{1}{k!}\left(\sum_{i=1}^s e_i \partial_{x_i}\right)^k (x^D w^\beta)\right]_{z^\gamma}$$

$$= \left[\binom{\beta}{\gamma} x^D w^\beta \prod_{i=1}^{n} \left(\frac{e_i}{x_i}\right)^{\gamma_i}\right]_{z^\gamma}$$

$$= \left[\binom{\beta}{\gamma} x^D w^\beta \prod_{i=1}^{n} \left(\frac{1}{w_i}(z_i + w_i \sum_{\ell=n+1}^{s} c_{\ell,i}\frac{e_\ell}{x_\ell}\right)^{\gamma_i}\right]_{z^\gamma}$$

$$= \binom{\beta}{\gamma} x^D w^{\beta-\gamma} \prod_{i=1}^{n} z_i^{\gamma_i}.$$

(A note to help see that the second line in the above computation follows from the first: Recall that each z_i is a homogenous linear combination of the e_j's. Hence, to find the z^γ-term, we only need to consider partial derivatives, ∂_{x_i}, with $i \leqslant n$.) Now note that since $\beta \leqslant \alpha$ and $|\alpha| = |\gamma|$, the final expression is zero unless $\beta = \alpha = \gamma$. Thus,

$$u_k(\mathrm{d}w^\alpha) = x^D z^\alpha$$

as required. □

2. Discussion

I. Taylor Series/Associated Maps

We define Taylor series maps with respect to the homogeneous coordinates on the toric variety, X, and compare them with the usual notion (cf. Sect. 0). Let $W \to \Gamma(X, \mathcal{O}_X(D))$ be a map of vector spaces over $\mathbb{C}$. By evaluating global sections, define the $\mathcal{O}_X$-linear map

$$\phi_k^h : W \otimes_{\mathbb{C}} \mathcal{O}_X \to \Gamma(X, \mathcal{O}_X(D)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{ev} \mathcal{O}_X(D) \xrightarrow{\delta_k} S^k\Upsilon(D).$$

There is a commutative diagram expressing the compatibility of ϕ_k^h with the usual truncated Taylor series map

$$\begin{tikzcd}
 & S^k\Upsilon(D) \\
W \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[ur, "\phi_k^h"] \arrow[r, "\phi_k"] & P^k(\mathcal{O}_X(D)) \arrow[u, "u_k"']
\end{tikzcd} .$$

If u_k is injective, then the ranks of ϕ_k^h and ϕ_k are the same at all points of X. Thus, ϕ_k^h can be used to measure inflections: if W maps to a set of globally generating sections of $\mathcal{O}_X(D)$, the kth order inflections of the corresponding map, $X \to \mathbb{P}^n$, are exactly the points where ϕ_k^h drops rank. The following proposition gives a sufficient condition for the injectivity of u_k.

PROPOSITION 2.1. *Using the notation of Section 1, the map* u_k *is injective provided that the classes,* $[D - D_{i_1} - \cdots - D_{i_\ell}]$ *are nonzero for all* $i_1, \ldots, i_\ell \in \{1, \ldots, s\}$ *and for* $\ell = 0, \ldots, k-1$.

Proof. From the definition of $S^\ell \Upsilon(D)$ and the fact that τ_ℓ is multiplication by degree, it follows that τ_ℓ is injective provided that $[D - D_{i_1} - \cdot - D_{i_\ell}]$ are nonzero for all $i_1, \ldots, i_\ell \in \{1, \ldots, s\}$.

Theorem 1.2 implies that u_ℓ is injective provided that $u_{\ell-1}$ and $\tau_{\ell-1}$ are injective. The result follows. □

For instance, on $\mathbb{P}^n$ with $\mathcal{O}_X(D) = \mathcal{O}_X(d)$, the map u_k is injective if $k \leqslant d$ or if $d < 0$. (For more on this point, see II, below.)

We can define variants of u_k and ϕ_k^h so that inflections can be measured using homogeneous coordinates even in the case where u_k is not injective. First, define the differential operator of order k:

$$\delta_{\leqslant k} = \bigoplus_{i=0}^{k} \delta_k \colon \mathcal{O}_X(D) \to \bigoplus_{\ell=0}^{k} S^\ell \Upsilon(D).$$

(Compare this with δ_k of Section 1, where we only took derivatives of order exactly k in the homogeneous coordinates.) The $\mathcal{O}_X$-linear map corresponding to $\delta_{\leqslant k}$ via the universal property of principal parts bundles is

$$u_{\leqslant k} = \bigoplus_{i=0}^{k} u_k \colon P^k(\mathcal{O}_X(D)) \to \bigoplus_{\ell=0}^{k} S^\ell \Upsilon(D).$$

Using the Euler formulas from Section 0, it is straightforward to check that $u_{\leqslant k}$ is always injective. Defining $\phi_{\leqslant k}^h := \oplus_{\ell=0}^k \phi_\ell^h$, we get a 'Taylor series' map and a commutative diagram

$$\begin{tikzcd}
 & \oplus_{\ell=0}^{k} S^\ell \Upsilon(D) \\
W \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, "\phi_k"] \arrow[ur, "\phi_{\leqslant k}^h"] & P^k(\mathcal{O}_X(D)) \arrow[u, "u_{\leqslant k}"']
\end{tikzcd} .$$

Now, the ranks of ϕ_k and $\phi_{\leqslant k}^h$ are the same at all points of X. This idea was used in [12] to define inflections of toric mappings using homogeneous coordinates.

