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# Special values of twisted symmetric square $L$ -functions and the trace formula<sup>\*</sup>

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**Abstract.** We use the trace formula to compute explicitly the trace, over a Hecke eigenbasis, of the algebraic part of the special values. The case of twisting holomorphic level one modular forms by a quadratic character modulo  $q$  is considered. The result involves both class numbers of binary quadratic forms with discriminant depending on  $q$ , and also the number of points on certain elliptic curves reduced modulo  $q$ .

## 0. Introduction

We let  $K = \mathbb{Q}(\sqrt{q})$ , with ring of integers  $\mathcal{O}$  for an odd prime  $q \equiv 1 \pmod{4}$ . We use  $\chi$  to denote  $(q/\ast)$ , the quadratic character modulo  $q$ . The integral kernel for the base change lifting  $f \rightarrow \tilde{f}$  from  $\mathrm{SL}(2, \mathbb{Z})$  cusp forms to those of  $\mathrm{SL}(2, \mathcal{O})$  is denoted  $\Omega(\tau, z, z')$  (see [12]).

Integrating  $\Omega$  against a Hilbert modular form  $F$  in the  $(z, z')$  variable gives a linear map from  $S_k(\mathrm{SL}(2, \mathcal{O}))$  to  $S_k(\mathrm{SL}(2, \mathbb{Z}))$ . It's easy to see this is the adjoint of the lift  $\Omega$ , so we denote this linear map  $\Omega^*$ . What can be said about this map? From a version of the Strong Multiplicity One theorem due to Ramakrishnan, we get that the lift is 1-1, and from this follows

**EASY LEMMA.** *Every Hecke eigenform  $f$  is also an eigenform of the map  $\Omega^*\Omega$ , with eigenvalue equal to  $\langle \tilde{f}, \tilde{f} \rangle / \langle f, f \rangle$ . If  $F$  is a Hilbert modular eigenform which is not a lift, then  $\Omega^*F = 0$ .*

As we will show below

$$\frac{\langle \tilde{f}, \tilde{f} \rangle}{\langle f, f \rangle} = \beta(f) \langle f, f \rangle, \quad \beta(f) \in \mathbb{Q}(f).$$

(Zagier did the analogous result for forms of nebentypus in [12], Corollary 1 to Theorem 5 by a different method.) Thus the map  $B$ , defined on eigenforms by

$$Bf = \langle f, f \rangle^{-1} \Omega^* \Omega f = \beta(f) f, \tag{0.1}$$

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is in the Hecke algebra. In this paper we will use the trace formula to get an explicit computation of the trace of  $B$ , and more generally the trace of  $T(n)B$  for any Hecke operator  $T(n)$ . The trace is a sum over  $l$  of class numbers  $H(q(l^2 - 4q^2n))$  and  $H(q(l^2 - 4n))$  of orders in complex quadratic fields. These class numbers are weighted, respectively, by the character  $\chi(l)$  and by the number of points

$$a(l, n) = \#E - q - 1, \quad E/\mathbb{F}_q: y^2 = x^3 + lx^2 + nx. \quad (0.2)$$

### 0.1. WHY IS THIS INTERESTING?

The method of computing the trace of the period is one Zagier suggested in Section 5 of [12]. He did the analogous trace formula for forms of level  $D$ , quadratic character in Section 6 of that paper; including the technically much more difficult case of weight 2 forms. (This is implicit in formulae (91) and (98). All that is missing is to subtract off an appropriate multiple of an Eisenstein series to get a cusp form.) His interest was in connection with intersection numbers of cycles on the Hilbert modular surface. Gordon [3] has considered higher weight analogs of intersection numbers; presumably  $\beta(f)$  has an interpretation in terms of these intersection numbers.

The value at  $s = k$  (in the standard normalization) of the twisted symmetric square  $L$ -function  $D(s, f, \chi)$  is  $\beta(f)\langle f, f \rangle$ . This is also the residue of the corresponding Asai  $L$ -function  $L(s, \tilde{f}, \rho \otimes \tilde{\chi})$ . By the work of Harder, Langlands and Rapoport, the Asai  $L$ -functions occur as factors of the Hasse–Weil zeta function of the Hilbert modular surface  $\mathcal{H}^2/\mathrm{SL}(2, \mathcal{O})$  (for weight 2 forms). There are lots of conjectures on the arithmetic significance of the special values of such zeta functions, in the context of higher  $K$ -theory and regulators for algebraic varieties. See [8] and [9] for an exposition.

