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The reduction number of an algebra

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Abstract. A complexity of an algebra A over a field k is a measure of comparison to a polynomial ring over k. Here we bring to the fore the reduction number of a graded algebra A, and study its relationship to the arithmetic degree of A. The relationship between the reduction number and the Castelnuovo–Mumford regularity has been object of previous studies, but a presumed relationship CM regularity and arithmetic degree breaks down.

Key words: Arithmetic degree, Castelnuovo–Mumford regularity, Cohen–Macaulay ring, Hilbert function, multiplicity, reduction number.

1. Introduction

Let A be a finitely generated, positively graded algebra over a field k,

$$A = k + A_1 + A_2 + \cdots = k + A_+,$$

where A_i denotes the space of homogeneous elements of degree i. We further assume that A is generated by its 1-forms, $A = k[A_1]$, in which case $A_i = {A_1}^i$. Such algebras are said to be standard. Among the *complexities* of A are the Castelnuovo–Mumford regularity $\operatorname{reg}(A)$ of A and various degrees – multiplicity $\deg(A)$ and arithmetic degree arith-deg(A). Another is the reduction number $\mathrm{r}(A)$ of A which occurs in the study of the distribution of the degrees of the generators of A as a module over its Noether normalizations (see below for all definitions). Here we introduce two techniques, one theoretical and the other computational, to examine the relative strengths of these numbers.

There have been several comparisons between these numbers. In addition, [2] is a far-flung survey of $\operatorname{reg}(A)$ and arith-deg(A) in terms of the degree data of a presentation $A = k[x_1, \ldots, x_n]/I$. Moreover, [12] carries out a detailed comparison between the degrees of an ideal and the degrees of some associated ideal of initial forms.

From the study of $\operatorname{reg}(A)$ (see [5], [9] and particularly [13]), one has

$$\mathrm{r}(A) \leqslant \operatorname{reg}(A). \tag{1}$$

Here we will show (Theorem 7) that for any standard algebra A

$$\mathrm{r}(A) < \text{arith-deg}(A), \tag{2}$$

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if k has characteristic zero. Actually, Theorem 7 deals with general affine algebras and the estimate above is one of its applications for standard algebras. The approach, an elementary combination of linear and homological algebras, may be suited to obtain other estimates for $\mathrm{r}(A)$.

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2. The degrees of a module

We recall the settings of three definitions, two of which may not be in basic references. If k is sufficiently large and $\dim A = d$, there are forms $x_1, \ldots, x_d \in A_1$, such that

$$R = k[x_1, \ldots, x_d] = k[\mathbf{z}] \hookrightarrow A = S/I, \; S = k[x_1, \ldots, x_d, x_{d+1}, \ldots, x_n]$$

is a Noether normalization, that is, the x_i are algebraically independent over k and A is a finite R-module. Let $b_1, b_2, \ldots b_s$ be a minimal set of homogeneous generators of A as an R-module

$$A = \sum_{1 \leqslant i \leqslant s} Rb_i, \quad \deg(b_i) = r_i.$$

We will be looking at the distribution of the r_i, particularly at the following integer.

DEFINITION 1. The reduction number $\mathrm{r}_R(A)$ of A with respect to R is the supremum of all $\deg(b_i)$. The (absolute) reduction number $\mathrm{r}(A)$ is the infimum of $\mathrm{r}_R(A)$ over all possible Noether normalizations of A.

One of our aims is to make predictions about these integers, but without availing ourselves of any Noether normalization. We emphasize this by saying that the Noether normalizations are invisible to us, and the information we may have about A comes from the presentation $A = S/I$.

