ON EXACT CONTROLLABILITY FOR THE NAVIER-STOKES EQUATIONS

O.YU. IMANUVILOV

Abstract. We study the local exact controllability problem for the Navier-Stokes equations that describe an incompressible fluid flow in a bounded domain $\Omega$ with control distributed in a subdomain $\omega \subset \Omega$. The result that we obtain in this paper is as follows. Suppose that $\tilde{v}(x)$ is a given steady-state solution of the Navier-Stokes equations. Let $v_0(x)$ be a given initial condition and $|\tilde{v}(x) - v_0(x)| < \varepsilon$ where $\varepsilon$ is small enough. Then there exists a locally distributed control $u$ such that the solution $v(t, x)$ of the Navier-Stokes equations

$$\partial_t v - \Delta v + (v, \nabla)v = -\nabla p + f, \quad \text{div} v = 0, \quad v|_{\partial \Omega} = 0, \quad v|_{\omega = 0} = v_0$$

coincides with $\tilde{v}(x)$ at instant $T$, $v(T, x) = \tilde{v}(x)$.

1. Introduction

This paper is concerned with the local exact controllability of the Navier-Stokes equations, defined on a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) with boundary $\partial \Omega \in C^m$. More precisely, the problem under study is as follows. Let us consider the nonstationary Navier-Stokes equations

$$\partial_t v(t, x) - \Delta v(t, x) + (v, \nabla)v + \nabla p = f(x) + \chi_{\omega} u \quad \text{in } \Omega, \quad \text{div} v = 0, \quad (1.1)$$

with initial and boundary conditions

$$v|_{t = 0} = 0, \quad v|_{\partial \Omega} = v_0(x), \quad (1.2)$$

where $v(t, x) = (v_1(t, x), \ldots, v_n(t, x))$ is the fluid velocity, $p$ the pressure, $f(x) = (f_1(x), \ldots, f_n(x))$ a density of external forces, $u(t, x)$ a control distributed in an arbitrary fixed subdomain $\omega$ of the domain $\Omega$ and $\chi_\omega$ is the characteristic function of the set $\omega$:

$$\chi_\omega(x) = \begin{cases} 1, & \text{for } x \in \omega \\ 0, & \text{for } x \in \Omega \setminus \omega. \end{cases}$$

Let $(\tilde{v}(x), \tilde{p}(x))$ be a steady-state solution of the Navier-Stokes equations

$$-\Delta \tilde{v} + (\tilde{v}, \nabla)\tilde{v} + \nabla \tilde{p} = f(x) \quad \text{in } \Omega, \quad \text{div} \tilde{v} = 0, \quad \tilde{v}|_{\partial \Omega} = 0 \quad (1.3)$$

Korea Institute for Advanced Study, 207-43 Cheongnyang-dong, Dongusan-myeon, Seoul 130-012, Korea. E-mail: oleg@kias.re.kr

Research partially supported by KIAS (Grant KIAS-A97001), GARC-KOSEF and KOSEF (K00103-040070).


© Société de Mathématiques Appliquées et Industrielles. Typeset by T\textsc{eX}.
close enough to the initial condition
\[ \| v_0 - \bar{v} \|_{V;\Omega} \leq \varepsilon, \]  
where the parameter \( \varepsilon \) is sufficiently small. We want to find a control \( u \) such that for given \( T > 0 \), the following equality holds:
\[ v(T, x) = \bar{v}(x). \]

We assume

**Condition 1.1.** The boundary \( \partial \Omega = \bigcup_{i=1}^n \Gamma_i \in C^\infty \), \( \Gamma_i \cap \Gamma_j = \emptyset \) for all \( i \neq j \) where \( \Gamma_i \) is a \( n-1 \)-dimensional connected manifold of class \( C^\infty \). For each \( \Gamma_i \), there exists a neighborhood \( \mathcal{U}_i \subset \mathbb{R}^n \) and a diffeomorphism \( \eta_i \in C^\infty(\mathcal{U}_i, \mathbb{R}^n) \) such that \( \eta_i(\Gamma_i) = S_i^\varepsilon \).

The main result of this paper is the following Theorem.

**Theorem 1.2.** Let \( v_0 \in V^1(\Omega) \) and pair \((\phi, \psi) \in (V^1(\Omega) \cap W^2_0(\Omega))^n \times W^2_0(\Omega)\) is a given steady state solution of the Navier-Stokes equations (1.3) such that \( \text{supp} \ \psi \subset \subset \Omega \). Then for sufficiently small \( \varepsilon \) there exists a solution \((v, p, u) \in V^{1-\varepsilon}(Q) \times L^2(0, T; W^2_0(\Omega)) \times (L^2(Q, \Omega))^n \) of problem (1.1), (1.2), (1.4), (1.5).

To explain this result, let us assume that \( v|_{\partial \Omega} = 0 \) and \( \bar{v} \) is an unstable singular point of the dynamical system generated by equation (1.1) in the phase space of solenoidal vector fields with adherence conditions on \( \partial \Omega \). Let \( v_0 \) be an initial condition in a neighborhood of the function \( \bar{v} \). This work shows that one can construct a locally distributed control such that the trajectory goes out of point \( v_0 \) and reaches \( \bar{v} \) in finite time. In other words, by means of the locally distributed control, one can suppress the generation of turbulence. This result clarifies the question of the connection between turbulence and controllability (see J.-L. Lions [26]).

The result we obtain in Theorem 1.2 is local. On the other hand, for the linearized Navier-Stokes system, we can prove global zero-controllability, see Theorem 1.3.

One important special case is the following controllability problem for the Stokes system:

\[ \partial_t v(t, x) - \Delta v(t, x) = \nabla p + f(t, x) + \chi_\varepsilon u, \quad \text{in } \Omega, \ \ \ \text{div} \ v = 0, \]  
\[ v|_{\Gamma} = 0, \ \ \ v|_{t=0} = v_0(x), \ \ \ v|_{t=T} = 0. \]

We have

**Theorem 1.3.** Let \( v_0 \in V^1(\Omega), f \in L^2(0, T; V^0(\Omega)) \) and there exists \( \varepsilon > 0 \) such that \( \int_0^T \| \sigma \|_{H^{-1}} \| \sigma \|_2 \mathrm{d}t < \infty \). Then there exists a solution \((v, p, u) \in V^{1-\varepsilon}(Q) \times L^2(0, T; W^2_0(\Omega)) \times (L^2(Q, \Omega))^n \) to problem (1.6), (1.7).

This paper is organized as follows. To prove Theorem 1.2 we use a variant of the implicit function theorem. The only nontrivial condition to be checked is to show that the derivative of the corresponding mapping at some
ON EXACT CONTROLLABILITY FOR THE NAVIER-STOKES EQUATIONS

point is an epimorphism. In our case, this problem is equivalent to the zero controllability of the linearization of the Navier-Stokes equations at point \( \bar{v} \), (see problem (1.1)-(1.3)). Sections 2-4 are devoted to this problem. One of the usual ways to solve the controllability problem for evolution equations is to reduce it to an observability problem for the adjoint equation. Thus, in section 2 we introduce a linear operator (see equation (2.1)) which after the change \( t \mapsto -t \) is formally adjoint to the derivative of the Navier-Stokes equations at point \( \bar{v} \). The observability problem for this operator is solved in three steps. First in Theorem 2.1, we get an appropriate estimate for the pressure \( p \). Then in Theorem 3.1, we obtain a Carleman estimate for the velocity \( y \) of the fluid via a weighted \( L^2 \)-norm of the density of external forces \( f \) and the pressure \( p \). Moreover, for the pressure, by Theorem 2.11, one can choose a weighted \( L^2 \) norm over \( (0,T) \times \omega \). And finally in Theorem 4.6, we prove an estimate (not of Carleman type) for the velocity where \( p \) and an initial condition are absent from the right-hand side. In section 4, this observability estimate is converted into a controllability result in Theorem 4.3. In section 5, all conditions for the implicit function theorem are checked.

We close this section by mentioning some of the previous results regarding our problems. The solvability of (1.1), (1.2),(1.5) was first proved in A.V. Fursikov, O.Yu. Immanuilov [15] in the case when (1.1) is Burgers’ equation. For a control distributed in a domain \( \omega \) such that \( \partial \Omega \subset \mathbb{R} \), this problem was studied in the case of the Navier-Stokes equations and \( \bar{v} \equiv 0 \) in A.V. Fursikov, O.Yu. Immanuilov [13] in dimension \( n = 2 \) and in A.V. Fursikov [16] when \( n = 3 \). The case of the Navier-Stokes equations and \( \bar{v} \neq 0 \) has been studied in A.V. Fursikov, O.Yu. Immanuilov [14], [17], O.Yu. Immanuilov [21] and for the Boussinesq system in [15] (see also [16]). On the other hand, in pioneering works [3]-[5], J.-M. Coron proved the global approximate controllability for the 2-D Euler equations and the 2-D Navier-Stokes equations with slip boundary conditions. In [6], combining results on global approximate and local exact controllability results, J.-M. Coron and A.V. Fursikov obtained the global exact controllability for the Navier-Stokes system on a 2-D manifold without boundary.

In [7], C. Fabre obtained an approximate controllability for \( \text{“cut off”} \) Navier-Stokes equations.

2. ESTIMATE FOR THE PRESSURE

Let us consider the system

\[
\frac{\partial y}{\partial t} - \Delta y + B^*(y, \bar{v}) + B^*(\bar{v}, y) = \nabla p + f \quad \text{in} \quad Q,
\]

\[
\partial \bar{v} \cdot y = 0, \quad p|\partial Q = 0, \quad y(0, x) = y_0(x),
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( \partial \Omega \subset C^{\infty}, Q = (0,T) \times \Omega \) and the operators \( B^*(y, \bar{v}), B^*(\bar{v}, y) \) are defined by the formulas:

\[
B^*(y, \bar{v}) = \left( y \frac{\partial \bar{v}}{\partial x_1}, \ldots, y \frac{\partial \bar{v}}{\partial x_n} \right), \quad B^*(\bar{v}, y) = \bar{v} (\bar{v}, \nabla) y.
\]
Denote \( Q_\omega = (0, T) \times \omega, \Sigma = (0, T) \times \partial \Omega \). Let \( \nu \) be the outward unit normal to \( \partial \Omega \). In this paper we use the following functional spaces. Recall that \( W^{1,p}_0(\Omega), k \geq 0, 1 \leq p < \infty \) is the Sobolev space of functions with finite norm
\[
\| u \|_{W^{1,p}_0(\Omega)} = \left( \sum_{k=1}^{\infty} \int_{\Omega} \left[ \frac{\partial^k u(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right]^p dx \right)^{1/p},
\]
where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), |\alpha| = \alpha_1 + \cdots + \alpha_n, \)
\[
W^{1,2}(Q) = \{ v(t,x) \in L^2(0,T;W^{1,2}_0(\Omega)), \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega)) \},
\]
\[
V^1(\Omega) = \{ v(x) = (v_1, \ldots, v_n) \in (L^2(\Omega))^n; \nabla v = 0, v|_{\partial \Omega} = 0 \},
\]
\[
V^2(\Omega) = \{ v(x) = (v_1, \ldots, v_n) \in (L^2(\Omega))^n; \nabla v = 0, (v, \nu)|_{\partial \Omega} = 0 \},
\]
\[
V^{-1}(\Omega) = (V^1(\Omega))^*,
\]
\[
V^{-2}(Q) = \{ v(t,x) \in W^{1,2}(Q); \nabla v = 0, v|_{\partial \Omega} = 0 \},
\]
\[
L^2(Q, \rho) = \{ v(t,x); \int_Q \rho v^2 \, dx \, dt < \infty \}.
\]
We have

**Proposition 2.1.** ([27]) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( \partial \Omega \in C^2 \). Then
\[
(L^2(\Omega))^n = V^0(\Omega) \oplus (V^0(\Omega))^*,
\]
where
\[
(V^0(\Omega))^* = \{ v(x) = (v_1, \ldots, v_n) \in (L^2(\Omega))^n; v = p, p \in W^{1,2}_0(\Omega) \}.
\]
Here and below we assume that the pair \((\dot{v}, \dot{v})\) satisfies (1.3) and
\[
(\dot{v}, \dot{v}) \in (V^1(\Omega))^n \cap (W^{1,2}_0(\Omega))^n \times W^{1,2}_0(\Omega) \quad \text{supp} \dot{v} \subset \subset \Omega.
\]
Let \( \omega \subset \subset \Omega \) be an arbitrary fixed subdomain and \( \eta_i \) be the mapping from Condition 1.1.