*Notation.* The prime  $q \equiv 1 \pmod{4}$  is fixed throughout, as is a positive integer  $n$  indexing a Fourier coefficient. The weight  $k$  of  $\mathrm{SL}(2, \mathbb{Z})$  cusp forms is fixed throughout, but  $k$  is also used sometimes as a subscript in sums.  $\delta(*)$  is 0 if the argument is not an integer, and  $e(*)$  denotes  $\exp(2\pi i *)$ .  $\Delta$  denotes a typical non-primitive discriminant, either  $l^2 - 4n$  or  $l^2 - 4q^2n$ , written as  $Df^2$  with  $D$  primitive. Subscripts  $q$  on  $L$ -functions indicate local factors at  $q$  omitted, while superscripts  $L^q$  are those local factors.

## 1. Special values of $L$ -series

It will be very useful to view the constant  $\beta(f)$  of (0.1) in terms of  $L$ -series. Write

$$f(\tau) = \sum_n a(n)n^{(k-1)/2} \exp(2\pi in\tau).$$

and let  $f_\chi(\tau)$  be the quadratic twist (with level  $q^2$  and trivial character.) Note we have a non-classical normalization of the Fourier coefficients. Letting  $\alpha + \alpha' = a(p)$

and  $\alpha\alpha' = 1$ , we write  $L(s, f \times f)$  and  $L(s, f \times f_\chi)$  as degree four Euler products; a computation then gives the splitting formula  $L(s, f \times f)L(s, f \times f_\chi) = L(s, \tilde{f} \times \tilde{f})$ . The poles of the Eisenstein series produce poles in the Rankin convolutions

$$\frac{\Gamma(k)\text{res}_{s=1}L(s, f \times f)}{(2\pi)^2} = (4\pi)^{k-1}\langle f, f \rangle,$$

$$\frac{q^2\Gamma(k)^2\text{res}_{s=1}L(s, \tilde{f} \times \tilde{f})}{(2\pi)^4} = (4\pi)^{2k-2}\langle \tilde{f}, \tilde{f} \rangle\text{res}_{s=1}\zeta_K(s),$$

so  $L(s, f \times f_\chi)$  is entire at  $s = 1$  and

$$\frac{q^2\Gamma(k)L(1, f \times f_\chi)}{(2\pi)^2} = (4\pi)^{k-1}\frac{\langle \tilde{f}, \tilde{f} \rangle}{\langle f, f \rangle}\text{res}_{s=1}\zeta_K(s).$$

We can also consider the twisted symmetric square  $L$  function

$$D(s, f, \chi) = \prod_p (1 - \chi(p)\alpha^2 p^{-s})^{-1} (1 - \chi(p)p^{-s})^{-1} (1 - \chi(p)\alpha^2 p^{-s})^{-1}.$$

Thus  $L(s, \chi)D(s, f, \chi) = L(s, f \times f_\chi)$ , and by the above

$$D(1, f, \chi) = C_k \frac{\langle \tilde{f}, \tilde{f} \rangle}{\langle f, f \rangle},$$

where  $C_k = (2\pi)^2(4\pi)^{k-1}(q^2\Gamma(k))^{-1}$ . We can build a cusp form  $\Phi_s(\tau)$  depending also on  $s$  which satisfies

$$\langle \Phi_s, f_i \rangle = C_k^{-1}D(s, f_i, \chi),$$

for each eigenform  $f_i$ . On the one hand, the eigenforms give an orthogonal basis, so

$$\Phi_s(\tau) = C_k^{-1} \sum_i \frac{D(s, f_i, \chi)}{\langle f_i, f_i \rangle} f_i(\tau).$$

This implies that the Fourier expansion of  $\Phi_s(\tau)$  looks like

$$\Phi_s(\tau) = C_k^{-1} \sum_{n=1}^{\infty} \left\{ \sum_i \frac{a_i(n)n^{(k-1)/2}D(s, f_i, \chi)}{\langle f_i, f_i \rangle} \right\} \exp(2\pi i n \tau).$$

Plugging in  $s = 1$  and using the above gives

$$\Phi_1(\tau) = \sum_{n=1}^{\infty} \left\{ \sum_i \frac{a_i(n)n^{(k-1)/2}\langle \tilde{f}_i, \tilde{f}_i \rangle}{\langle f_i, f_i \rangle^2} \right\} \exp(2\pi i n \tau).$$