The integer $\mathrm{r}(A)$ may be understood as a measure of complexity of the algebra A. Taken this way, it has been compared to another index of complexity of A:

DEFINITION 2. Let S be a polynomial ring, let $A = S/I$ and let

$$0 \to \bigoplus_{j=1}^{b_g} S(-d_{gj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} S(-d_{1j}) \longrightarrow S \longrightarrow A = S/I \to 0$$

be a minimal graded free resolution of A. The *Castelnuovo–Mumford regularity* of A is the integer

$$\mathrm{reg}(A) \;=\; \max\{|d_{ij} - i| \; \forall j\}. \tag{3}$$

Another way to size A is given in [2] and [8] through the notion of the *arithmetic degree* of a module. Let A be a Noetherian ring and let M be a finitely generated R-module. For each prime ideal $\mathfrak{p}$ of A, let $\Gamma_{\mathfrak{p}}(M)$ denote the set of elements of the localization $M_{\mathfrak{p}}$ annihilated by some power of $\mathfrak{p}$. Note that $\Gamma_{\mathfrak{p}}(M)$ is an Artinian module over $A_{\mathfrak{p}}$ that vanishes unless $\mathfrak{p}$ is an associated prime of M. We denote the length of $\Gamma_{\mathfrak{p}}(M)$ by $\mathrm{mult}_M(\mathfrak{p})$.

DEFINITION 3. The *arithmetic degree* of M is the integer

$$\text{arith-deg}(M) = \sum_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathrm{mult}_M(\mathfrak{p}) \cdot \deg A/\mathfrak{p}. \tag{4}$$

The definition (see how these sums are calculated in Proposition 5, even though the individual summands are not always available) applies to arbitrary modules, not just graded modules, although its main use is for graded modules over a standard algebra A. If all the associated primes of M have the same dimension, then arith-deg(M) is just the multiplicity $\deg(M)$ of M, which is obtained from its Hilbert polynomial.

EXAMPLE 4. In the following families of examples, we explore possible relations between arith-deg(A) and $\mathrm{reg}(A)$.

(a) In dimension 1, we have $\mathrm{reg}(A) <$ arith-deg(A). Indeed, let $R = k[x]$ be a Noether normalization of A and let

$$A \simeq \bigoplus_{i=1}^{e} R(-a_i) \bigoplus_{j=1}^{f} (R/(x^{c_j}))(-b_j)$$

be the decomposition of A as the direct sum of cyclic R-modules.

We have

$$\begin{aligned} \mathrm{r}(A) &= \sup\{a_i, b_j\}, \\ \mathrm{reg}(A) &= \sup\{a_i, b_j + c_j - 1\}, \\ \text{arith-deg}(A) &= e + \sum_{j=1}^{f} c_j, \end{aligned}$$

while the minimal number of generators of A as an R-module is

$$\nu(A) = e + f.$$

Because A is generated by elements of degree 1, there must be no gaps in the sequence of degrees of its module generators, which implies $b_j < \nu(A)$, and no gaps either in the degrees of its torsion-free part so that $a_i < e$. This proves that for each integer ℓ,

$$b_\ell \leqslant \nu(A) - 1 \leqslant e + \sum_{j\neq \ell} c_j,$$

and therefore

$$\mathrm{r}(A) \leqslant \mathrm{reg}(A) < \text{arith-deg}(A).$$

(b) In dimension greater than 1, the inequality $\mathrm{reg}(A) < \text{arith-deg}(A)$ does not always hold, according to the following example of Bernd Ulrich. Let $A = k[x, y, u, v]/I$, with $I = ((x, y)^2, xu^t + yv^t)$. If $t \geqslant 3$,

$$\mathrm{r}(A) = 1 \leqslant \text{arith-deg}(A) = 2 \leqslant \mathrm{reg}(A) = t.$$

(c) Despite the preceding example, it is significant that the inequality $\mathrm{reg}(A) < \text{arith-deg}(A)$ has been established for large classes of algebras (see [12]).

We now show how a program with the capabilities of *Macaulay* ([3]) can be used to compute the arithmetic degree of a graded module M without availing itself of any primary decomposition. Let $S = k[x_1, \dots, x_n]$ and suppose $\dim M = d \leqslant n$. It suffices to construct graded modules M_i, $i = 1 \dots n$, such that

$$e_1(M_i) = e_i(M).$$

For each integer $i \geqslant 0$, denote $L_i = \mathrm{Ext}_S^i(M, S)$. By local duality ([4]), a prime ideal $\mathfrak{p} \subset S$ of height i is associated to M if and only if $(L_i)_\mathfrak{p} \neq 0$; furthermore $\ell((L_i)_\mathfrak{p}) = \mathrm{mult}_M(\mathfrak{p})$.