Without loss of generality we can assume that \( \eta_i(\partial_\omega \Omega) \subset B_1 \). (Otherwise we can make the change \( x \rightarrow x|e| \).) Set \( \bar{u}_i = \eta_i^{-1}(\{ x \in \mathbb{R}^n; e < |x| < 1 \}) \), where \( e \in (0,1) \). For all sufficiently small \( e \), the set \( \bar{u}_i \) is correctly defined and
\[
\bar{u}_i \cap \bar{u}_j = \emptyset \quad \text{for all } i \neq j, \quad \partial \bar{u}_i = \Gamma_i \cup \gamma_i,
\]
where \( \gamma_i = \eta_i^{-1}(S_{\infty}^{\omega_i}) \) and
\[
(\overline{\mathbb{N}} \cup \text{supp} \dot{v}) \cap \overline{\mathbb{N}} = \emptyset \quad \forall i = 1, \ldots, K.
\]
Let \( G \subset \mathbb{R}^n \) be a domain which satisfies the following condition:

**ESAIM:** Ocean, Mar. 1998, vol. 3, p. 41-50
CONDITION 2.2. The domain G is diffeomorphic to the cylinder $\Gamma \times [0, T_\alpha]$, where $T_\alpha > 0$ is a number and $\Gamma \subset \mathbb{R}^n$ is a closed $(n-1)$-dimensional manifold of class $C^\infty$.

This condition implies immediately that $\partial G \in C^\infty$ and $\partial G = \sigma_u \cup \sigma_1$, where $\sigma_i$ is a $(n-1)$-dimensional connected manifold of class $C^\infty$.

Let $w(x)$ be a harmonic function in $G:
\begin{equation}
\Delta w = 0 \text{ in } G,
\end{equation}
such that
\begin{equation}
\frac{\partial w}{\partial n}|_{\partial G} = 0,
\end{equation}
\begin{equation}
\int_{\partial G} |w| d\sigma \leq (2M)^2, \int_{\partial G} |\nabla w|^2 d\sigma \leq (2\varepsilon)^2.
\end{equation}

Let $\theta(x) \in C^\infty(\Gamma)$ be a function satisfying the conditions
\begin{equation}
0 < C_1 \leq |\nabla \theta(x)| \forall x \in G; \quad \theta_{\partial_G} = 0, \quad \theta|_{\partial_G} = 1.
\end{equation}

Then the set
\begin{equation}
\kappa_i = \{x \in G; \theta(x) = i, t \in [0, 1]\}
\end{equation}
is a smooth manifold diffeomorphic to $\sigma_0$ and $\sigma_1$. We have:

**THEOREM 2.3.**\footnote{ESAIM: Control, Optim. and Calc. Var., vol. 3, 1997} There exist a constant $C_2 > 0$ and a function $\theta(x) \in C^\infty(\Gamma)$, satisfying condition (2.9), such that for any function $w \in W_2^2(G)$ for which (2.6)-(2.8) are satisfied
\begin{equation}
\int_{\partial G} |w| d\sigma \leq C_2 \|w\|_{L^2(G)} \|w\|_{L^2(\kappa_0)} \leq C_2 (2\varepsilon)^2 (1-\varepsilon)(2M)^2,
\end{equation}
where $\kappa_0$ is the manifold (2.10).

Obviously the domain $\Omega_i$ satisfies Condition 2.2 for all $i \in \{1, \ldots, K\}$. Denote by $\theta_i(x)$ the function from Theorem 2.3 which corresponds to the domain $\Omega_i$. Similar to (2.10) we set
\begin{equation}
k_i(\tilde{t}) = \{x \in \Omega_i; \theta_i(x) = \tilde{t}\}.
\end{equation}

For all $r \in (0, 1)$ we introduce the auxiliary domains
\begin{equation}
\Omega_r(x) = \{x \in \Omega_i; r < \theta_i(x) < 1\} \quad \Omega_r = \bigcup_{i=1}^{K} \Omega_r(x).
\end{equation}

Let $w_0 \in w$ be an arbitrary subdomain. We have:

**LEMMA 2.4.**\footnote{ESAIM: Control, Optim. and Calc. Var., vol. 3, 1997} There exists a function $\psi(x) \in C^\infty(\Omega)$ such that
\begin{equation}
\psi(x) = 1 - \theta(x) \forall x \in \Omega, \quad |\nabla \psi(x)| > 0 \forall x \in \Omega \setminus \partial \Omega.
\end{equation}

**Proof.** First we construct an auxiliary function $\beta(x) \in C^\infty(\Omega)$ such that
\begin{equation}
\beta(x) = 1 - \theta(x) \forall x \in \Omega_i, \quad \beta(x) > 0 \text{ in } \Omega.
\end{equation}
To do this, we consider the sequence of domains $U_i \subset \mathbb{R}^n$, $i = \{1, \ldots, K\}$ with the following properties

$$
\partial U_i = \Gamma_i \cup \gamma_i \in C^\infty, \quad U_i \subset U_i \cup \gamma_i \subset \widetilde{U_i}, \quad \overline{U_i} \cap \overline{U_j} = \{\emptyset\} \text{ for } i \neq j,
$$

where $\gamma_i$ is a connected $(n-1)$-dimensional $C^\infty$ manifold. (For example we can choose $U_i$ as $U_i = \mathbb{R}^n \sim \{(x \in \mathbb{R}^n) | -\varepsilon < |x| < 1\}$ for some $\varepsilon$.)

Since by assumption $\gamma_i$ is a $C^\infty$ surface, one can extend the function $\theta_i(x)$ to a smooth function of $C^\infty(\widetilde{U_i})$ such that $\theta_i = 0$ in some neighborhood of $\gamma_i$.

Set $\beta(x) = 1 - \theta_i(x)$ and $\beta = 0$ in $\Omega \setminus \cup_{i=1}^K U_i$.

Let $\rho(x) \in C^\infty(\Omega \setminus \cup_{i=1}^K U_i)$ be a nonnegative function such that

$$
\rho(x) > 0 \quad \forall x \in \{x \in \Omega \setminus \cup_{i=1}^K U_i | \beta(x) = 0\}.
$$

Then the function $\beta(x) = \beta + \varepsilon \rho(x)$ satisfies (2.15) for all $\varepsilon > 0$ sufficiently large.

Now let us show that the function $\beta(x)$ which satisfies (2.15) can be chosen as a Morse function. Since by (2.9), (2.15)

$$
[\nabla \beta(x)] > 0 \quad \forall x \in \cup_{i=1}^K U_i
$$

there exists a sequence of domains $\widetilde{U_i} \subset \mathbb{R}^n$ such that

$$
\partial \widetilde{U_i} = \Gamma_i \cup \gamma_i \in C^\infty, \quad U_i \subset \widetilde{U_i} \cup \gamma_i \subset \widetilde{U_i}, \quad \overline{U_i} \cap \overline{U_j} = \{\emptyset\} \text{ for } i \neq j,
$$

where $\gamma_i$ is a connected $(n-1)$-dimensional $C^\infty$ manifold, and

$$
[\nabla \beta(x)] > 0 \quad \forall x \in \cup_{i=1}^K \widetilde{U_i}.
$$

Let $\rho \in C^\infty(\Omega \setminus \cup_{i=1}^K \widetilde{U_i})$ such that

$$
\rho(x) = 1 \quad \forall x \in \Omega \setminus \cup_{i=1}^K \widetilde{U_i}, \quad \rho(x) = 0 \quad \forall x \in \cup_{i=1}^K \widetilde{U_i}.
$$

For every $\varepsilon > 0$ there exists a Morse function $\beta_\varepsilon$ such that $\|\beta - \beta_\varepsilon\|_{C^0} \leq \varepsilon$.

Set $\psi_\varepsilon(x) = (1 - \rho(x))\beta(x) + \rho(x)\beta_\varepsilon(x)$. Obviously

$$
\psi_\varepsilon(x) = 1 - \theta_i(x) \quad \forall x \in U_i,
$$

and for all sufficiently small $\varepsilon > 0$

$$
\psi_\varepsilon(x) > 0 \quad \text{in } \Omega.
$$

Let us show that for all small $\varepsilon$, $\psi_\varepsilon$ is a Morse function. Actually, in $\cup_{i=1}^K \widetilde{U_i}$ function $\psi_\varepsilon$ has no critical points. In $\Omega \setminus \cup_{i=1}^K \widetilde{U_i}$, $\psi_\varepsilon$ coincides with the Morse function $\beta_\varepsilon$. Short calculations give the inequality

$$
[\nabla \psi_\varepsilon(x)] = [\nabla \beta(x)] - \nabla \rho(x) (\beta - \beta_\varepsilon(x)) + \rho(x) [\nabla \beta - \nabla \beta_\varepsilon] (x) \geq C - 2\|\rho\|_{C^1} \|\beta - \beta_\varepsilon\|_{C^0} \geq C - 2\|\rho\|_{C^1} \|\beta - \beta_\varepsilon\|_{C^0},
$$

where \( C > 0, x \in \Omega \setminus \mathring{\Omega} \).

Since for all sufficiently small \( \varepsilon \), the right-hand side of this inequality is positive, the function \( \psi_{\varepsilon} \) has no critical points in \( \Omega \setminus \mathring{\Omega} \).

Denote by \( \mathcal{M} \) the set of critical points of the function \( \psi_{\varepsilon} \). Exactly in the same way as it was done in [2], [20], one can construct a diffeomorphism \( r : \Omega \to \Omega \) such that

\[
r(x) = x \quad \forall x \in \Omega \setminus \mathring{\Omega}, \quad r^{-1}(\mathcal{M}) \subset \omega_{0}.
\]

Thus, the function \( \psi(x) = \psi_{\varepsilon}(r(x)) \) satisfies all the conditions of our lemma.

We set

\[
\varphi(t, x) = e^{\lambda \varphi(t)} / (t(T - t))^2, \tag{2.36}
\]

\[
\alpha(t, x) = |x|^{4} e^{\lambda \varphi(t)} / (t(T - t))^2, \tag{2.17}
\]

\[
\mathcal{M}(t) = \alpha(t, x_0), \quad \mathcal{E}(t) = \varphi(t, x_0),
\]

where \( \lambda > 1 \), function \( \psi \) from Lemma 2.4 and \( x_0 \in \partial \Omega \). By (2.9), (2.14) the functions \( \mathcal{M}, \mathcal{E} \) are independent of the selection of \( x_0 \in \partial \Omega \). Note that (2.9), (2.14) imply the obvious inequality

\[
0 > \alpha(t, x) \geq \mathcal{M}(t); \quad \varphi(t, x) \geq \mathcal{E}(t) \quad \forall (t, x) \in Q.
\]

Let us introduce a function \( \ell(t) \in C = [0, T] \) by the formula

\[
\ell(t) > 0, \quad t \in (0, T); \quad \ell(t) = \begin{cases} 1, & t \in (0, 1/4) \\ T, & t \in (1/4, T). \end{cases} \tag{2.38}
\]

In this section, our aim is to get an estimate for the function \( p \) using the trace of \( p \) on \( \partial \Omega \) and the restriction of \( p \) on \( [0, T] \times \omega_{0} \). To prove this estimate, we need to recall some previous results on Carleman inequalities for the Laplace operator.

Let us consider the analog of problem (2.6)-(2.8) in the domain \( \Omega_{\varepsilon} \):

\[
\Delta w = 0 \quad \text{in} \quad \Omega_{\varepsilon}, \tag{2.29}
\]

\[
\frac{\partial w}{\partial \nu} = 0, \tag{2.30}
\]

Set

\[
A = A(\lambda) = \max_{x \in \Omega_{\varepsilon}} \frac{\psi^{(1 - \tau)}(x) - 1}{1 - \tau}, \tag{2.21}
\]

By (2.9), (2.14) there exists \( \lambda_0 > 1 \) such that

\[
1 + A(\lambda) \varphi(x) \geq e^{\lambda \varphi(x)} \quad \forall \lambda > \lambda_0, x \in \Omega_{1/4},
\]

\[
e^{\lambda} = \left[ e^{\lambda \varphi(x)} \right]_{1/4} > 1 + A(\lambda) \varphi(x) > e^{\lambda \varphi(x)} \quad \forall \lambda > \lambda_0, x \in \Omega_{1/4} \setminus \Omega_{3/4} \tag{2.22}
\]
We have:

**Lemma 2.5.** Let the function \( w \in W^2_0 (\Omega) \) be a solution of problem (2.19), (2.20). Then there exist \( \delta \in (0, 1) \) and \( \bar{\lambda} > 1 \) such that for \( \lambda > \bar{\lambda} \)

\[
\frac{\kappa}{(\Omega-\Omega)^2} \int_{\Omega-\Omega} |w|^{2} \frac{1}{|\nabla w|^{n+2}} \ dx \leq C \int_{\Omega-\Omega} |w|^{2} |\nabla w^{2} dz + \int_{\partial \Omega} |w^{2} | dz, \tag{2.23}
\]

where the domain \( \Omega (r) \) is defined in (2.13), \( r \in \{1, \ldots, K\} \).