On the other hand we can define the usual the Poincare series  $G_r(\tau)$  which satisfies

$$\langle G_r, f \rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} a(r) r^{(1-k)/2},$$

and use the identity (formula (0.2) of [10])

$$D(s, f, \chi) = \zeta(2s)_q \sum_{m=1}^{\infty} \chi(m) a(m^2) m^{-s}.$$

This gives

$$\Phi_s(\tau) = \frac{(k-1)q^2}{(2\pi)^2} \zeta(2s)_q \sum_{m=1}^{\infty} \chi(m) m^{-s+k-1} G_{m^2}(\tau). \quad (1.1)$$

The series converges absolutely and uniformly on compact subsets of  $\{\operatorname{re}(s) > 1\} \times \mathcal{H}$ . Writing out the Fourier expansion of the Poincare series  $G_{m^2}(\tau)$ , interchanging the order of summation, and continuing to  $s = 1$  will give an explicit computation of the Fourier series coefficients

$$\Phi_s(\tau) = \sum_{n=1}^{\infty} b(n, s) \exp(2\pi i n \tau), \quad (1.2)$$

and  $b(n, 1)$  gives an explicit computation of the (algebraic) expression

$$\sum_i \frac{a_i(n) n^{(k-1)/2} \langle \tilde{f}_i, \tilde{f}_i \rangle}{\langle f_i, f_i \rangle^2} = \sum_i a_i(n) n^{(k-1)/2} \beta(f_i) = \operatorname{trace}(T(n)B).$$

## 2. Poisson summation

In computing the Fourier expansion of  $\Phi_s(\tau)$ , we will suppose first  $1 < \operatorname{re}(s) < k-1$  and then find analytic continuation to include  $\operatorname{re}(s) = 1$ . It is well known the Fourier expansion of the Poincare series  $G_r(\tau)$  is

$$\sum_{n=1}^{\infty} \delta_{r,n} + 2\pi(-1)^{k/2} \left(\frac{n}{r}\right)^{(k-1)/2} \left\{ \sum_{c=1}^{\infty} \frac{1}{c} K_c(r, n) J_{k-1} \left( \frac{4\pi\sqrt{rn}}{c} \right) \right\} e(n\tau).$$

Plugging this into (1.1) with  $r = m^2$  we get

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \chi(m) m^{-s+k-1} \left( \delta_{m^2, n} + 2\pi(-1)^{k/2} \left(\frac{n}{m^2}\right)^{(k-1)/2} \right. \\ & \quad \left. \times \left\{ \sum_{c=1}^{\infty} \frac{1}{c} K_c(m^2, n) J_{k-1} \left( \frac{4m\pi\sqrt{n}}{c} \right) \right\} \right) e(n\tau). \end{aligned}$$

Letting

$$S = 2 \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} \frac{\chi(m)}{c} m^{-s} K_c(m^2, n) J_{k-1} \left( \frac{4m\pi\sqrt{n}}{c} \right),$$

we see the  $n$ th Fourier coefficient  $b(n, s)$  of  $\Phi_s(\tau)$  in (1.2) satisfies

$$\frac{(2\pi)^2}{(k-1)q^2\zeta(2s)_q} b(n, s) = \pi(-1)^{k/2} n^{(k-1)/2} S + \delta(\sqrt{n})\chi(\sqrt{n})n^{(-s+k-1)/2}.$$

PROPOSITION. The ‘main term’,  $S$  is equal

$$\begin{aligned} & \sum_{l \in \mathbb{Z}} \sum_{(c,q)=1} c^{-1}(cq)^{-s} I(l, nq^2, s) \left\{ \sum_{r(cq)} \chi(r) K_c(r^2, n) e\left(\frac{lr}{cq}\right) \right\} \\ & + \sum_{a=1}^{\infty} (cq^a)^{-1-s} I(l, n, s) \left\{ \sum_{r(cq^a)} \chi(r) K_{cq^a}(r^2, n) e\left(\frac{lr}{cq^a}\right) \right\}. \end{aligned} \tag{2.1}$$

where the special function  $I(l, n, s)$  is defined by (2.2), and  $K_c(r^2, n)$  is a Kloosterman sum.