This gives what is required: Compute for each L_i its degree $e_1(L_i)$ and codimension c_i. Then choose M_i according to

$$M_i = \begin{cases} 0 & \text{if } c_i > i \\ L_i & \text{otherwise.} \end{cases}$$

In other words:

PROPOSITION 5. *For a graded S-module M and for each integer i denote by c_i the codimension of* $\mathrm{Ext}_S^i(M, S)$. *Then*

$$\text{arith-deg}(M) = \sum_{i=0}^{n} \left[\frac{i}{c_i}\right] \deg(\mathrm{Ext}_S^i(M, S)).$$

Equivalently,

$$\text{arith-deg}(M) = \sum_{i=0}^{n} \deg(\mathrm{Ext}_S^i(\mathrm{Ext}_S^i(M, S), S)).$$

The amusing of this formula (one sets $\left[\frac{0}{0}\right] = 1$) is that it gives a sum of sums of terms some of which may not be available.

One consequence of this formulation of the arithmetic degree is the following. Let T be a term order on $S = k[x_1, \dots, x_n]$ and let I be a homogeneous ideal of S. Denote by $in(I)$ the initial ideal of I for the term order. For each integer r, $\mathrm{Ext}_S^r(S/I, S)$ is a submodule of a quotient of $\mathrm{Ext}_S^r(S/in(I), S)$. This arises from the spectral sequence induced by the attached filtration (see [6], [11]). Consequently, we have

$$\deg(\mathrm{Ext}_S^r(S/I, S)) \leqslant \deg(\mathrm{Ext}_S^r(S/in(I), S)),$$

giving Theorem 2.3 of [12].

3. Integrality equations

From now on A is a standard graded ring and $R = k[\mathbf{z}] \hookrightarrow A$ is a fixed Noether normalization. To determine $\mathrm{r}(A)$, we look for equations of integral dependence of the elements of A with respect to R.

A simple approach is to find graded R-modules on which A acts as endomorphisms (e.g. A itself). The most naive path to the equation is through the Cayley–Hamilton theorem working as follows. Let E be a finitely generated R-module and let $f: E \to E$ be an endomorphism. Map a free graded module over E and lift f:

$$\begin{array}{ccc} F & \xrightarrow{\pi} & E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow[\pi]{} & E\,. \end{array}$$

Let

$$P_\varphi(t) = \det(tI + \varphi) = t^n + \cdots + a_n$$

be the characteristic polynomial of φ, $n = \mathrm{rank}(F)$. It follows that $P_\varphi(f) = 0$. The drawback is that n, which is at least the minimal number of generators of E, may be too large. One should do much better using a trick of [1]. Lift f to a mapping from a projective resolution of E into itself:

$$\begin{array}{ccccccccccc} 0 & \to & F_s & \to & \cdots & \to & F_1 & \to & F_0 & \to & E & \to 0 \\ & & \downarrow{\scriptstyle\varphi_s} & & & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle f} & \\ 0 & \to & F_s & \to & \cdots & \to & F_1 & \to & F_0 & \to & E & \to 0\,. \end{array}$$

Define

$$P_f(t) = \prod_{i=0}^{s} (P_{\varphi_i}(t))^{(-1)^i}.$$

This rational function is actually a polynomial in $R[t]$ ([1]). If E is a graded module and f is homogeneous, then $P_f(t)$ is a homogeneous polynomial, $\deg E = \deg P_f(t)$.

PROPOSITION 6 (Cayley-Hamilton theorem). *If the rank of E over R is e, $P_f(t)$ is a monic polynomial of degree e. Moreover, if E is torsion-free then $P_f(f)\cdot E = 0$. Furthermore, if E is a faithful A-module and f is homogeneous of degree* 1, *then* $f^e \in (\mathbf{z})A_+$.