**Proof.** Set

\[
(2M)^2 = \int_{\Omega} |w^{2} | dz, \quad (2e)^2 = \int_{\partial \Omega} |w^{2} | dz.
\]

Let \( \lambda > 1 \) and \( e_{\nu} (i) \) be the manifold defined by (2.12). Note that, by (2.14)

\[
\Lambda = \max_{x \in \Omega \Omega (i)} \frac{e^{\lambda (\nu-1)} - 1}{1 - \nu} = \max_{x \in \Omega \Omega (i)} \frac{e^{\lambda (\nu-1)} - 1}{1 - \nu} \leq \lambda e^{\Lambda} = \Lambda e^{\Lambda} = \Lambda (e^{\Lambda} z), \tag{2.24}
\]

Using inequality (2.11), we obtain

\[
\int_{\Omega \Omega (i)} |w|^{2} e^{2 \lambda k} \ dx \leq C_{e} \int_{\Omega \Omega (i)} \int_{\partial \Omega} |w|^{2} e^{2 \lambda k} |1 - \nu| d\nu dz \leq C_{e} \int_{0}^{1} |w|^{2} e^{2 \lambda k} |1 - \nu| \ d\nu dz \leq C_{e} \int_{0}^{1} (e^{\lambda k})^{2} |1 - \nu| M^{2} dz = I. \tag{2.25}
\]

Short calculations give the equality

\[
I = \frac{M^{2}}{2} \left[ \left( \frac{e^{\lambda k}}{M} \right)^{2} - 1 \right] \int_{0}^{1} \left( \frac{e^{\lambda k}}{M} \right) dz. \tag{2.26}
\]

Obviously, by (2.24) there exist \( \delta \in (0, 1) \) and \( \bar{\lambda} > 1 \) such that for all \( \lambda > \bar{\lambda} > 1 \)

\[
A + 1 \leq \lambda e^{\Lambda} = 1 + \delta \min_{x \in \Omega \Omega (i)} \frac{e^{\lambda (\nu-1)} - 1}{1 - \nu} = \delta e^{\Lambda} - 1. \tag{2.27}
\]

Let us consider two cases.

A) Let

\[
\frac{\epsilon^{(\nu+1)} e^{\lambda k}}{M} \leq 1. \tag{2.28}
\]

Thus \( \ln \left( \frac{\epsilon^{\lambda k}}{M} \right) \leq 0 \) and

\[
-\ln \left( \frac{\epsilon^{\lambda k}}{M} \right) = s - \ln \left( \frac{\epsilon^{(\nu+1)} e^{\lambda k}}{M} \right) \geq s. \tag{2.29}
\]
Inequalities (2.26), (2.28) and (2.29) imply

\[ I \leq \frac{M^2}{2s}. \]

By this inequality and (2.25), keeping in mind that \( e^{j \lambda} \big|_{\partial \Omega} = 1 \), we obtain

\[ s \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx \leq s \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx. \] (2.30)

By this inequality and (2.28), keeping in mind that \( e^{j \lambda} \big|_{\partial \Omega} = 1 \), we obtain

\[ s \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx \leq C_{20} \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx. \] (2.31)

Inequalities (2.30), (2.31) after the change \( s \rightarrow s/(T - t)^2 \) imply (2.23).

We have:

**Lemma 2.6.** Let \( p \in W_2^2(\Omega) \) be a harmonic function in \( \Omega \). Then there exists \( \lambda > 1 \) such that for \( \lambda > \lambda \) there exists \( s_0(\lambda) > 0 \) such that

\[ \frac{s}{(T - t)^2} \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx \]

\[ \leq C \left( \int_{\Omega} \frac{\partial}{\partial n} e^{j \lambda} + \frac{A}{2} \partial_n e^{j \lambda} dx \right) \] (2.32)

for some \( \delta \in (0, 1) \).

**Proof.** Let \( \delta \) be defined in Lemma 2.5 and \( \lambda \) be the maximum of the corresponding parameter from Lemma 2.5 and \( \lambda_0 \) from (2.22). We are looking for a function \( p \) of the form: \( p = z_1 + z_2 \),

\[ \Delta z_1 = 0 \quad \text{in} \quad \Omega, \quad z_1 \big|_{\partial \Omega} = 0, \quad \frac{\partial z_1}{\partial n} \big|_{\partial \Omega} = \frac{\partial p}{\partial n} \big|_{\partial \Omega}, \] (2.33)

and

\[ \Delta z_2 = 0 \quad \text{in} \quad \Omega, \quad z_2 \big|_{\partial \Omega} = p, \quad \frac{\partial z_2}{\partial n} \big|_{\partial \Omega} = 0, \] (2.34)
where \( \mathbf{n}(i) \) is a unit outward normal to \( \partial \Omega \). Note that

\[
\varphi|_{\partial \Omega} = \frac{e^2}{(T - t)^{2}} \quad \forall \, i \in \{1, \ldots, K\}.
\]

Also by (2.22) there exists \( \delta \in (0, 1), \delta > 0 \) such that

\[
1 + A \max_{x \in \Omega} \psi(x) < \delta \quad \frac{e^2}{(T - t)^{2}} = \delta \varphi|_{\partial \Omega}.
\]

From this inequality, (2.11) and (2.33) we obtain

\[
\int_{\gamma} |z_1|^2 e^{2z_i} e^2 dx + \beta \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx \leq C_{\infty} \int_{\gamma} \left( \frac{\partial \varphi}{\partial n} \right)^{2} e^{2z_i} e^2 dx.
\]

Then, by Lemma 2.5, the function \( z_2 \) satisfies the estimate

\[
\frac{\beta}{(T - t)^{2}} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx \leq C_{\infty} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx + \int_{\gamma} |z_2|^2 e^{2z_i} e^2 dx \leq C_{\infty} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx + \int_{\gamma} \left( |z| + \beta \right) e^{2z_i} e^2 dx.
\]

By (2.35), (2.36) we have

\[
\frac{\beta}{(T - t)^{2}} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx \leq C_{\infty} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx + \int_{\gamma} \left( |z| + \beta \right) e^{2z_i} e^2 dx \leq C_{\infty} \int_{\Omega} |z_2|^2 e^{2z_i} e^2 dx + \int_{\gamma} \left( |z| + \beta \right) e^{2z_i} e^2 dx.
\]

Inequalities (2.35), (2.37) imply (2.32).

Let us consider the Dirichlet boundary value problem for the Laplace operator

\[
\Delta z(x) = f(x) \quad x \in \Omega, \quad z|_{\partial \Omega} = 0.
\]

We have:

**Lemma 2.7.** Let \( f(x) \in L^2(\Omega) \). There exists \( \lambda > 0 \) such that for \( \lambda > \lambda \), there exists \( s_0(\lambda) > 0 \) such that for any \( s > s_0 \) the solution \( z(x) \in W^2_0(\Omega) \) of problem (2.38) satisfies the estimate

\[
\int_{\Omega} \left( \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + s \lambda^2 z + s \lambda^2 z^2 \right) e^{2z_i} e^2 dx \leq C_{\infty} \int_{\Omega} f^2 e^{2z_i} e^2 dx + \int_{\Omega} s \lambda^2 z^2 e^{2z_i} e^2 dx.
\]
where $C_i > 0$ does not depend on $s, t$.

The Carleman inequality (2.30) can be proved in the same way as the corresponding Carleman estimate for a parabolic equation in [20], [17]. Note that for the case $\partial \Omega \subset \mathbb{R}^n$, this estimate was proved in [19].

Now to continue estimating the pressure $p$, we have to use equations (2.4), (2.22). Applying the operator $\text{div}$ to both parts of equation (2.1) we obtain

$$\Delta p = \text{div}(B[p, y] + B^*(y, t)) \in \Omega$$

for a.e. $t \in [0, T]$.

The following lemma gives an estimate of the $L^2$-norm of the pressure $p$.

**Lemma 2.8.** Let $f \in L^2(0, T; V^1(\Omega))$ and $p(t, \cdot) \in L^2(\Omega)$ satisfies (2.40). There exists $\lambda > 1$ such that for all $\lambda > \lambda_0$, there exists $\sigma_0(\lambda)$ such that

$$\int_\Omega s^{1/2} \varphi^2 \varphi' e^{2\sigma_0} dx \leq C_m(\lambda) \left( \int_\Omega s^{1/2} \varphi^2 \varphi' e^{2\sigma_0} dx ight)$$

$$+ \int_\Omega s \varphi^2 \varphi' e^{2\sigma_0} dx + \int_\Omega \left( \frac{|\nabla p|^2}{s \lambda^2 \varphi^2} + s \lambda^2 \varphi^2 |\nabla f|^2 \right) e^{2\sigma_0} dx \quad \forall \sigma > \sigma_0(\lambda),$$

where the constant $C_m$ is independent of $s$ and $t$.

**Proof.** Let $\lambda$ be a maximum of the corresponding parameters from Lemma 2.6 and Lemma 2.7 and $\lambda_0$ from (2.22). Note that $\text{supp} \text{div}(B^*(v, y) + B^*(y, t)) \subset \text{supp} \psi$. By (2.5)

$$\text{supp} \psi \cap \Gamma_i = \emptyset$$

for all $i \in \{1, \ldots, K\}$.

So there exists a neighborhood of $\gamma_{ij}$, a domain $G_i$ such that $\psi$ is the harmonic function in $G_i$ and $\bigcup_{i=1}^K G_i \subset \Omega \setminus \bigcup_{i=1}^K \Omega(\gamma_{ij}) = \emptyset$. Note also that

$$\varphi_i = e^{\frac{1}{(1 + s)n}} \quad \forall i \in \{1, \ldots, K\}.$$

Thus, by the properties of interior regularity of solutions of elliptic equations, there exists a constant $C$ such that

$$\sum_{i=1}^K \int_{G_i} \nabla \psi \varphi_i e^{2\psi} dx \leq C \int_{\bigcup_{i=1}^K G_i} |\nabla \psi| e^{2\psi} dx,$$

where $\delta \in (0, 1)$ is defined in Lemma 2.6.

By (2.18), (2.14)

$$\lim_{s \to +\infty} \min_{x \in \Omega(\psi^{-\delta}(1)) \setminus \Omega(\psi^\delta)} \psi(x) = \lim_{s \to +\infty} \max_{x \in \Omega(\psi^{-\delta}(1)) \setminus \Omega(\psi^\delta)} \psi(x) = \frac{1}{4}.$$

Thus, by (2.22), there exists $\varepsilon_0 \in (0, 1)$ such that

$$1 + A \inf_{x \in \Omega(\psi^{-\delta}(1)) \setminus \Omega(\psi^\delta)} \psi(x) > \sup_{x \in \Omega(\psi^{-\delta}(1)) \setminus \Omega(\psi^\delta)} e^{\psi_0},$$

where $

\psi_0 

Again, using the property of interior regularity of harmonic functions, by (2.43) we have

\[
\int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} (|\nabla p|^2 + p^2) e^{2r \gamma} dx \\
\leq C \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} p^2 e^{2r \gamma} dx \quad \forall s > 1.
\]  

(2.44)

Let \(\rho(x)\) be a function such that

\[
\rho(x) = 1 \quad \forall x \in \Omega \setminus \mathcal{D}(\frac{3}{4} (1 - c_0)) \ ; \ \rho \geq 0 \text{ in } \Omega, \quad \rho|_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} = 0.
\]

Then by (2.40) the function \(z = \rho \psi\) satisfies the equation

\[
\Delta z = 2(\nabla \psi, \nabla \rho) + \rho \Delta \rho + \rho \| \text{div} (B'((\cdot), y) B'(y, \cdot)) \| \text{ in } \Omega, \quad z|_{\partial \Omega} = 0. \quad (2.45)
\]

Note that

\[
\text{supp}(2(\nabla \psi, \nabla \rho) + \rho \Delta \rho) \subset \mathcal{D}(\frac{3}{4} (1 - c_0)) \setminus \mathcal{D}(\frac{3}{4} (1 - c_0)).
\]

Short calculations give the inequality:

\[
\left| \text{div} (B'((\cdot), y) B'(y, \cdot)) (t, x) \right| \\
\leq C \left( \| \nabla \psi(t, x) \| + |\psi(t, x)| \| \nabla \psi(t, x) \| + |\psi(t, x)| \right) \quad \text{ in } Q. \quad (2.46)
\]

By (2.44), (2.46), (2.39) we obtain from (2.45)

\[
s^2 \int_{Q} \phi^2 p^2 e^{2r \gamma} dx \leq s^2 \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} \phi^2 p^2 e^{2r \gamma} dx \\
\leq C(\lambda) \left( \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} \frac{1}{s^2} |\nabla \psi|^2 + |\psi|^2 e^{2r \gamma} dx \\
+ \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} |\nabla \psi|^2 dx + \frac{1}{s^2} \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} p^2 e^{2r \gamma} dx \right) \quad \forall s > s_0(\lambda). \quad (2.47)
\]

On the other hand, by (2.32), (2.41) we have

\[
s^2 \int_{\mathcal{D}(1/2)} \phi^2 p^2 (e^{2r \gamma} + e^{\frac{1}{2} 4^{-1} m^2 s^4}) dx \\
\leq C \left( \int_{\mathcal{D}(1/2)} s \phi |\nabla \psi|^2 dx + \int_{\mathcal{D}(1/2) \cap \mathcal{D}(1/3)} p \psi |\nabla \psi|^2 dx \right). \quad (2.48)
\]

Since \(\mathcal{D}(1/2) \cap \mathcal{D}(1/3) \subset \Omega \setminus \mathcal{D}(\frac{3}{4} (1 - c_0))\) and \(\mathcal{D}(\frac{1}{2} - c_0) \setminus \mathcal{D}(\frac{1}{2}) \subset \mathcal{D}(\frac{1}{2})\) inequalities (2.47), (2.48) imply (2.41).