Proof. Put

$$A(x) = |x|^{-s} J_{k-1}\left(\frac{4\pi\sqrt{n}|x|}{c}\right), \quad A(0) = 0,$$

then  $S$  equals

$$\sum_{c=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{\chi(m)}{c} K_c(m^2, n) A(m).$$

Before applying Poisson summation we need to relate the modulus  $q$  of  $\chi$  to the modulus  $c$  of the Kloosterman sum  $K_c$ . We first break the sum on  $c$  into a double sum of  $c$  prime to  $q$  and a sum on prime powers  $q^a$

$$\sum_{(c,q)=1} \sum_{a=0}^{\infty} \sum_{m \in \mathbb{Z}} (cq^a)^{-1} \chi(m) K_{cq^a}(m^2, n) A(m).$$

Now  $\chi(m) K_{cq^a}(m^2, n)$  depends only on  $m$  modulo  $cq^{a+\varepsilon}$  with

$$\varepsilon = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \geq 1. \end{cases}$$

We can then replace the sum on  $m$  with a double sum over  $r$  modulo  $cq^{a+\varepsilon}$  and  $m$  in  $\mathbb{Z}$ ,  $m \equiv r$ . We then consider

$$\sum_{m \equiv r} A(m) = (cq^{a+\varepsilon})^{-s} \sum_{j \in \mathbb{Z}} \left| j + \frac{r}{cq^{a+\varepsilon}} \right|^{-s} J_{k-1}\left(4\pi\sqrt{n}q^\varepsilon \left| j + \frac{r}{cq^{a+\varepsilon}} \right| \right),$$

and let

$$B(x) = \sum_{j \in \mathbb{Z}} |j+x|^{-s} J_{k-1}(4\pi\sqrt{n}q^\varepsilon |j+x|).$$

Write the Fourier expansion of  $B(x)$  as  $\sum_l c_l e(lx)$ . By Poisson summation we have

$$c_l = \int_{-\infty}^{\infty} |x|^{-s} J_{k-1}(4\pi\sqrt{n}q^\varepsilon|x|)e(-lx) dx = (\text{definition}) I(l, nq^{2\varepsilon}, s). \quad (2.2)$$

Thus

$$\begin{aligned} \sum_{m \equiv r} A(m) &= (cq^{a+\varepsilon})^{-s} B\left(\frac{r}{cq^{a+\varepsilon}}\right) \\ &= (cq^{a+\varepsilon})^{-s} \sum_{l \in \mathbb{Z}} I(l, nq^{2\varepsilon}, s) e\left(\frac{lr}{cq^{a+\varepsilon}}\right). \end{aligned}$$

which proves the proposition.  $\square$

### 3. Character sums

We will work towards getting the character sums in braces in (2.1) into a closed form, to realize the  $n$ th Fourier coefficient  $b(n, s)$  as an infinite sum over  $l$  of Dirichlet series times the special functions  $I(l, n, s)$ . Consider first when  $a \geq 1$

$$\sum_{r(cq^a)} \chi(r) \sum_{(x, cq^a)=1} e\left(\frac{r^2 x^{-1} + nx}{cq^a}\right) e\left(\frac{lr}{cq^a}\right). \quad (3.1)$$

After interchanging the sums and changing variables  $r \rightarrow rx$ , one sees the character sum depends on the behavior of the two counting functions (choosing either  $+$  or  $-$  and fixed  $t$ )

$$\# \left\{ r \bmod cq^a \mid \chi(r) = \pm 1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod cq^a \right\},$$

and particularly their difference. Each of these counting functions can be written as a product of a term depending only on  $c$  and a term depending only on  $a$ . In particular the former can be written

$$N(t, x^2 + lx + n, c) = \# \left\{ r \bmod c \mid r^2 + lr + n \equiv t \bmod c \right\},$$

while for the latter, the relevant term is the difference

$$\begin{aligned} R(t, q^a) &= \# \left\{ r \bmod q^a \mid \chi(r) = +1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod q^a \right\} \\ &\quad - \# \left\{ r \bmod q^a \mid \chi(r) = -1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod q^a \right\}, \end{aligned}$$

and in this case the dependence on the quadratic is suppressed in the notation. By direct computation one sees that the character sum (3.1) simplifies to

$$\begin{aligned} cq^{a-1/2} \sum_{d|c} \chi\left(\frac{c}{d}\right) \mu\left(\frac{c}{d}\right) \chi(d) N(0, x^2 + lx + n, d) \\ \times \sum_{j=1}^q \chi(j) R(jq^{a-1}, q^a). \end{aligned}$$