Proof. Most of these properties are proved in [1]. Passing over to the field of fractions of R, the characteristic polynomial of the vector space mapping

$$f \otimes K\colon E \otimes K \to E \otimes K$$

is precisely $P_f(t)$.

The existence of an equation

$$f^e + c_1 f^{e-1} + \cdots + c_e = 0,\ c_i \in R,$$

and the fact that $\mathrm{Hom}_R(E, E)$ is graded implies that there is a similar equation where $c_i \in (\mathbf{z})^i$.

Without the torsion-free hypothesis the assertions may fail. For instance, if $A = k[x,y]/(xy, y^2)$, f is multiplication by y on A, then $P_f(t) = t$, but $P_f(f) \neq 0$.

In case the module E is A itself, we do not need the device of Proposition 6, as we can argue directly as follows. For $u \in A_1$, $R[u] \simeq R[t]/I$, where $I = f \cdot J$, height$(J) \geqslant 2$. But if A is torsion-free over R, $R[u]$ will have the same property and necessarily $J = (1)$. This means that the rank of $R[u]$, which is the degree of f, is at most $e(A)$.

A question of independent interest is to find R-modules of small multiplicity that afford embeddings

$$A \hookrightarrow \mathrm{Hom}_R(E, E).$$

For example, the relationship between their multiplicities may be as large as $\deg(E) = n$ and $\deg(R) = [\frac{n^2}{2}] + 1$ ([10]). There are however certain restrictions to be overcome: If the Cohen–Macaulay type of the localization of A at its minimal primes is at most 3, then $\deg(E) \geqslant \deg(A)$ (see [7]).

4. Arithmetic degree of a module and the reduction number of an algebra

Let A be an affine algebra, not necessarily standard. We now bound the degrees of the equations satisfied by the elements of A with respect to any of its 'optimal' Noether normalizations. These simply mean those normalizations that can be used to read the degrees.

THEOREM 7. *Let A be an affine algebra over an infinite field k, let $k[\mathbf{z}]$ be an optimal Noether normalization of A, and let M be a finitely generated graded, faithful A-module. Then every element of A satisfies a monic equation over $k[\mathbf{z}]$ of degree at most* arith-deg(M).

Proof. Let

$$(0) = L_1 \cap L_2 \cap \cdots \cap L_n$$

be an equidimensional decomposition of the trivial submodule of M, derived from an indecomposable primary decomposition by collecting together the components of the same dimension. If I_i = annihilator (M/L_i), then each ring A/I_i is unmixed, equidimensional and

$$\dim A/I_i > \dim A/I_{i+1}.$$

Since k is infinite, there exists a Noether normalization $k[z_1, \ldots, z_d]$ of A such that for each ideal I_i, a subset of the $\{z_1, \ldots, z_d\}$ generates a Noether normalization for A/I_i.

First, we are going to check that arith-deg(M) can be determined by adding the arithmetic degrees of the factors of the filtration

$$M \supset L_1 \supset L_1 \cap L_2 \supset \cdots \supset L_1 \cap L_2 \cap \cdots \cap L_n = (0),$$

at the same time that we use the Cayley–Hamilton theorem.

We write the arithmetic degree of M as

$$\text{arith-deg}(M) = e_1 + e_2 + \cdots + e_n,$$

where e_i is the contribution of the prime ideals minimal over I_i. (Warning: This does not mean that $e_i = \text{arith-deg}(M/L_i)$.) We first claim that (set $L_0 = M$)

$$\text{arith-deg}(M) = \sum_{i=1}^{n} \text{arith-deg}(L_1 \cap \cdots \cap L_{i-1}/L_1 \cap \cdots \cap L_i).$$

Indeed, there is an embedding

$$F_i = L_1 \cap \cdots \cap L_{i-1}/L_1 \cap \cdots \cap L_i \hookrightarrow M_i = M/L_i,$$

showing that F_i is equidimensional of the same dimension as M_i. If $\mathfrak{p}$ is an associated prime of I_i, localizing we get $(L_1 \cap \cdots \cap L_i)_{\mathfrak{p}} = (0)$ which shows that

$$\Gamma_{\mathfrak{p}}(M_{\mathfrak{p}}) \subset \Gamma_{\mathfrak{p}}((L_1 \cap \cdots \cap L_{i-1})_{\mathfrak{p}}),$$

while the converse is clear. This shows that the geometric degree of the module F_i is exactly the contribution of e_i to arith-deg(M).