Now in right-hand side of (2.41), we have to estimate the integral on \(\Sigma\) containing pressure \(p\).
Lemma 2.9. Let $p \in L^3(Q)$ satisfies (2.1), (2.2), and $\lambda > \lambda$, $s > s_0(\lambda)$ where $\lambda, s_0(\lambda)$ are defined in Lemma 2.8. Then the following estimate holds

$$
\int_\Sigma s \lambda^2 p^2 e^{2\Delta s} d\Sigma
\leq C_3 \left( \int_Q s^2 \lambda^2 \phi^2 p^2 e^{2\Delta s} d\Sigma + \int_Q s^2 \lambda^2 \phi^2 |F e^{2\Delta s} d\Sigma
\right)
+ \int_Q \lambda \left( \frac{\partial y}{\partial t} + e^{2\Delta s} d\Sigma + \int_0^T \lambda (\phi e^{2\Delta s} d\Sigma)dt
\right),
$$

where the constant $C_3$ is independent of $s$.

Proof. Let us introduce a function $\gamma(t,x)$ by the formula

$$
\Delta \gamma(t, \cdot) = 0 \text{ in } \Omega, \quad \frac{\partial \gamma}{\partial t} |_{\partial \Omega} = p - a_0 \int_{\partial \Omega} pdx \text{ for a.e. } t \in (0,T),
$$

where $a_0 = (\int_{\Omega_1} 1 dx)^{-1}$. Note that solutions of (2.50) satisfy the estimate

$$
\|\gamma\|_{W_2^2(\Omega)} \leq C_{14}\|p\|_{L^2(\Omega_1)}.
$$

Taking the scalar product of (2.1) with $\nabla \gamma$ in $(L^2(\Omega))'$ and integrating by parts we have

$$
\int_\Omega p^2 dx = - \int_\Omega \frac{\partial \gamma}{\partial t} \nabla \gamma ds + a_0(\int_\Omega pdx)^2
\quad + \int_\Omega (B(\gamma, y) + B(\gamma, \hat{v}) - f, \nabla \gamma) dx.
$$

Multiplying (2.52) by $s \lambda |\nabla \gamma|^{2s}$ and integrating on the segment $(0,T)$ we obtain

$$
\int_\Sigma s \lambda^2 p^2 e^{2\Delta s} d\Sigma = - \int_\Sigma s \lambda^2 \phi \nabla \gamma e^{2\Delta s} d\Sigma
+ s \lambda \int_0^T \frac{\partial y}{\partial t} \nabla \gamma e^{2\Delta s} d\Sigma
+ s \lambda \int_\Sigma \frac{\partial y}{\partial t} \nabla \gamma e^{2\Delta s} d\Sigma
\leq C_{15} \left( \int_{\Omega} s \lambda^2 \|\nabla \gamma \|^{1+2s} d\Sigma
\right)
+ s \lambda \int_0^T \frac{\partial y}{\partial t} \nabla \gamma e^{2\Delta s} d\Sigma
+ s \lambda \int_0^T \nabla \gamma e^{2\Delta s} d\Sigma
\leq C_{15} \left( \int_{\Omega} s \lambda^2 \|\nabla \gamma \|^{1+2s} d\Sigma
\right)
+ s \lambda \int_0^T \frac{\partial y}{\partial t} \nabla \gamma e^{2\Delta s} d\Sigma
+ s \lambda \int_0^T \nabla \gamma e^{2\Delta s} d\Sigma
\right).
$$

After estimating of functions $B'(\hat{v}, \hat{y}), B'(y, \hat{v})$, we deduce from (2.51) and this inequality

\[
\int_{\Omega} s\lambda_{2}^{2}e^{2\gamma_2(t)}\,d\Sigma \leq C_{16} \left( \int_{\Omega} s\lambda_{2}^{2}\frac{\partial y}{\partial \nu}e^{2\gamma_2(t)}\,d\Sigma + s\right) \int_{\Omega} \frac{\partial y}{\partial \nu}e^{2\gamma_2(t)}\,dt + s\lambda \int_{Q} |\nabla y|^2 + |b|^2 + |y|^2 e^{2\gamma_2(t)}\,dt dt + \frac{\lambda}{2} \int_{\Omega} s\lambda_{2}^{2}e^{2\gamma_2(t)}\,d\Sigma.
\]

(2.53)

Let us introduce the function $\Delta\gamma(t, \cdot) = \alpha \chi_{\Omega}, \quad \frac{\partial \gamma}{\partial \nu} = 1,$

where $\alpha = \int_{\Omega_{0}} df/\int_{\Omega_{0}} dx.$ Multiplying (2.1) by $\nabla \gamma_1$ scalarly in $(L^2(\Omega))^n$ and integrating by parts for a.e. $t \in (0, T)$ we have

\[
\int_{\Omega} p(t, x)\,dt = -\int_{\Omega} \left( \frac{\partial y}{\partial \nu} \nabla \gamma_1 \right)\,dt + \int_{\Omega} \alpha p\,dt + \int_{\Omega} \left( B'(\hat{v}, \hat{y}) + B'(y, \hat{v}) - f, \nabla \gamma_1 \right)\,dx.
\]

Thus we obtain

\[
\int_{\Omega} p(t, x)\,dt \leq C_{17} \left( \int_{\Omega} \left| \frac{\partial y}{\partial \nu} \right|^2\,dx \right)^{\frac{1}{2}} + \int_{\Omega} \alpha p\,dt + \int_{\Omega} \left| \nabla y \right| + |b| + |y|\,dx.
\]

(2.54)

Taking the scalar product of equation (2.1) with $s\lambda_{2}^{2}e^{2\gamma_2(t)}$ in $(L^2(\Omega))^n$ and using Gronwall’s inequality we have

\[
\int_{Q} s\lambda_{2}^{2}\left| \nabla y \right|^2 e^{2\gamma_2(t)}\,dx dt \leq C_{18} \left( \int_{Q} s\lambda_{2}^{2}\left| \nabla y \right|^2 e^{2\gamma_2(t)}\,dx dt + \int_{Q} s\lambda_{2}^{2}\left| \nabla e^{2\gamma_2(t)}\,dx dt \right)
\]

\[
\leq C_{18} \left( \int_{Q} s\lambda_{2}^{2}\left| \nabla e^{2\gamma_2(t)}\,dx dt + \int_{Q} s\lambda_{2}^{2}\left| \nabla e^{2\gamma_2(t)}\,dx dt \right)
\]

(2.55)

By (2.54) - (2.55) we obtain (2.49).

The following lemma plays a decisive role in the estimation of the pressure.

**Lemma 2.10.** Let $y \in L^2(0, T; V^4(\Omega))$ and $p \in L^2(Q)$ satisfy (2.1), (2.2), $f \in (L^2(Q))^n$, $\lambda > \lambda, s > s_0(\lambda)$ where $\lambda, s_0(\lambda)$ are defined in Lemma 2.8.

Then the following estimate holds

\[
\int_{\Sigma} \frac{1}{(T - t)^{\lambda}} \left| \frac{\partial y}{\partial \nu} \right|^2 e^{2\gamma_2(t)}\,d\Sigma \leq C_{18} \int_{Q} \frac{\partial y}{\partial \nu} e^{2\gamma_2(t)}\,dx dt + \int_{Q} \frac{1}{(T - t)^{\lambda}} e^{2\gamma_2(t)}\,dx dt
\]

(2.56)
where the constant $C_{39}$ is independent of $s$.

Proof. Set $u(t, x) = ye^{st \cdot \phi}_t |(T - t)\beta_1$, $q(t, x) = pe^{st \cdot \phi}_t |(T - t)\beta_2$, $m = \{ f = B^2(y, \phi) - B^2(y, \psi) e^{st \cdot \phi}_t |(T - t)\beta_1 \}$. Then the pair $(u, q)$ satisfies the equations

$$
\frac{\partial u}{\partial t} - \Delta u + l(t) u = \nabla q + m \text{ in } Q,
$$

$$
\text{div } u = 0, \quad u \big|_{\Gamma_2} = 0, \quad u(0, x) = u(T, x) = 0,
$$

where $l(t) = -\frac{s - m}{\alpha} + \frac{T - t}{T}$. Obviously the estimate holds

$$
\|l(t)\| \leq C_{39}(\lambda) \frac{s}{(T - t)\beta_1}, \quad \frac{d\|l(t)\|}{dt} \leq C_{39}(\lambda) \frac{s}{(T - t)\beta_1}.
$$

Taking the scalar product of equation (2.57) with $\frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u$ in $(L^2(Q))^n$ and integrating by parts with respect to variables $x$ and $t$ we have:

$$
\int_Q (\mu, \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u) \, dx\, dt
= \int_Q (\frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u)^2 - (\Delta u, \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u)
+ l(t) u - (\nabla \psi, \nabla) u, \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u - (\nabla q, \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u) \, dx\, dt
= \int_Q (\frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u)^2 + (\nabla u, \nabla (\nabla \psi, \nabla) u) + q \text{div} ((\nabla \psi, \nabla) u)
+ l(t) u - (\nabla \psi, \nabla) u, \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u) \, dx\, dt
- \int_Q \frac{\partial u}{\partial t}, (\nabla \psi, \nabla) u) \, dx\, dt
- \int_{\Sigma} q(\psi, (\nabla \psi, \nabla) u) \, d\Sigma. \tag{2.59}
$$

Since $u \big|_{\Gamma_2} = 0$, we can rewrite equation $\text{div } u = 0$ on the boundary as follows: $(\mu, \frac{\partial u}{\partial n}) = 0$. Thus

$$
\int_{\Sigma} q(\psi, (\nabla \psi, \nabla) u) \, d\Sigma = -\int_{\Sigma} q(\nabla \psi) \big( \psi, \frac{\partial u}{\partial n} \big) \, d\Sigma = 0. \tag{2.60}
$$

Also, by (2.9) and Lemma 2.4 $x_2 (x) = -\nabla (\psi, \psi) |(\nabla \psi, \psi)|$, so

$$
- \int_{\Sigma} \frac{\partial u}{\partial n}, (\nabla \psi, \nabla) u) \, d\Sigma
= \int_{\Sigma} \nabla |(\nabla \psi)|^2 \frac{\partial u}{\partial n} \, d\Sigma. \tag{2.61}
$$
Taking into account (2.60), (2.61), we can deduce from (2.59)

\[
\int_Q \left( \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u \right) \, dx \, dt
\]

\[
= \int_Q \left( \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u \right) \, dx \, dt + \int_{\Sigma} \left[ \nabla u \, \frac{\partial u}{\partial n} + \nabla \psi - \frac{\partial}{\partial t} \right] \, d\Sigma
\]

\[
+ \int_{\Sigma} \nabla \psi \cdot \nabla u \, d\Sigma
\]

\[
= \int_Q \left( \frac{\partial u}{\partial t} + (\nabla \psi, \nabla) u \right) \, dx \, dt + \int_{\Sigma} \left[ \nabla u \, \frac{\partial u}{\partial n} + \nabla \psi - \frac{\partial}{\partial t} \right] \, d\Sigma
\]