The character sum  $\sum_{j=1}^q \chi(j)R(j, q)$  is equal  $\#E - q - 1$ , where  $\#E$  is the number of projective points over  $\mathbb{Z}/q\mathbb{Z}$  of the curve  $E: y^2 = x^3 + lx^2 + nx$  with discriminant  $(4n)^2(l^2 - 4n)$ . We denote this character sum  $a(l, n)$ . In particular if the curve is singular, the character sum is  $-\chi(l/2)$ . For the general  $a$ , it is convenient to introduce the generating function  $F_{l,n}(T)$  defined by

$$\sum_{a=1}^{\infty} \sum_{j=1}^q \chi(j)R(jq^{a-1}, q^a)T^a = a(l, n) \times \begin{cases} T & \text{if } q \nmid \Delta \\ T - \sum_{\beta=1}^{\alpha-1} (q-1)q^{\beta}T^{2\beta+1} + q^{\alpha}T^{2\alpha+1} & \text{if } q \nmid D, \alpha > 0 \\ T - \sum_{\beta=1}^{\alpha} (q-1)q^{\beta}T^{2\beta+1} + \chi\left(\frac{D}{q}\right)q^{\alpha+1}T^{2\alpha+2} & \text{if } q \mid D \\ T - \sum_{\beta=1}^{\infty} (q-1)q^{\beta}T^{2\beta+1} & \text{if } \Delta = 0. \end{cases}$$

Here the discriminant of the quadratic  $l^2 - 4n$  is written  $\Delta = Df^2$  with  $D$  a fundamental discriminant. In the first three cases  $q^{\alpha} \parallel f$ , while in the last three cases the factor  $a(l, n)$  reduces to  $-\chi(l/2)$ . By summing the geometric series one can write this in the alternative form (with  $\chi = \chi(D/q)$  when  $q \mid D$ )

$$F_{l,n}(T) = \frac{a(l, n)T}{1 - qT^2} \times \begin{cases} 1 - qT^2, \\ 1 - q^2T^2 + q^{\alpha+1}T^{2\alpha}(1 - T^2), \\ 1 - q^2T^2 - \chi q^{\alpha+1}T^{2\alpha+1}(1 + \chi T)(1 - \chi qT), \\ 1 - q^2T^2. \end{cases} \tag{3.2}$$

For the other sum in braces in (2.1), coming from ‘ $a = 0$ ’, similar methods show that

$$\begin{aligned} & \sum_{r(cq)} \chi(r)K_c(r^2, n)e\left(\frac{lr}{cq}\right) \\ &= \chi(l)cq^{1/2} \sum_{d|c} \chi\left(\frac{c}{d}\right) \mu\left(\frac{c}{d}\right) \chi(d)N(0, qx^2 + lx + qn, d). \end{aligned} \tag{3.3}$$

### 4. Zeta functions

We now sum the Dirichlet series

$$\sum_{(c,q)=1} c^{-s} \sum_{d|c} \chi\left(\frac{c}{d}\right) \mu\left(\frac{c}{d}\right) \chi(d)N(0, q^{\epsilon}x^2 + lx + q^{\epsilon}n, d), \tag{4.1}$$



where  $\varepsilon$  is 0 or 1. This is a convolution of two Dirichlet series, so we have

$$L(s, \chi)^{-1} \sum_{c=1}^{\infty} \chi(c) N(0, q^\varepsilon x^2 + lx + q^\varepsilon n, c) c^{-s}.$$

We will write  $N_{c,\Delta}$  instead of  $N(0, q^\varepsilon x^2 + lx + q^\varepsilon n, c)$ , where  $\Delta = l^2 - 4q^{2\varepsilon}n$  is the discriminant. Elementary considerations tell us that in this case  $N_{c,\Delta}$  is multiplicative. And for  $k > 0$ ,  $N_{p^k,\Delta} = N_{p,\Delta} = 1 + (\Delta/p)$  whenever the discriminant  $\Delta$  is prime to  $p$ . More generally write  $\Delta = Df^2$  with  $D$  the discriminant of  $\mathbb{Q}(\sqrt{\Delta})$ . One can compute

$$\begin{aligned} L(s, \chi)^{-1} \sum_{c=1}^{\infty} \chi(c) N_{c,\Delta} c^{-s} \\ = \begin{cases} \frac{L(s, (D/*)\chi)}{\zeta(2s)_q} \sum_{\substack{c|f \\ (c,q)=1}} \mu(c) \left(\frac{D}{c}\right) \chi(c) c^{-s} \sigma_{1-2s}(fc^{-1}) \\ \frac{\zeta(2s-1)_q}{\zeta(2s)_q}, \end{cases} \end{aligned}$$

if  $\Delta \neq 0$  (resp.  $\Delta = 0$ ).