We are now ready to use Proposition 6 on the modules F_i. Let $f \in A$ act on each F_i. For each integer i, we have a polynomial

$$P_i(t) = t^{e_i} + c_1 t^{e_i - 1} + \cdots + c_{e_i},$$

with $c_j \in (\mathbf{z})^j$, and such that $P_i(f) \cdot F_i = (0)$. Consider the polynomial

$$P_f(t) = \prod_{i=1}^{n} P_i(t),$$

and evaluate it on f from left to right. As

$$P_i(f) \cdot F_i = 0,$$

meaning that $P_i(f)$ maps $(L_1 \cap \cdots \cap L_{i-1})$ into $(L_1 \cap \cdots \cap L_i)$, a simple inspection shows that $P_f(f) = 0$, since M is a faithful module.

Observe that if A is a standard algebra and f is an element of A_1, then $P_f(t)$ gives an equation of integrality of f relative to the ideal generated by $\mathbf{z}$.

COROLLARY 8. *Let A be a standard graded algebra and denote*

$$\mathrm{edeg}(A) = \inf\{\, \text{arith-deg}(M) \mid M \textit{ faithful graded module}\}.$$

For any standard Noether normalization R of A, every element of A satisfies an equation of degree $\mathrm{edeg}(A)$ *over R.*

5. Reduction equations from integrality equations

If A is an standard algebra, for a given element $u \in A_1$, a typical equation of reduction looks like

$$u^e \in (\mathbf{z}){A_1}^{e-1},$$

which is less restrictive than an equation of integrality. One should therefore expect these equations to have lower degrees. Unfortunately we do not yet see how to approach it.

The following argument shows how to pass from integrality equation to some reduction equations, but unfortunately injects the issue of characteristic into the fray.

PROPOSITION 9. *Let $A = k[A_1]$ be a standard algebra over a field k of characteristic zero. Let $R = k[\mathbf{z}] \hookrightarrow A$ be a Noether normalization, and suppose that every element of A_1 satisfies a monic equation of degree e over $k[\mathbf{z}]$. Then* $\mathrm{r}(A) \leqslant e - 1$.

Proof. Let $u_1, \ldots, u_n$ be a set of generators of A_1 over k, and consider the integrality equation of

$$u = x_1u_1 + \cdots + x_nu_n,$$

where the x_i are elements of k. By assumption, we have

$$u^e = (x_1u_1 + \cdots + x_nu_n)^e = a_1u^{e-1} + \cdots + a_e,$$

where $a_i \in (\mathbf{z})^i$. Expanding u^e we obtain

$$\sum_{\alpha} a_\alpha m_\alpha u^\alpha \in (\mathbf{z})A_1^{e-1},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an exponent of total degree e, a_α is the multinomial coefficient $\binom{e}{\alpha}$, and m_α is the corresponding 'monomial' in the x_i. We must show

$$u^\alpha \in (\mathbf{z})A_1^{e-1}$$

for each α.

To prove the assertion, it suffices to show that the span of the vectors $(a_\alpha m_\alpha)$, indexed by the set of all monomials of degree e in n variables, has the dimension of the space of all such monomials. Indeed, if these vectors lie in a hyperplane

$$\sum_\alpha c_\alpha T_\alpha = 0,$$

we would have a homogeneous polynomial

$$f(X_1, \ldots, X_n) = \sum_\alpha c_\alpha a_\alpha X^\alpha$$

which vanishes on k^n. This means that all the coefficients $c_\alpha a_\alpha$ are zero, and therefore each c_α is zero since the a_α do not vanish in characteristic zero.

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