From this equality, taking into account that, by (2.14), \( \min_{x \in \Omega} |\nabla \psi(x)| > 0 \), we obtain the estimate:

\[
\int_Q \left| \frac{\partial u}{\partial t} \right|^2 \, dx \leq C_{22} \int_Q q^2 \, dx + \int_{\Sigma} \left( \left| \nabla u \right|^2 + \left( \left| \frac{\partial u}{\partial n} \right| + 1 \right) |f|^2 + n^2 \right) \, d\Sigma.
\]

Multiplying equation (2.57) by \( u \) scalarly in \( (L^2(\Omega))^n \), by Gronwall's inequality, we have

\[
\int_Q |\nabla u|^2 \, dx \leq C_{21} \left( \int_Q \left| \frac{\partial u}{\partial t} \right|^2 \, dx + \int_Q \frac{|f|^2}{(\tau - \tau')^2} \, dx \right),
\]

By (2.62), (2.63) and Lemma 2.4 we deduce

\[
\int_{\Sigma} \left[ \nabla u \, \frac{\partial u}{\partial n} + \nabla \psi - \frac{\partial}{\partial t} \right] \, d\Sigma \leq 0
\]

After returning in (2.64) to the variables \( y, p, f \) we get (2.56). \( \Box \)

Now we can prove main theorem of this section:

**Theorem 2.11.** Let \( y \in L^2(0, T; V^{-1}(\Omega)), p \in L^2(\Omega) \) satisfies (2.1), (2.2), \( f \in L^2(0, T; V^2(\Omega)), \lambda \geq \rho, s > m(\lambda) \) where \( \lambda, m(\lambda) \) are defined in Lemma

**Proof.**...
2.8. Then the following estimate holds

\[ \int_Q s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt \]

\[ \leq C_{25} \left( \int_{Q_{\omega}} s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt + \int_Q s^2 \lambda^2 \varphi^2 |\nabla y|^2 e^{2+\alpha} \, dxdt \right. \]

\[ + \int_Q \frac{\nabla y^2}{s \lambda^2 \varphi} e^{2+\alpha} \, dxdt + \int_Q s \lambda \varphi |\nabla y|^2 e^{2+\alpha} \, dxdt \bigg), \]

where the constant \( C_{25} \) is independent of \( s \).

Proof. Inequalities (2.41), (2.49) imply that

\[ \int_Q s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt \]

\[ \leq C_{21}(\lambda) \left( \int_{Q_{\omega}} s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt + \int_Q s \lambda \varphi \left[ \frac{\partial \varphi}{\partial t} + \frac{\partial y}{\partial x} \right] e^{2+\alpha} \, dxdt \right. \]

\[ + \int_Q \left( \frac{\nabla y^2}{s \lambda^2 \varphi} + s^2 \lambda^2 \varphi^2 |\nabla y|^2 \right) e^{2+\alpha} \, dxdt + \int_Q s \lambda \varphi |\nabla y|^2 e^{2+\alpha} \, dxdt \bigg). \]

Note that

\[ \varphi(t, x) \bigg|_{Q_{\omega}} = ((T-t)t)^{-\alpha}. \]

Then, estimating the normal derivative of the function \( y \) by (2.56), we have

\[ \int_Q s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt \]

\[ \leq C_{21}(\lambda) \left( \int_{Q_{\omega}} s^2 \lambda^2 \varphi^2 p^2 e^{2+\alpha} \, dxdt + \int_Q \left( \frac{\nabla y^2}{s \lambda^2 \varphi} + s^2 \lambda^2 \varphi^2 |\nabla y|^2 \right) e^{2+\alpha} \, dxdt \right. \]

\[ \left. + \int_Q s \lambda \varphi |\nabla y|^2 e^{2+\alpha} \, dxdt \bigg). \]

Hence, increasing the parameter \( s \) if necessary in (2.66), we obtain (2.65).

Remark 2.12. In system (2.1), (2.2), the pressure \( p \) is defined up to an arbitrary constant. In the next section we will fix it by setting

\[ p(t, x_0) = 0 \quad \forall t \in (0, T) \]

for some \( x_0 \in \omega_0 \).

3. Carleman estimate for the Stokes system

In this section, our aim is to solve the observability problem for system (2.1), (2.2). In other words, we would like to obtain an a priori estimate for solutions of (2.1), (2.2) via function \( f \) and restriction of \( y \) on \( Q_{\omega_0} \).
Let $\omega_1$ be an arbitrary subdomain of $\Omega$ such that

$$\omega_0 \subset \omega_1 \subset \omega.$$  

We start from the following theorem.

**Theorem 3.1.** Let the pair $(y, p)$ satisfy $(2.1), (2.2), f \in (L^2(Q))^n$. Then there exists a $\lambda > 1$ such that for any $\lambda > \lambda_0$ one can find $s_\lambda(\lambda)$ such that the following inequality holds

$$\int_Q \left( \frac{1}{2} \left( \frac{\partial^2 y}{\partial t^2} \right)^2 + \sum_{i,j=1}^n \left( \frac{\partial^2 y}{\partial x_i \partial x_j} \right)^2 + s \lambda \lambda^2 \phi \psi \right) e^{\lambda \phi \psi} \, dx \, dt$$

$$\leq C_1 \left( \int_Q |f|^n \, dx \, dt + \int_Q s \lambda \lambda^2 \phi \psi e^{\lambda \phi \psi} \, dx \, dt ight) \quad \forall \lambda \geq \lambda_0,$$  

(3.1)

where the constant $C_1$ is independent of $s$.

**Proof.** Let us denote $w(t, x) = y(t, x)e^{\lambda \phi \psi}, q(t, x) = p(t, x)e^{\lambda \phi \psi}$.

By [2.14], [2.17] we have

$$w(0, t) = 0 \quad \text{in} \quad \Omega; \quad w|_{\Omega} = 0.$$  

(3.2)

We define the operator $P$ by formula

$$Pw = e^{\lambda \phi \psi}(\partial_t - \Delta)e^{\lambda \phi \psi} w.$$  

(3.3)

The operator $P$ can be written explicitly as follows:

$$Pw = \frac{\partial w}{\partial t} - \Delta w + 2 s \lambda \phi \psi \nabla \phi \nabla w + s \lambda \lambda^2 \phi \psi \nabla^2 w$$

$$- s \lambda \lambda^2 \phi \psi \nabla^2 \phi \psi \nabla w + s \lambda \lambda^2 \phi \psi \nabla w.$$  

Let us introduce the operators $L_1, L_2$ as follows:

$$L_1(w, q) = -\Delta w - s \lambda \lambda^2 \phi \psi \nabla^2 w - \nabla q + s \lambda \phi \psi \nabla w,$$

(3.4)

$$L_2w = \frac{\partial w}{\partial t} + 2 s \lambda \phi \psi \nabla \phi \nabla w + 2 s \lambda \lambda^2 \phi \psi \nabla^2 w.$$  

(3.5)

It follows from $(2.1), (2.2), (3.4), (3.5)$ that

$$L_1(w, q) + L_2 w = f_1 \text{ in } Q,$$  

(3.6)

$$\text{div } w = s \lambda \lambda^2 \phi \psi \nabla \phi \psi w,$$  

(3.7)

where

$$f_1 = (f - B^* (y, \psi) - B^* (\phi, \psi)) e^{\lambda \phi \psi} + s \lambda \lambda^2 \phi \psi \nabla^2 \phi \psi w - s \lambda \lambda^2 \phi \psi \nabla w.$$  

Taking the $L_p$-norm of both sides of (3.6), we obtain
\[
\|f\|_{L_p(Q)} = \|f_1\|_{L_p(Q)} + \|f_2\|_{L_p(Q)} + 2\|f_3\|_{L_p(Q)}.
\]
By (3.4) and (3.5) we have the following equality:
\[
\begin{align*}
\mathcal{L}_{(w', q)}(\mathcal{L}_{(w', q)}) &= (L_1(w, q), 2\lambda \partial_q \nabla \psi, \nabla w)_{L_p(Q)}^p \\
+ \langle L_1(w, q), \frac{\partial w}{\partial t} \rangle_{L_p(Q)} + 2\lambda^2 \partial_q \nabla (\psi | w)_{L_p(Q)}^p &= A_0 + A_1.
\end{align*}
\]
Integrating by parts in the first term of right-hand side of (3.9) we obtain
\[
\begin{align*}
A_0 &= (-\Delta w - s^2 \lambda^2 \nabla \psi, \nabla w)_{L_p(Q)} + 2\lambda \partial_q \nabla \psi, \nabla w)_{L_p(Q)} \\
&= \iint_D \left\{ 2\lambda^2 \nabla \psi, \nabla (\psi, w) \\
+ 2\lambda \partial_q \nabla \psi, \nabla (\psi, w) + 2s^2 \lambda^2 \partial_q \nabla (\psi, w)_{L_p(Q)}^p = A_0 + A_1.
\end{align*}
\]
Now let us transform some terms in (3.10):
\[
\begin{align*}
2\lambda q \partial_q \nabla (\psi, w) &= 2\lambda^2 \psi, (\psi, w) + 2\lambda \psi q \partial_q (\psi, w) \\
&= 2\lambda^2 \psi, (\psi, w) + 2\lambda \psi q \partial_q (\psi, w) \\
&= \sum_{i,j} \left( \psi \frac{\partial^2 \psi}{\partial x_i \partial x_j} \frac{\partial \psi}{\partial x_j} + 2\lambda \psi q \frac{\partial (\psi, w)}{\partial x_j} \right) \\
&= 2\lambda^2 \psi, (\psi, w) + 2\lambda \psi q \partial_q (\psi, w)
\end{align*}
\]

\[
\begin{align*}
2\lambda \psi q \partial_q (\psi, w) &= 2\lambda^2 \psi, (\psi, w) + 2\lambda \psi q \partial_q (\psi, w) \\
&= \sum_{i,j} \left( \psi \frac{\partial^2 \psi}{\partial x_i \partial x_j} \frac{\partial \psi}{\partial x_j} + 2\lambda \psi q \frac{\partial (\psi, w)}{\partial x_j} \right) \\
&= 2\lambda^2 \psi, (\psi, w) + 2\lambda \psi q \partial_q (\psi, w)
\end{align*}
\]
Similar to (2.61)

$$-\int_\Omega 2\alpha \partial w \frac{\partial w}{\partial \nu} (\nabla \psi, \nabla w) d\Omega = \int_\Omega 2\alpha \lambda \partial \psi \frac{\partial w}{\partial \nu} d\Omega. \quad (3.12)$$

By virtue of (3.2), (3.7) we have

$$\frac{\partial w}{\partial \nu} \bigg|_{\partial \Omega} = 0.$$

Hearing this equality in mind, we deduce that the last term in (3.10) equals zero:

$$\int_\Omega 2\alpha \partial \psi (\nabla \psi, \nabla w) d\Omega = -\int_\Omega 2\alpha \lambda \partial \psi \frac{\partial w}{\partial \nu} d\Omega = 0. \quad (3.13)$$

Integrating by parts in equation (3.10) and taking into account equations (3.11) - (3.13) we get

$$A_\alpha = \int_\Omega \left\{2\alpha \lambda \phi \frac{|\nabla \psi|^2}{\partial w} + 2\alpha \lambda \partial \psi \sum_{i=1}^{n} \left( \nabla w \cdot \sum_{j=1}^{n} \frac{\partial w}{\partial x_j} \frac{\partial \psi}{\partial x_j} \right) \right\} + s \lambda \phi \frac{|\nabla \psi|^2}{\partial w} - s \lambda \partial \psi \frac{|\nabla \psi|^2}{\partial w} + 3s \lambda \phi \frac{|\nabla \psi|^2}{\partial w}$$

$$+ \partial \psi \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} \frac{\partial w}{\partial x_i} \nabla \psi \cdot \nabla w + 3s \lambda \phi \frac{|\nabla \psi|^2}{\partial w} + 3s \lambda \phi \frac{|\nabla \psi|^2}{\partial w} + s \lambda \phi \frac{|\nabla \psi|^2}{\partial w} + s \lambda \phi \frac{|\nabla \psi|^2}{\partial w}$$

$$+ 2s \lambda \phi \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} \frac{\partial w}{\partial x_i} \nabla \psi \cdot \nabla w + 2s \lambda \phi \frac{|\nabla \psi|^2}{\partial w}$$

$$+ 2s \lambda \phi \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} \frac{\partial w}{\partial x_i} \frac{\partial \psi}{\partial x_i} \frac{\partial w}{\partial x_i} + 2s \lambda \phi \frac{|\nabla \psi|^2}{\partial w}$$