The term  $\zeta(2s)_q^{-1}$  in the lemma will be canceled out by the corresponding  $\zeta(2s)_q$  in (1.1). What remains will simplify if for each  $\Delta = Df^2$ , we write  $q\Delta = D'f'^2$ . The term on the right side above is very nearly the one associated to the non-fundamental discriminant  $q\Delta$

$$\begin{aligned} L(s, q\Delta) \\ = (\text{definition}) L\left(s, \left(\frac{D'}{*}\right)\right) \sum_{c|f'} \mu(c) \left(\frac{D'}{c}\right) c^{-s} \sigma_{1-2s}(f'c^{-1}), \quad (4.2) \end{aligned}$$

the only local factor missing is the one at  $q$ ,  $L^q(s, q\Delta)$ . If  $(q, \Delta) = 1$ , this local factor is 1. We can always assume this is the case for  $\Delta$  of the form  $l^2 - 4q^2n$ , since otherwise  $q | \Delta$  implies  $q | l$ . The relevant character sum (3.3) is then zero; these terms will disappear from the trace. In the other case  $\Delta = l^2 - 4n$ ; the missing Euler factor will be obtained from the sum on  $a$  in (2.1). We introduce a fudge factor  $\Gamma(s, \Delta)$  (which turns out to be related to the  $q$ -adic  $\Gamma$  function) so that

$$F_{l,n}(q^{-s}) = a(l, n) q^{-s} \Gamma(s, \Delta) L^q(s, q\Delta). \quad (4.3)$$

Then (3.2) implies

- (i) If  $(q, \Delta) = 1$ ,  $\Gamma(s, \Delta) = 1$  for all  $s$ .
- (ii) If  $q|D$  and  $\chi(Dq^{-1}) = 1$ , then  $\Gamma(s, \Delta)$  vanishes at  $s = 1$ .
- (iii) If  $\Delta = 0$ ,  $\Gamma(s, \Delta) = 1 - q^{2-2s}$  vanishes at  $s = 1$ .

### 5. Special functions

After a change of variables the special function

$$I(l, n, s) = 2^s \pi^{s-1} \int_0^\infty x^{-s} J_{k-1}(2\sqrt{n}x) \cos(|l|x) dx,$$

is found in the tables to have an analytic continuation to  $\frac{1}{2} < \text{re}(s) < k$ . For example, if  $n$  is a square,  $\Delta$  will be 0 when  $l^2 = 4n$  or  $l^2 = 4q^2n$ , and we see from [1], vol. 2 (19.2.24) on p. 342 that  $I(2\sqrt{n}, n, s)$  is a quotient of Gamma functions, and has a simple zero at  $s = 1$ .

The general case of this special function is found in [4], (6.561.14) on p. 684 and (6.699.2) on p. 747, in terms of hypergeometric functions. In particular if  $l^2 > 4n$ , [2] vol. 2, 2.8 (6) shows that  $I(l, n, s)$  has a simple zero at  $s = 1$ . If  $4n > l^2$  we can get the value at  $s = 1$  from [7], (1.13.12), p. 67

$$I(l, n, 1) = (-1)^{(k-2)/2} \frac{(4n - l^2)^{1/2}}{(k-1)\sqrt{n}} C_{k-2}^1 \left( \frac{|l|}{(\sqrt{4n})} \right). \tag{5.1}$$

Here  $C_{k-2}^1$  is a Gegenbauer polynomial.