II. Principal parts on projective space

Let $X = \mathbb{P}^n$ and $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(d)$ for some integer d. The commutative diagram in Theorem 1.2 can be written

$$\begin{tikzcd}
0 \arrow[r] & S^k\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d) \arrow[r] & \mathcal{O}_{\mathbb{P}^n}(d-k)^{\binom{b+k}{k}} \arrow[r, "\sigma_k"] & \mathcal{O}_{\mathbb{P}^n}(d-k+1)^{\otimes\binom{n+k-1}{k-1}} \arrow[r] & 0 \\
 & & & \mathcal{O}_{\mathbb{P}^n}(d-k+1)^{\otimes\binom{n+k-1}{k-1}} \arrow[u, "\cdot(d-k+1)"] & . \\
0 \arrow[r] & S^k\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d) \arrow[r] \arrow[uu, "="] & P^k(\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r, "\pi_k"] \arrow[uu, "u_k"] & P^{k-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] \arrow[u, "u_{k-1}"] & 0
\end{tikzcd}$$

The surjectivity of the upper row can be checked in local coordinates. The map τ_{k-1} is multiplication by degree, $d-k+1$. Since u_0 is an isomorphism, it follows from the five-lemma that all the u_k are isomorphisms for $k = 1, \ldots, d$. When $k = d+1$, the map τ_d is multiplication by zero and u_{d+1} is not an isomorphism. If d is negative, then u_k is an isomorphism for all k.

We have identified $P^k(\mathcal{O}_{\mathbb{P}^n}(d))$ as a direct sum of line bundles in the case where $k \leqslant d$ or $d < 0$. In the case where $k > d$, it follows from Theorem 1.2 that

$$P^k(\mathcal{O}_{\mathbb{P}^n}(d)) = Q^k \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+d}{d}},$$

where the bundle Q^k is given as

$$Q^k = \ker\left(\mathcal{O}_{\mathbb{P}^n}(d-k)^{\oplus\binom{n+k}{k}} \xrightarrow{\sigma_k\circ\cdots\circ\sigma_{d+1}} \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+d}{d}}\right).$$

In particular, $Q^{d+1} = S^{d+1}\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d)$. It also follows that for $k > d$ there are exact sequences

$$0 \to S^k\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d) \to Q^k \to Q^{k-1} \to 0$$

It would be nice to know more about these bundles, Q^k.

Note. In [10] it had previously been noted that $P^k(\mathcal{O}(d)) \cong \mathcal{O}(d-k)^{\oplus(k+1)}$ on $\mathbb{P}^1$ for $\leqslant k \leqslant d$.

III. Principal parts of projective bundles

In [11], a description of principal parts on projective bundles over arbitrary schemes was given. We recall this description here in an extended form. Let E be a vector bundle of rank $n+1$ over a noetherian scheme S. Let $P = \mathbb{P}(E)$ be the projective bundle of one-dimensional quotients of E with projection $u\colon P \to S$ and universal quotient bundle $\mathcal{O}(\infty)$. (For example, if E is a trivial bundle over a field $S = k$, then $P \cong \mathbb{P}^n_k$.)

To describe the principal parts of $\mathcal{O}(\lceil)$, we use the Euler sequence on P, which we write as

$$0 \to \Omega_{P/S} \otimes \mathcal{O}(\infty) \to \mathcal{E}_{\mathcal{P}} \xrightarrow{\varepsilon} \mathcal{O}(\infty) \to /,$$

where $E_P := u^* E$. Define a map

$$\begin{aligned} &\varepsilon_k\colon S^k E_P \to \mathcal{O}(\infty) \otimes \mathcal{S}^{||-\infty}\mathcal{E}_{\mathcal{P}} := \mathcal{S}^{||-\infty}\mathcal{E}(\infty), \\ &v_1 \cdots v_k \mapsto \sum_i (\varepsilon(v_i) \otimes v_1 \cdots \hat{v}_i \cdots v_k). \end{aligned}$$

Tensoring by $\mathcal{O}(\ell)$ gives a map which we also denote by ε_k

$$\varepsilon_k\colon S^k E_P(\ell) \to S^{k-1} E_P(\ell + 1).$$

(This map is essentially σ_k from Section 1.)

THEOREM 2.1. *Let $k \geqslant 0$ be an integer, and assume that the characteristic of the residue field at each point of S is zero or greater than k; then there is a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & S^k\Omega_{P/S} \otimes \mathcal{O}(d) \arrow[r] & S^k E_P(d-k) \arrow[r, "\varepsilon_k"] & S^{k-1} E_P(d-k+1) \arrow[r] & 0 \\ 0 \arrow[r] & S^k\Omega_{P/S} \otimes \mathcal{O}(d) \arrow[r] \arrow[u, "="] & P^k(\mathcal{O}(d-k)) \arrow[r, "\pi_k"] \arrow[u, "\nu_k"] & P^{k-1}(\mathcal{O}(d-k+1)) \arrow[r] \arrow[u, "\nu_{k-1}"] & 0 \end{tikzcd}.$$

The bottom row is the fundamental sequence for principal parts bundles. The map ν_k is an isomorphism when $k \leqslant d$ or when $d < 0$.

In [11], this theorem is proved only for the case $k \leqslant d$. Using the ideas presented in this paper, the result can be extended to the case $d < 0$.

IV. Differential Operators on Toric Varieties

We have given a description of $P^k(\mathcal{O}_X(D))$ on a toric variety. Thus, taking duals – applying $\mathcal{H}om(\,\cdot\,, \mathcal{O}_X)$ to Theorem 1.2 – should give a description of the differential operators $D\colon \mathcal{O}_X(D) \to \mathcal{O}_X$. I. Musson, [7], has described these 'twisted' differential operators on a toric variety, and it would be interesting to compare our results.

Acknowledgments

I would like to thank the Matematisk Institutt of the University of Oslo, Norway for its hospitality, and thank Reed College for providing time and support through a Vollum research grant.

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