Let us now transform the second term in the right hand side of (3.9) keeping (3.7) in mind.
\[ A_3 = (L_1(w, q), \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla \psi|)_{L^2(Q)} \]

\[ = \int_Q \left( \frac{1}{2} \frac{\partial}{\partial t} |w|^2 - \frac{1}{2} \frac{\partial}{\partial t} (\lambda^2 \varphi^2 |\nabla \psi|^2) \right) \, dx \, dt + \int_Q \left( q \text{div} \frac{\partial w}{\partial t} + s\varphi \lambda q |\nabla \psi| \frac{\partial w}{\partial t} \right) \, dx \, dt \]

\[ + \int_Q \left\{ 2s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 + 2s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 - 2s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 |w|^2 \right\} \, dx \, dt \]

\[ + 2s\lambda^2 \sum_{i=1}^n \left( \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \]

Hence, by virtue of (3.8), (3.9), (3.14) and (3.15), we finally obtain

(3.15)

\[ \|f\|_{L^2(Q)} \leq \|[L_1, \varphi] \|_{L^2(Q)} + \|[L_2, w] \|_{L^2(Q)} + 2 \int_Q (2s^2 \lambda^2 \varphi |\nabla \psi|)^2 \, dx \, dt \]

\[ + s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 + s^2 \lambda^2 \varphi^2 |\nabla \psi|^2 |w|^2 \]

\[ + 2s^2 \lambda^2 \varphi |\nabla \psi|^2 \sum_{i=1}^n \left( \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \]

(3.16)

\[ = s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 + s^2 \lambda^2 \varphi^2 \sum_{i=1}^n \left( \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \]

\[ + 2s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 \]

\[ + 2s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 \]

where

\[ X_1 = \int_Q s^2 \lambda^2 \varphi |\nabla \psi|^2 |w|^2 \, dx \, dt, \]
and
\[ X_2 = \int_Q \left( q 2s^2 \lambda \phi |\nabla \psi| J_2 w + q s \lambda \phi (\nabla \psi, w) + 2s^2 \lambda^2 s^2 q |\nabla \phi| |\nabla \psi| w \right) + 2s^2 \lambda^2 q \sum_{i=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial^2 \psi}{\partial x_j} \frac{\partial w}{\partial x_i} \right) dx \right) dt. \]

where $C$ is some independent constant. Therefore
\[ |X_2| \leq C_4 \int_Q \left( s^3 \lambda^2 s^2 |\nabla \phi| |\nabla \phi| |\nabla \psi| |\nabla \psi| + s^3 \lambda^2 s^2 |\nabla \phi| |\nabla \psi| |\nabla \psi| \right) dx dt. \]

To estimate $X_3$, we observe that by definition (2.16) of the function $\varphi$, $|X_3| \leq C_2 |\varphi| \forall (t, x) \in Q$, where $C_2$ is some independent constant. Therefore
\[ |X_3| \leq C_2 \int_Q \left( s^3 \lambda^2 s^2 |\nabla \phi| |\nabla \phi| |\nabla \psi| |\nabla \psi| \right) dx dt. \]

We have that by Lemma 2.4
\[ |\nabla \psi(x)| > \beta \quad \forall x \in \bar{\Omega}\backslash \omega. \]

Thus, there exists $\lambda > 1$ independent on $w$ such that for any $\lambda > \lambda$ and $s > 1$
\[ \int_{\partial \Omega \times (0, T)} \left( 2s^2 \lambda^2 s^2 |\nabla \psi| |\nabla \psi| |\nabla \psi| |\nabla \psi| \right) + \frac{1}{4} s^2 \lambda^2 s^2 |\nabla \phi| |\nabla \psi| |\nabla \psi| \right) \right) dx dt. \]

Combining (3.21) and (3.16), we deduce
\[ \|J_2 w\|_{L^p(Q)} \|
abla \psi\|_{L^p(Q)} + \|J_2 w\|_{L^p(Q)} \|
abla \psi\|_{L^p(Q)} \]
\[ + \int_Q \left[ (s^2 \lambda^2 s^2 \|\psi\|_{L^p(Q)} + s^2 \lambda^2 s^2 |\nabla \psi| |\nabla \psi| \right) dx dt + 2 \int_Q \left( s^2 \lambda^2 \|\psi\|_{L^p(Q)} + s^2 \lambda^2 s^2 |\nabla \psi| \right) \right) dx \]
\[ \leq 2|X_1| + 2|X_2| + \|J_2 w\|_{L^p(Q)} + C_2 \int_Q \left( (s^2 \lambda^2 s^2 |\nabla \psi| + s^2 \lambda^2 s^2 |\nabla \psi| \right) dx dt, \]

\[ \text{ESAIM: COPR, May 2008, vol. 3, w*<30.} \]
for all \( \lambda > \lambda_0, s > 1 \). Let now \( \lambda \geq \lambda_0 \) be fixed. By (3.19), (3.20), (3.22) there exists \( s_1(\lambda) \) such that

\[
\|f_i\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \frac{1}{2}\|f_2\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \frac{1}{2} \int_\Omega \langle s \lambda^2 \phi^2 \rangle \|\nabla v\|^2 \, dx \leq \|f_1\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \int_\Omega \langle s \lambda^2 \phi^2 \rangle \|\nabla v\|^2 \, dx + \int_\Omega s^2 \lambda^2 \phi^2 \phi^2 \, dx dt
\]

(3.23)

for all \( s \geq s_1(\lambda) \).

Let \( \rho \in C_0^\infty(\omega_0), \rho = 1 \) \( \forall x \in \omega_0 \). Taking the scalar product of (3.6) with \( s\lambda^2 \phi \rho \) in \( (L^2(\Omega))^n \) and integrating by parts we have

\[
\int_{Q_\omega} s\lambda^2 \phi \rho \nabla \cdot \nabla \, \rho \, dx dt \leq C_1 \left( \int_{Q_\omega} s\lambda^2 \phi^2 \phi \|\nabla v\|^2 \, dx dt + \|f_1\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \int_\Omega s^2 \lambda^2 \phi^2 \phi^2 \, dx dt \right).
\]

(3.24)

The inequalities (3.23), (3.24) imply the estimate

\[
\|f_i\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \frac{1}{2}\|f_2\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \frac{1}{2} \int_\Omega \langle s \lambda^2 \phi^2 \rangle \|\nabla v\|^2 \, dx + 2 \int_\Omega \lambda \phi^3 \nabla \cdot \phi \, dx dt \leq \|f_1\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} + \int_\Omega s\lambda^2 \phi \rho \nabla \cdot \nabla \phi \, dx dt + \int_\Omega s^2 \lambda^2 \phi \phi \phi \phi \, dx dt
\]

(3.25)

for all \( s \geq s_2(\lambda) \).

We observe that

\[
|u_i(t,x)| \leq C_4(\lambda)|x(t,x)|^2 \quad \forall (t,x) \in Q.
\]

Then the definition of the function \( f_i \) implies

\[
\|f_i\|_{\mathcal{L}(\mathbb{R};L^2(\Omega))} \leq C_4(\lambda) \left( \int_Q |\nabla u|^2 + |\nabla v|^2 + |\phi|^2 \|u\|^2 + \int_\Omega \langle s \lambda^2 \phi^2 \rangle \|\nabla v\|^2 \, dx dt \right)
\]

(3.26)

Then, estimating the term with \( f_i \) in (3.25) by (3.26), we obtain (3.1). □

Combining the statements of Theorem 2.11 and Theorem 3.1 we obtain.
Corollary 3.2. Let the pair $(y, p) \in V^{1/2}(Q) \times L^2(0, T; W^2_2(\Omega))$ satisfy (2.1), (2.2), $f \in L^2(0, T; V^1(\Omega))$. Then there exists a $\lambda > 1$ such that for any $\lambda > \lambda_1$ there exists $s_0(\lambda)$ such that the following inequality holds:

$$\int_Q \left( \frac{1}{\sigma^2} \left( \frac{\partial y}{\partial t} \right)^2 + \sum_{j=1}^n \left( \frac{\partial y}{\partial x_j} \right)^2 \right) + \alpha s^2 \| \nabla y \|^2 + s^2 \lambda^2 \| f \|^2 e^{-\sigma} \, dx \, dt$$

$$\leq C_{in} \left( \int_Q \left( s^2 \lambda^2 \| f \|^2 e^{-\sigma} \right) \, dx \, dt + \int_Q s \lambda \| f \|^2 e^{-\sigma} \, dx \, dt \right) \forall s \geq s_0(\lambda),$$

(3.27)

where the constant $C_{in}$ is independent of $s$.

Proof. Since all assumptions of Theorem 3.1 are fulfilled, there exists $\lambda_1$ such that for every $\lambda > \lambda_1$, there exists $s_0(\lambda)$ such that for $s > s_0(\lambda), \lambda > \lambda_1$, inequality (3.1) holds. For $\lambda$ sufficiently large, we can estimate the last term in (3.1) by the right part of inequality (2.65). Then, increasing magnitude of the parameter $\lambda$, we can get (3.27).

Remark 3.3. In particular, estimate (3.27) implies the $s$-controllability for the system formally adjoint to (2.1), (2.2), as well as the unique continuation property for the Navier-Stokes equations. Such results were obtained for the Stokes system in [18] and for the Navier-Stokes equations in [8] under weak regularity assumptions of the function $u$.

Let us consider the system of partial differential equations which is obtained from (2.1), (2.2) by the change of variables $t \to -t$:

$$L^* z = \frac{\partial z}{\partial t} - \Delta z + B^*(z, \cdot) + B^*(\cdot, z) = \nabla q + f \text{ in } Q, \quad (3.28)$$

$$\text{div } z = 0, \quad z|_{\partial \Omega} = 0, \quad z(T, \cdot) = \tilde{z}_0, \quad (3.29)$$

where the operators $B^*(\cdot, \cdot), B^*(\cdot, \cdot)$ are defined in (2.3). Short calculations show that $L^*$ is formally adjoint to the operator which is the linearization of the Navier-Stokes equations at point $\tilde{z}$.

Using the energy method, we can prove

Lemma 3.4. Let $\tilde{z}_0 \in V^1(\Omega)$. Then the solutions of problem (3.28), (3.29) satisfy the estimate:

$$\left\| \frac{\partial z}{\partial t} \right\|_{L^p(0, T; H^1(\Omega))} + \left\| z \right\|_{L^p(0, T; H^2(\Omega))} \leq C_1 \left( \left\| z|_{\partial \Omega, T} \right\|_{L^p(\partial \Omega, T)} + \left\| f \right\|_{L^p(0, T; L^2(\Omega))} \right).$$

(3.30)

Let us consider the boundary value problem for the stationary Stokes system:

$$\Delta v = \nabla q + g \text{ in } \Omega, \quad \text{div } v = 0, \quad v|_{\partial \Omega} = 0.$$

(3.31)

The following lemma is proved in [27].
Lemma 3.5. For any $g \in V^{-1}(\Omega)$ there exists a unique solution $v \in V^3(\Omega)$ of problem (3.31) and this solution satisfies the estimate

$$
\|v\|_{V^3(\Omega)} \leq C_1 \|g\|_{V^{-1}(\Omega)}.
$$

(3.32)

Let us introduce the function $\kappa$ by formula

$$
\kappa(t, x) = \left( e^{\beta t} - e^{\beta T}\right) \left( \|f(t, x)\|_{\ell(Y)} \right)^2,
$$

$$
\hat{\kappa}(t) = \min_{x \in \Omega} \kappa(t, x), \quad \hat{\kappa}(t) = \max_{x \in \Omega} \kappa(t, x),
$$

(3.33)

where the function $f(t)$ is defined in (2.18). The parameter $\lambda$ is such that

$$
\tilde{\lambda} > \lambda,
$$

where $\lambda$ is defined in Corollary 3.2 and

$$
\max_{x \in \Omega} \kappa(t, x) < \frac{9}{10} \min_{x \in \Omega} \kappa(t, x) \quad \forall t \in [0, T],
$$

(3.34)

Note that

$$
\kappa(t, x) = \alpha \kappa(t, x) \quad \forall (t, x) \in \left[ \frac{3}{4} T, T \right] \times \Omega.
$$

We have:

Theorem 3.6. Let the pair $(z, \psi) \in V^3(\Omega) \times L^2(0, T; W^2(\Omega))$ satisfy (3.28), (3.29), $f \in L^2(0, T; V^3(\Omega))$. Then there exists $\varepsilon > 1$ such that the following inequality holds

$$
\|z(t, \cdot)\|_{L^2(\Omega)} + \int_0^T \|f - \psi\|_{L^2(\Omega)} \, dx \, dt
$$

$$
\leq C_{34} \left( \int_0^T \|f\|_{L^2(\Omega)} \, dx \, dt + \int_0^T \|f\|_{L^2(\Omega)} \, dx \, dt \right).
$$

(3.35)

Proof. Let us introduce the functions $r, g, \tilde{f}$ by formulas

$$
r(t, x) = \int_0^t z(t, x) \, dx, \quad g(t, x) = \int_0^t q(t, x) \, dx, \quad \tilde{f}(t, x) = \int_0^t f(t, x) \, dx.
$$