### 6. Evaluate at $s = 1$

We return to consideration of the Fourier expansion. Combine the proposition in Section 2 with formula (4.1) and (4.3) to see that the Fourier coefficient  $b(n, s)$  is written in closed form as

**THEOREM.**

$$\begin{aligned} & \frac{(-1)^{k/2} (2\pi)^2 b(n, s)}{(k-1)q^{3/2}} \\ &= \pi \sum_{l \in \mathbb{Z}} \chi(l) q^{(1-s)n^{(k-1)/2}} I(l, q^2n, s) L(s, q(l^2 - 4q^2n)) \\ & \quad + \pi \sum_{l \in \mathbb{Z}} a(l, n) q^{-sn^{(k-1)/2}} I(l, n, s) L(s, q(l^2 - 4n)) \Gamma(s, l^2 - 4n) \\ & \quad + \delta(\sqrt{n}) (-1)^{k/2} \sqrt{q} \zeta(2s)_q \chi(\sqrt{n}) n^{(-s+k-1)/2}. \end{aligned}$$

For the reader inclined to skim, we recall that  $L(s, q(l^2 - 4n))$  is defined by formula (4.2) to be the Dirichlet series associated to a (non-fundamental) discriminant, and  $a(l, n)$  is defined by (0.2). The special functions  $I(l, n, s)$  are defined by (2.2), and  $\Gamma(s, l^2 - 4n)$  by (4.3).

The infinite sum converges uniformly and absolutely in the strip  $\frac{1}{2} < \text{re}(s) < k - 1$ , which by Fubini justifies changing the order of summation in Section 2.

This is as far as we will go with the variable  $s$  present; the expression will simplify when we evaluate at  $s = 1$ . In terms of the Eichler–Selberg trace formula, this is the ‘Selberg principle’ that the orbital integrals coming from hyperbolic conjugacy classes should not contribute to the trace.

**PROPOSITION.** *The terms in the theorem with  $\Delta \geq 0$  all vanish at  $s = 1$ .*

*Proof.* The discriminant  $\Delta$  can only be zero if  $n$  is a square. We observed in Section 5 that  $I(2\sqrt{n}, n, s)$  has a simple zero at  $s = 1$ , while  $L(s, 0)$  has a simple pole at  $s = 1$ . However if  $l^2 - 4q^2n$  is 0, then  $l$  is 0 modulo  $q$  and so  $\chi(l) = 0$ . And (4.3 (iii)) shows that  $\Gamma(s, 0)$  has a simple zero at  $s = 1$ . This completes the  $\Delta = 0$  case. We know from Section 5 that  $I(l, n, 1) = 0$  when  $\Delta = l^2 - 4n > 0$ . These terms contribute nothing unless  $L(s, q\Delta)$  has a pole at  $s = 1$ . This happens exactly when  $\Delta = qf^2$  for some  $f \neq 0$ , then  $L(s, q^2f^2)$  is  $\zeta(s)$  times a Dirichlet polynomial. If  $l^2 - 4q^2n = qf^2$ , then  $\chi(l) = 0$ ; these terms drop out. On the other hand  $\Delta = l^2 - 4n = qf^2$  terms are in one to one correspondence with divisors of  $n = \nu\nu'$ ,  $\nu \in \mathcal{O}$ ,  $\nu \notin \mathbb{Z}$ . (There will be infinitely many such, if any.) But the simple zero of  $I(l, n, s)$  will cancel the pole of  $\zeta(s)$ , and (4.3 (ii)) shows that  $\Gamma(s, qf^2)$  has a simple zero at  $s = 1$ , since  $Dq^{-1}$  is a square in this case.  $\square$

There are finitely many  $l$  such that  $\Delta < 0$ . We use (5.1) in this case to evaluate the special function  $I(l, n, s)$  at  $s = 1$  in terms of Gegenbauer polynomials (see also [2], vol. 1, Sect. 3.15.1)

$$n^{(k-2)/2} C_{k-2}^1 \left( \frac{|l|}{\sqrt{4n}} \right) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}, \quad \rho + \bar{\rho} = |l|, \quad \rho\bar{\rho} = n,$$

which is  $P_{k,1}(l, n)$  in the notation of Zagier ([12] formula (18).) The value of  $L(1, q\Delta) = \pi H(q\Delta) / \sqrt{|q\Delta|}$  is classical,  $H(q\Delta)$  being the number of equivalence classes of binary quadratic forms  $\phi$  of discriminant  $q\Delta$  weighted by  $1/\text{Aut}(\phi)$ . Equivalently,  $H(q\Delta)$  is the sum of class numbers of orders of discriminant  $q\Delta/f^2$  in  $\mathbb{Q}(\sqrt{q\Delta})$ .

