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STABILITY RATES FOR PATCHY VECTOR FIELDS

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Abstract. This paper is concerned with the stability of the set of trajectories of a *patchy* vector field, in the presence of impulsive perturbations. Patchy vector fields are discontinuous, piecewise smooth vector fields that were introduced in Ancona and Bressan (1999) to study feedback stabilization problems. For patchy vector fields in the plane, with polygonal patches in generic position, we show that the distance between a perturbed trajectory and an unperturbed one is of the same order of magnitude as the impulsive forcing term.

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1. Introduction

Let g be a bounded vector field, and consider the Cauchy problem with impulsive perturbations

$$\dot{y} = g(y) + \dot{w}. \tag{1.1}$$

Here $w = w(t)$ is a left continuous function with bounded variation. By a solution of (1.1) with initial condition

$$y(t_0) = y_0, \tag{1.2}$$

we mean a measurable function $t \mapsto y(t)$ such that

$$y(t) = y_0 + \int_{t_0}^{t} g\big(y(s)\big)\,\mathrm{d}s + \big[w(t) - w(t_0)\big]. \tag{1.3}$$

If $w(\cdot)$ is discontinuous, the forcing term in (1.1) will have impulsive behavior, and the solution $y(\cdot)$ will be discontinuous as well. We choose to work with (1.1) because it provides a simple and general framework to study stability properties. Indeed, consider a system with both inner and outer perturbations, of the form

$$\dot{x} = g\big(x + e_1(t)\big) + e_2(t). \tag{1.4}$$

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Then, the map $y = y(t) \doteq x(t) + e_1(t)$ satisfies the impulsive equation

$$\dot{y} = g(y) + e_2(t) + \dot{e}_1(t) = g(y) + \dot{w},$$

where

$$w(t) = e_1(t) + \int_{t_0}^{t} e_2(s)\,\mathrm{d}s.$$

Therefore, from the stability of solutions of (1.1) under small BV perturbations w, one can immediately deduce a result on the stability of solutions of (1.4), when Tot.Var.$\{e_1\}$ and $\|e_2\|_{\mathbf{L}^1}$ are suitably small.

Our main concern is how much a trajectory is affected by the presence of the impulsive perturbation. More precisely, we wish to estimate the distance, in the $\mathbf{L}^\infty$ norm, between solutions of the two Cauchy problems

$$\begin{cases} \dot{x} = g(x), \\ x(0) = x_0, \end{cases} \qquad\qquad \begin{cases} \dot{y} = g(y) + \dot{w}, \\ y(0) = x_0. \end{cases} \tag{1.5}$$

Consider first the special case where g is a continuous vector field with Lipschitz constant L. It is then well known that the Cauchy problems (1.5) have unique solutions, obtained by a fixed point argument (see [3]). Their distance can be estimated as

$$\big|y(t) - x(t)\big| \le \int_0^t \mathrm{e}^{L(t-s)}\,\big|\mathrm{d}w(s)\big| \le \mathrm{e}^{Lt}\cdot \text{Tot.Var.}\{w\}\cdot \tag{1.6}$$

In other words, on a fixed time interval, this distance grows linearly with Tot.Var.$\{w\}\cdot$

In this paper, we will prove a similar estimate in the case where g is a discontinuous, *patchy vector field*. These vector fields were introduced in [1] in order to study feedback stabilization problems. We recall the main definitions:

Definition 1.1. By a *patch* we mean a pair $\big(\Omega,\, g\big)$ where $\Omega \subset \mathbb{R}^n$ is an open domain with smooth boundary $\partial\Omega$, and g is a smooth vector field defined on a neighborhood of the closure $\overline{\Omega}$, which points strictly inward at each boundary point $x \in \partial\Omega$.

Calling $\mathbf{n}(x)$ the outer normal at the boundary point x, we thus require

$$\big\langle g(x),\ \mathbf{n}(x)\big\rangle < 0 \qquad \text{for all} \quad x \in \partial\Omega. \tag{1.7}$$

Definition 1.2. We say that $g : \Omega \mapsto \mathbb{R}^n$ is a *patchy vector field* on the open domain Ω if there exists a family of patches $\big\{(\Omega_\alpha,\ g_\alpha);\ \ \alpha \in \mathcal{A}\big\}$ such that
– $\mathcal{A}$ is a totally ordered set of indices;
– the open sets Ω_α form a locally finite covering of Ω, *i.e.* $\Omega = \cup_{\alpha\in\mathcal{A}}\Omega_\alpha$ and every compact set $K \subset \mathbb{R}^n$ intersect only a finite number of domains Ω_α, $\alpha \in \mathcal{A}$;
– the vector field g can be written in the form

$$g(x) = g_\alpha(x) \qquad \text{if} \qquad x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta. \tag{1.8}$$

By setting

$$\alpha^*(x) \doteq \max\big\{\alpha \in \mathcal{A}\ ;\ \ x \in \Omega_\alpha\big\}, \tag{1.9}$$

we can write (1.8) in the equivalent form

$$g(x) = g_{\alpha^*(x)}(x) \qquad \text{for all} \quad x \in \Omega. \tag{1.10}$$

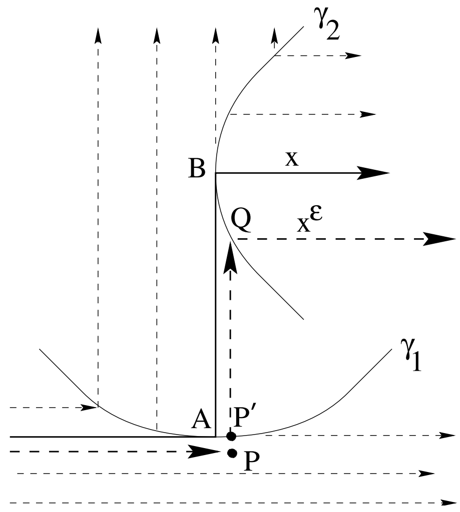

FIGURE 1.