Short calculations show that the pair $(r, g)$ satisfies the equations

$$
L^* r = \nabla g - z(0, \cdot) + \tilde{f} \text{ in } Q,
$$

$$
\text{div } r = 0, \quad r|_{\partial Q} = 0.
$$

(3.36)

(3.37)

Let us show that the function $g$ satisfies the estimate

$$
\|g(t, \cdot)\|_{L^2(\Omega)} \leq C_{33} \left( \|z(0, \cdot)\|_{L^2(\Omega)} + \|z(t, \cdot)\|_{L^2(\Omega)} + \|f(t, \cdot)\|_{L^2(\Omega)} + \|f(t, \cdot)\|_{L^2(\Omega)} \right),
$$

(3.38)

where $C_{13}$ is independent of $t$. Using the definition of the function $r$ we can rewrite equation (3.36) as follows

$$-\Delta r = \nabla g - z(0, r) + z(t, r) - B'(r, v) - B''(v, r) + f \quad \text{in } \Omega. \quad (3.39)$$

Note that the function $g$ in (3.39) is defined up to an arbitrary constant. To fix it, we set

$$p(t, x_0) = 0 \quad \forall t \in [0, T]$$

for some $x_0 \in \omega_0$. This equality implies

$$g(t, x_0) = 0 \quad \forall t \in [0, T]. \quad (3.40)$$

We are looking for functions $r, g$ in the form $r = r_1 + r_2, g = g_1 + g_2$, where

$$-\Delta r_1 = \nabla g_1 - z(0, r) + z(t, r) - B'(r, v) - B''(v, r) + f \quad \text{in } \omega,$$

$$\text{div } r_1 = 0, \quad r_1|_{\omega_0} = 0, \quad g_1(t, x_0) = 0 \quad \forall t \in [0, T]. \quad (3.41)$$

By Lemma 3.5, the unique solution of problem (3.41) exists and satisfies the estimate

$$\|r_1\|_{W^{1, 2}(\omega)} + \|g_1\|_{L^2(\omega)} \leq C_{14}(\|z(0, r)\|_{L^2(\omega)} + \|z(t, r)\|_{L^2(\omega)} + \|z(t, r)\|_{L^2(\omega)} + \|f\|_{L^2(\omega)}). \quad (3.42)$$

By virtue of (3.29), (3.37), (3.41) the functions $r_2, g_2$ should satisfy the equations

$$\Delta r_2 = \nabla g_2 \quad \text{in } \omega, \quad \text{div } r_2 = 0. \quad (3.43)$$

Applying the Laplace operator $\Delta$ to this equation, we have $\Delta^2 r_2 = 0$. Thus, by (3.42) and well-known estimates for interior regularity of solutions of elliptic equations (see [23]), we have:

$$\|r_2(t)\|_{L^2(\omega)} \leq C\|r(t) - r_1(t)\|_{L^2(\omega)},$$

$$\leq C_{15}(\|z(0, r)\|_{L^2(\omega)} + \|z(t, r)\|_{L^2(\omega)} + \|z(t, r)\|_{L^2(\omega)} + \|f\|_{L^2(\omega)}). \quad (3.44)$$

By (3.44) equality (3.43) implies the estimate

$$\|\nabla g_2(t, \cdot)\|_{L^2(\omega)} \leq C_{14}(\|z(0, \cdot)\|_{L^2(\omega)} + \|z(t, \cdot)\|_{L^2(\omega)} + \|z(t, \cdot)\|_{L^2(\omega)} + \|f\|_{L^2(\omega)}) \quad (3.45)$$

By (3.40), (3.41)

$$g_2(t, x_0) = 0 \quad \forall t \in [0, T].$$

Thus inequality (3.45) yields

$$\|g_2(t, \cdot)\|_{L^2(\omega)} \leq C_{15}(\|z(0, \cdot)\|_{L^2(\omega)} + \|z(t, \cdot)\|_{L^2(\omega)} + \|z(t, \cdot)\|_{L^2(\omega)} + \|f\|_{L^2(\omega)}). \quad (3.46)$$

This inequality and (3.42) imply (3.38).
Applying the Carleman inequality (3.27) to equations (3.36) and (3.37), we have:
\[
\int_Q \left( \frac{1}{s^2}\left| T^P + \sum_{i=1}^n \frac{\partial^2 P}{\partial x_i \partial x_j}\right|^2 + s^2 \phi \nabla \phi + s^4 \phi^2 |P|^4 \right) e^{\gamma t} \, dx \, dt \\
\leq C_{16}(\lambda) \int_{Q_{\gamma_1}} s^2 \phi^2 \mu^2 \nabla e^{\gamma t} + \int_{Q_{\gamma_1}} s^2 \phi^2 \mu^2 \phi e^{\gamma t} \, dx \, dt \\
+ s^2 \phi \lambda \left( |F|^4 + |f(0, \cdot)|^4 e^{\gamma t} \right) e^{\gamma t} \, dx \, dt ,
\]
(3.47)
where \( s \geq s_0(\lambda) \).

The parameter \( s_0(\lambda) \) is defined in Corollary 3.2. Set \( s = s_0(\lambda) \). Using the a priori estimate (3.30) for system (3.22), (3.23) in the right-hand side of inequality (3.47), we can replace the function \( \alpha \) by \( \kappa \), the function \( \phi \) by \( (T - t)^{-\alpha} \) and the constant \( C \) by \( C(s) \):
\[
\int_Q (T - t)^{\alpha} |F e^{\gamma t} + \|z(0, \cdot)\|^2_{V_0}) \\
\leq C_{18}(s) \left( \int_{Q_{\gamma_1}} \frac{s^2}{(T - t)^{\alpha}} \mu^2 e^{\gamma t} \, dx \, dt \\
+ \int_{Q_{\gamma_1}} \frac{|F|^4 + |f(0, \cdot)|^4}{(T - t)^{\alpha}} e^{\gamma t} \, dx \, dt \right) \left( ||z(0, \cdot)\||_{V_0} \right) ,
\]
(3.48)
where \( s \geq \delta \). Using estimate (3.38), we can rewrite (3.18) as follows:
\[
\int_Q (T - t)^{\alpha} |F e^{\gamma t} + \|z(0, \cdot)\|^2_{V_0}) \\
\leq C_{18}(s) \left( \int_{Q_{\gamma_1}} (T - t)^{-\alpha} |F e^{\gamma t} \, dx \, dt \\
+ \int_{Q_{\gamma_1}} \frac{|F|^4 + |f(0, \cdot)|^4}{(T - t)^{\alpha}} e^{\gamma t} \, dx \, dt \right) \left( ||z(0, \cdot)\||_{V_0} \right) ,
\]
(3.49)
Note that by (3.34)
\[
\int_{Q_{\gamma_1}} \frac{|F|^4 + |f(0, \cdot)|^4}{(T - t)^{\alpha}} e^{\gamma t} \, dx \, dt \leq C_{21}(\int_Q |F e^{\gamma t} \, dx \, dt + \int_{Q_{\gamma_1}} \frac{|F|^4}{(T - t)^{\alpha}} e^{\gamma t} \, dx \, dt) ,
\]
where \( C_{21} \) is an independent constant. By this inequality, we deduce from (3.49)
\[
\int_Q (T - t)^{\alpha} |F e^{\gamma t} + \|z(0, \cdot)\|^2_{V_0}) \\
\leq C_{21}(\int_Q |F e^{\gamma t} \, dx \, dt + \int_{Q_{\gamma_1}} \frac{|F|^4}{(T - t)^{\alpha}} e^{\gamma t} \, dx \, dt) ,
\]
(3.50)
Let us finish the proof by contradiction. If the estimate (3.35) is not true, then by (3.50) there exists a sequence \((z_k, q_k, f_k)\) such that

\[
L^* z_k = \nabla q_k + f_k \text{ in } Q, \quad \text{div } z_k = 0, \quad z_k|_{\Gamma_1} = 0, \quad \|z(0, \cdot, \cdot)|_{L^2(Q_2)}^* = 1,
\]

(3.51)

\[
f_k \to 0 \text{ in } (L^2(Q, e^{\frac{n}{n+1}}))^n, \quad \int_{Q_n} |z_k|^2 e^{\frac{n}{n+1}} \text{d}x \text{d}t \to 0 \text{ as } k \to \infty,
\]

(3.52)

for all \(z \in (0, T)\). Passing to the limit in (3.51), taking into account (3.52) we obtain

\[
L^* z = \nabla q \text{ in } Q, \quad \text{div } z = 0, \quad z|_{\Gamma_1} = 0, \quad z_{\Gamma_2} = 0, \quad \|z(0, \cdot, \cdot)|_{L^2(Q_2)}^* = 1.
\]

(3.53)

By (3.27), (3.53) we obtain

\[
z \equiv 0,
\]

but this is impossible by virtue of (3.54). The proof of the lemma is complete. \(\Box\)

4. SOLVABILITY OF THE LINEAR CONTROLLABILITY PROBLEM

Let us consider the problem of exact controllability of the linearized Navier-Stokes equations:

\[
Ly = \frac{\partial y}{\partial t} + \Delta y + B(y, v) + B(v, y) = \nabla p + f + \chi_w u \text{ in } Q,
\]

(4.1)

\[
\text{div } y = 0, \quad y|_{\Gamma_1} = 0, \quad y(0, x) = y_0(x),
\]

(4.2)

\[
y(T, x) = 0,
\]

(4.3)

where the functions \(y_0, f\) are given and \(u\) is a control from the space \((L^2(Q_\omega))^n\). Before studying the solvability of problem (4.1)-(4.3), let us recall some existence theorems for boundary value problem (4.1), (4.2), assuming that \(u\) is a fixed function.

**Lemma 4.1.** Let \(v \in V^1(\Omega)^\ast \cap (W^1_0(\Omega))^n\). Then for any \(f \in L^2(0, T; V^1(\Omega))\), \(w \in (L^2(Q))^n\), and \(y_0 \in V^1(\Omega)\) there exists a solution \((y, p) \in V^{1, 2}(Q) \times L^2(Q)\) of problem (4.1), (4.2). Moreover this solution is unique in the space \(C([0, T]; V^{1, 2}(\Omega)) \times L^2(0, T; V^1(\Omega))\) and satisfies the estimate

\[
\|y\|_{V^{1, 2}(Q)} \leq C_u(\|f\|_{L^2(0, T; V^1(\Omega))} + \|w\|_{(L^2(Q))^n} + \|y_0\|_{V^1(\Omega)}).
\]

(4.4)

Set

\[
\eta(t, x) = -\delta \kappa(t, x),
\]

(4.5)

where the function \(\kappa\) is defined in (3.33), (3.34) and the parameter \(s\) from Theorem 3.6. Since the function \(\kappa(t, x)\) is negative, \(\eta(t, x)\) is positive. Moreover \(\lim_{t \to -\infty} \eta(t, x) = +\infty\).
We use below the following weight function:

$$\theta(t, x) = (1 - \chi_\omega) e^\frac{\epsilon n^t}{(T-t)^2} + \chi_\omega. \quad (4.6)$$

To formulate our results, we need to introduce some nonstandard functional spaces

$$F(Q, \theta) = \{ f \in (L_2(Q))^n; \exists f_1 \in (L_2(Q, \theta))^n, \ \exists f_2 \in L_2(0, T; W_2^2(\Omega)) \text{ such that } f = f_1 + \nabla f_2 \}. $$

The norm in $F(Q, \theta)$ is defined by the relation

$$\|f\|_{F(Q, \theta)} = \inf_{f = f_1 + \nabla f_2} (\|f_1\|_{L_2(Q, \theta)}^2 + \|\nabla f_2\|_{L_2(Q)}^2)^{1/2}.$$ 

We are looking for solutions of the controllability problem in the following space:

$$Y(Q) = \{ y \in V^{1,2}(Q); Ly \in F(Q, \theta), e^{-\frac{\theta}{2}} y \in V^{1,2}(Q) \}$$

with the norm

$$\|y\|_{Y(Q)} = \|Ly\|_{F(Q, \theta)} + \|e^{-\frac{\theta}{2}} y\|_{V^{1,2}(Q)}.$$ 

**Remark 4.2.** The space of solutions $Y(Q)$ depends, at least formally, on the function $\theta$. To construct a suitable Banach space of solutions of (4.1), (4.3) independent of $\theta$, we have to prove sharper estimates on the rate of convergence of $y(t, \cdot)$ to zero near $t = T$ than those we obtain below. This is probably possible.