**THEOREM.** *For the map  $B$  defined in (0.1), the trace of the Hecke operator  $T(n)B$  is  $b(n, 1)$  where*

$$\begin{aligned} b(n, 1) &= -\frac{1}{4} \sum_{l^2 < 4q^2n} \chi(l) q^{2-k} P_{k,1}(l, q^2n) H(q(l^2 - 4q^2n)) \\ &\quad - \frac{1}{4} \sum_{l^2 < 4n} a(l, n) P_{k,1}(l, n) H(q(l^2 - 4n)) \Gamma(1, l^2 - 4n) \\ &\quad + \delta(\sqrt{n}) \chi(\sqrt{n}) n^{(k-2)/2} \frac{k-1}{24} (q^2 - 1). \end{aligned}$$

**COROLLARY.** *Let  $\beta(f) = \langle \tilde{f}, \tilde{f} \rangle / \langle f, f \rangle^2$ , then  $\beta(f) \in \mathbb{Q}(f)$ .*

*Proof.* This follows from the fact that the Fourier coefficients  $b(n, 1)$  of  $\Phi_1(\tau)$  are rational in the theorem above, and Lemma 4, p. 792 of [11].  $\square$

## 7. Examples

To convince ourselves the computations above are correct, we did some examples using *Mathematica*. In the case of weight  $k = 10$ , there are no cusp forms so  $\Phi_s(\tau) = 0$  for all  $s$ . With  $q = 5$  we verified  $b(n, 1) = 0$  for  $1 \leq n \leq 20$ . In the case of weight  $k = 12$ , the space of cusp forms is spanned by the discriminant cusp form, so the Fourier coefficient  $b(n, 1) = \tau(n)b(1, 1)$ , with  $\tau(n)$  Ramanujan's tau function, and again the above relation holds for  $1 \leq n \leq 20$ . The coefficient  $b(1, 1) = \beta$  was then computed (still weight  $k = 12$ ) for some small primes  $q$

$$\begin{aligned} q = 5 & \quad \beta = 2^{12} \cdot 3^6 \cdot 7 / 5^8, \\ q = 13 & \quad \beta = 2^{12} \cdot 3^9 \cdot 5^3 \cdot 7 \cdot 563 / 13^{10}, \\ q = 17 & \quad \beta = 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 2389 / 17^{10}, \\ q = 29 & \quad \beta = 2^{12} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 13 \cdot 17 \cdot 317 / 29^{10}, \\ q = 37 & \quad \beta = 2^{12} \cdot 3^{10} \cdot 5^6 \cdot 7 \cdot 89 \cdot 3889 / 37^{10}, \\ q = 41 & \quad \beta = 2^{23} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 117413 / 41^{10}. \end{aligned}$$

The formula for  $b(n, s)$  also reduces to a finite sum for  $s = 3, 5, \dots, k - 1$ , with a different Gegenbauer polynomial and Cohen's function instead of the Hurwitz–Kronecker class number. The computations are analogous to those in [12]. In this case the Dirichlet series  $D(s, f, \chi)$  converges absolutely, and we computed the first 100 terms of the series with  $s = 3, 5, 7$

$$\begin{aligned} \frac{D(3, f, \chi)}{C_k \langle f, f \rangle} &\approx 22.5795 & b(1, 3) &= \frac{2^{12} \cdot 3^2 \cdot 7 \cdot 2851}{5^{12} \cdot 13} \pi^4 \\ & & &\approx 22.5794729896, \\ \frac{D(5, f, \chi)}{C_k \langle f, f \rangle} &\approx 20.30838094 & b(1, 5) &= \frac{2^{15} \cdot 3 \cdot 1511599}{5^{17} \cdot 7 \cdot 13} \pi^8 \\ & & &\approx 20.3083809367, \\ \frac{D(7, f, \chi)}{C_k \langle f, f \rangle} &\approx 19.903417716 & b(1, 7) &= \frac{2^{13} \cdot 3^2 \cdot 521 \cdot 295387}{5^{22} \cdot 13 \cdot 17} \pi^{12} \\ & & &\approx 19.9034177155. \end{aligned}$$

Here again  $q = 5$  and  $k = 12$ . The value  $1.03536205679 \times 10^{-6}$  for the square of the norm of the discriminant function was taken from [12].

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