We have:

**Theorem 4.3.** Let $f \in F(Q, \theta)$, $y_0 \in V^{1,2}(\Omega)$. Then there exists a solution of problem (4.1)-(4.3) $(y, p, u) \in Y(Q) \times L^2(0, T; W_2^2(\Omega)) \times (L_2(Q, \theta))^n$ which satisfies the estimate

$$\|(y, p, u)\|_{Y(Q) \times L^2(0, T; W_2^2(\Omega)) \times (L_2(Q, \theta))^n} \leq C \|\theta\|_{[0, T]} + \|f\|_{F(Q, \theta)}. \quad (4.7)$$

**Proof.** We first assume that $f \in L^2(Q, \theta)$ and $f_{L^2} = 0$. Let us consider the extremal problem

$$\mathcal{J}(y, u) = \frac{1}{2} \int_Q \rho_0|u|^2 \, dx dt + \frac{1}{2} \int_{\Omega} m_1|u|^2 \, dx \rightarrow \inf,$$

$$Ly = u + \nabla p + f \quad \text{in} \ Q, \ \div y = 0, \ y_{L^2} = 0, \ y(0, x) = y_0(x), \ y(T, x) = 0, \quad (4.9)$$

where

$$\rho_0(t) = e^{\frac{-\theta t}{(T-t)^2}}, \quad m_1(t, x) = \begin{cases} e^{\frac{-\theta t}{(T-t)^2}}, & x \in \Omega, \\ k, & x \in \Omega \setminus \overline{\Omega}. \end{cases}$$

ESAIM: Contr., vol. 1599, no. 3, 97–128
Obviously the functions $\rho_k, m_k$ are bounded in $Q$ for every $k > 0$.

By Proposition 2.1 and Lemma 4.1, there exists an admissible element to the problem (4.8), (4.9). So it is easy to prove (see [21], [25]) that the problem (4.8)-(4.9) has a unique solution, which we denote by $(\tilde{y}_k, \tilde{u}_k) \in V^{1/2}(Q) \times (L^2(Q))^n$.

Thus, applying the Lagrange principle to problem (4.8) - (4.9) (see [1], [9]), we obtain

$$L \tilde{y}_k = \tilde{f} + \nabla \tilde{p}_k + \tilde{u}_k \text{ in } Q, \quad \text{div} \tilde{y}_k = 0, \quad \tilde{y}_k |_{\partial Q} = 0, \quad \tilde{y}_k(T, \cdot) = 0, \quad \tilde{y}_k(0, \cdot) = y_0,$$

where the operator $L$ to the problem (4.8)-(4.9) is easily to prove (see [2], [4]) that the

$$z_k = \nabla \tilde{q}_k + \rho_k \tilde{y}_k \text{ in } Q, \quad \text{div} z_k = 0, \quad z_k = -m_k \tilde{u}_k \text{ in } Q,$$

where the operator $L^*$ defined in (3.28) is formally conjugate to the operator $L$.

Since the function $\rho_k$ depends only on the variable $t$, by Lemma 4.1 $\rho_k \tilde{y}_k \in L^2(0, T; V^1(\Omega))$. So we can apply estimate (3.35) to equation (4.11):

$$\int_Q (T - t)^{k/2} e^{\text{Re}(\mu_0)} |z_k| \, dx \, dt + \|z_k(0, \cdot)\|_{L^2(\Omega)}^2 \leq C_{L^2(0, T; V^1(\Omega))} (t) \left( \int_Q (T - t)^{k/2} e^{\text{Re}(\mu_0)} |z_k| \, dx \, dt + \|z_k(0, \cdot)\|_{L^2(\Omega)}^2 \right).$$

(4.12)

We observe that $|y_0(t) e^{\text{Re}(\mu_0)}| \leq 1$ for all $(t, x) \in Q$ and $|m_k(t, x) e^{\text{Re}(\mu_0)}| \leq 1$ for all $(t, x) \in Q_*$. Actually,

$$\frac{9k^2}{10(T - t)^2} + \frac{9}{10} < 0,$$

where the function $f(t)$ is defined in (2.18). Keeping in mind these inequalities and substituting $z_k$ by the right-hand side of (4.11), in the last integral of equality (4.12) we have

$$\int_\Omega z_k(0, x) \, dx + \int_Q (T - t)^{k/2} e^{\text{Re}(\mu_0)} dx \leq C_{L^2(0, T; V^1(\Omega))} (t) \left( \int_Q (T - t)^{k/2} e^{\text{Re}(\mu_0)} |z_k| \, dx \, dt + \|z_k(0, \cdot)\|_{L^2(\Omega)}^2 \right).$$

Taking the scalar product of (4.11) with $\tilde{y}_k$ in $(L^2(Q))^n$ and integrating by parts with respect to $t$ and $x$, bearing in mind (4.10), after simplifications we have

$$0 = (Lz_k - \nabla \tilde{q}_k - \rho_k \tilde{y}_k, \tilde{y}_k)_{(L^2(Q))} =$$

$$- \int_Q \rho_k |\tilde{y}_k| \, dx \, dt + (z_k, \tilde{y}_k)_{(L^2(Q))} + (z_k(0, \cdot), \tilde{y}_k(0, \cdot))_{L^2(\Omega)},$$

$$= - \int_Q \rho_k |\tilde{y}_k| \, dx \, dt + \int_Q m_k |\tilde{u}_k| \, dx \, dt + \int_Q (f, \tilde{z}_k) \, dx \, dt + (z_k(0, \cdot), \tilde{y}_k)_{L^2(\Omega)}.$$
Hence,  
\[ \mathcal{J}_k(y_k, u_k) = \frac{1}{2} \int_Q \left( \rho u_k \left| \nabla u_k \right|^2 + m_1 \left| u_k \right|^2 \right) \, dx  
\]
\[ \text{subject to } \left\{ \begin{array}{l}
  u_k(0, \cdot) = y_k, \\
  u_k(T, \cdot) = \left| \nabla \phi \right| u_k(T, \cdot), \\
  u_k \in L^2(Q) \end{array} \right. \]  

(4.14)  

Note that  
\[ \left\| \int_Q (f, z_k) \, dx \, dt \right\| \leq C_4 \left( \left\| f \right\|_{L^2(Q,T)} + \left\| T - \theta \right\|_{L^2(Q)} \right). \]  

(4.15)  

By (4.13), (4.14), (4.15), we obtain  
\[ \mathcal{J}_0(y_0, u_0) \leq C_4 \left( \left\| f \right\|_{L^2(Q,T)} + \left\| T - \theta \right\|_{L^2(Q)} \right). \]  

(4.16)  

By virtue of (4.16), (4.4), we have a subsequence \( \{ (y_k, u_k) \}_{k=1}^\infty \) such that  
\[ (y_k, u_k) \rightarrow (y, u) \text{ weakly in } V^{1,2}(Q) \times (L^2(Q))^{n} \]  
\[ u_k \rightarrow 0 \text{ in } (L^2(0,T) \times (\Omega \setminus A^{\varepsilon}))^{n} \]  

(4.17)  

By (4.13), (4.17) it also follows from (4.11) that  
\[ \left\| m_k \sqrt{\varepsilon} \right\|_{L^2(Q,T)} \leq C_{\varepsilon} \]  

for all \( \varepsilon \in (0, T) \). Hence, without loss of generality, we can assume  
\[ u_k(t, x) \rightarrow u(t, x) \text{ almost everywhere in } Q \]  

(4.18)  

Using (4.17), we pass to the limit in (4.10) to obtain that the pair \((y, u)\) is a solution of problem (1.1)-(1.3). The relations (4.16), (4.17) and Lemma 4.1 imply the estimate  
\[ \left\| (y, u) \right\|_{V^{1,2}(Q) \times (L^2(Q))^{n}} \leq C_4 \left( \left\| f \right\|_{L^2(Q,T)} + \left\| T - \theta \right\|_{L^2(Q)} \right). \]  

(4.19)  

Furthermore, by (4.17), (4.18) and Fatou's theorem (see [22, p. 207]), we have  
\[ \left\| (y,u) \right\|_{W^{1,n}(Q,\omega \times T^n)} \leq C_4 \]  

(4.20)  

Now, to prove (1.7), we need only to estimate the norm of the function \( e^{-\beta s} y \) in the space \( V^{1,2}(Q) \). Let us make the change in (4.1). Set \( \tilde{y} = e^{-\beta s} y \) and
Then there exists \( A \) holds and the derivative condition

\[ A^\prime(x) = z \]

holds and the derivative \( A^\prime(x) : X \to Z \) of the map \( A \) at \( x_0 \) is an epimorphism. Then there exists \( \varepsilon > 0 \) such that for any \( z \in Z \) which satisfies the condition

\[ \| z - z_0 \|_Z < \varepsilon \]
there exists a solution \( x \in X \) of the equation

\[
A(x) = z.
\]

In our case the space

\[
X = Y(Q) \times L^2(0, T; W^1_2(\Omega)) \times (L^2(Q_\omega))^{n^2}
\]

and

\[
Z = F(Q, \theta) \times V^1(\Omega).
\]

The operator \( A \) defined by formula

\[
A(y, q, u) = \langle A(y, q, u), y(0, \cdot) \rangle.
\]

We have:

**Lemma 5.2.** Let \( \dot{v} \in V^1(\Omega) \cap W^1_\infty(\Omega)^n \), then \( A \in C^1(X, Y) \).

**Proof.** It follows directly from the definitions of the spaces \( X, Z \) that the operator

\[
(y, q, u) \rightarrow (\partial_t y(t, x) - \Delta y + B(\dot{v}, y) + B(y, 0) - \nabla q - \chi_{\omega} v y(0, \cdot)) : X \rightarrow Z
\]

is continuous and by virtue of linearity continuously differentiable. The operator \( B \) is bilinear. So, to prove this theorem, it is sufficient to establish the continuity of the bilinear operator

\[
B : Y(Q) \times Y(Q) \rightarrow (L^2(Q, \theta))^n,
\]

where the function \( \theta \) is defined in (4.6). Note that by [2.4], [3.33], (4.5)

\[
\|y(t, x)\| \leq \varepsilon y(t, x) \quad \forall (t, x) \in Q.
\]

Then (5.10) and simple transformations give the estimate

\[
\begin{align*}
\|B[y_1, y_2]\|_{L^2(Q, \Omega)}^2 & \leq C_1 \sum_{i, j=1}^2 \left( \int_{Q_{\omega}} \|y_i F \nabla y_j F e^{\nu} \|_{(T - \frac{n}{2})^2}^2 dx dt + \int_{Q_{\omega}} \|y_i F \nabla y_j F \|^2 dx dt \right) \\
& \leq C_2 \sum_{i, j=1}^2 \left( \int_{Q_{\omega}} \|y_i F \nabla y_j F \|^2 \left( \frac{T - n}{2} \right)^2 \right) dx dt \\
& \leq C_3 \|y_1 F \|_{L^2(Q, \Omega)} \|y_2 F \|_{L^2(Q, \Omega)}.
\end{align*}
\]

This inequality proves the theorem. \( \blacksquare \)

**Proof of Theorem 1.2.** First, we apply the inverse operator theorem to problem (5.2)-(5.5). Let \( A \) be defined by formulas (5.9), (5.2), and the spaces \( X, Z \) defined in (5.7), (5.8). Set \( x_0 = (0, 0, 0), z_0 = (0, 0) \). Then equation (5.6) obviously holds. By Lemma 5.2, \( A \in C^1(X, Z) \) and by Theorem 4.3, \( \text{Im} A'(0) = Z \). So all necessary conditions needed to apply the theorem on
the inverse operator are fulfilled. Therefore there exists \( \varepsilon > 0 \) such that for any initial data \((y, p, u)\) satisfying the inequality

\[
\|y\| \leq \varepsilon
\]

problem (5.2)-(5.5) has a solution \((y, p, u) \in X\). Then the triple \((y + \hat{v}, p + p, u)\) is a solution of problem (1.1)-(1.3), (1.5).

Remark 5.3. If we assume that \( \hat{v} \in (C^\infty(\bar{Q}))^n, supp \hat{v} \subseteq [0, T] \times \Omega \) is solution of the nonstationary Navier-Stokes system:

\[
\partial_t \hat{v}(t, x) - \Delta \hat{v}(t, x) + (\hat{v}, \nabla) \hat{v} + \nabla p = f(t, x) \quad in \Omega, \quad \text{div} \hat{v} = 0
\]

and interchange inequality (1.4) on

\[
\|\hat{v}(0, \cdot) - \omega_l\| \vee \|\varphi_l\| \leq \varepsilon
\]

the statement of Theorem 1.2 holds true. Some small changes have to be done in the proof of Theorem 3.6.

References

ON EXACT CONTROLLABILITY FOR THE NAVIER-STOKES EQUATIONS