We shall occasionally adopt the longer notation $\big(\Omega,\ g,\ (\Omega_\alpha,\ g_\alpha)_{\alpha\in\mathcal{A}}\big)$ to indicate a patchy vector field, specifying both the domain and the single patches. If g is a patchy vector field, the differential equation

$$\dot{x} = g(x) \tag{1.11}$$

has many interesting properties. In particular, in [1] it was proved that the set of Carathéodory solutions of (1.11) is closed (in the topology of uniform convergence) but possibly not connected. Moreover, given an initial condition $x(t_0) = x_0$, the corresponding Cauchy problem has at least one forward solution, and at most one backward solution, in the Carathéodory sense. For every Carathéodory solution $x(\cdot)$ of (1.11), the map $t \mapsto \alpha^*(x(t))$ is left continuous and non-decreasing.

Since the Cauchy problem for (1.11) does not have forward uniqueness and continuous dependence, one clearly cannot expect that a single solution can be stable under small perturbations. Instead, one can establish the following stability property referring to the whole set of solutions.

Proposition 1.3. *([2], Cor. 1.1) Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$. Given any compact set $K \subset \Omega$, and any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $y : [0,T] \mapsto K$ is a solution of the perturbed system (1.1) with Tot.Var.$(w) < \delta$, then there exists a solution $x : [0,T] \mapsto \Omega$ of the unperturbed equation (1.11) such that*

$$\big\|x - y\big\|_{\mathbf{L}^\infty([0,T])} < \varepsilon. \tag{1.12}$$

The relevance of this result for the robustness of discontinuous feedback controls is discussed in [2].

In connection with Proposition 1.3, it is interesting to study how the distance $\|y - x\|_{\mathbf{L}^\infty}$ can depend on the perturbation w. For a general BV function w, the derivative $\dot{w}$ is a Radon measure whose total mass coincides with the total variation of w. It is thus natural to use the BV norm $\|w\|_{BV}$ as a measure of the strength of the perturbation. In the case of a Lipschitz continuous field g, we have seen in (1.6) that this distance grows linearly with $\|w\|_{BV}$. In the case of patchy vector fields, one cannot expect a linear dependence, in general.

Example 1.4. Consider a patchy vector field on $\mathbb{R}^2$, as in Figure 1. Assume $g = (1,0)$ below the curve γ_1 and to the right of the curve γ_2, while $g = (0,1)$ above the curve γ_1. Observe that there exists a Carathéodory solution $x(\cdot)$ of (1.11) going through the points A and B. Next, consider a perturbed solution x^ε, following the vector field horizontally up to P, jumping from P to P', then moving vertically to Q and horizontally

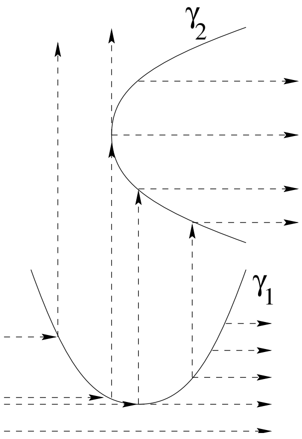

FIGURE 2.

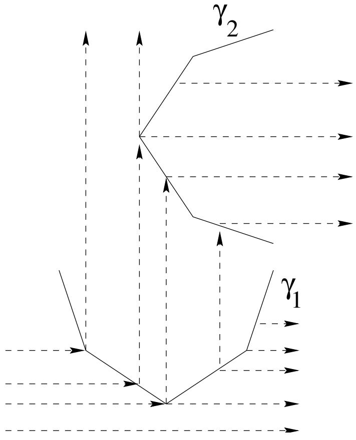

FIGURE 3.

afterwards. To fix the ideas, assume that

$$A = (0,0), \quad B = (0,1), \quad P = (\varepsilon, -\varepsilon^\alpha), \quad P' = (\varepsilon, \varepsilon^\alpha),$$

$$\gamma_1 = \{x_2 = |x_1|^\alpha\}, \qquad \gamma_2 = \{x_1 = |x_2 - 1|^\beta\}.$$

In this case the trajectory x^ε is a solution of a perturbed system where $\dot w$ is a single Dirac mass of strength $|P' - P| = 2|\varepsilon|^\alpha$. On the other hand, after both trajectories have switched to the right of the curve γ_2 their distance is $\|x^\varepsilon - x\| = \varepsilon^{1/\beta}$. In this example, the distance between solutions grows much worse than linearly w.r.t. the strength of the perturbation, Indeed, the only estimate available is

$$\big\|y - x\big\|_{\mathbf{L}^\infty} = \mathcal{O}(1) \cdot \big(\text{Tot.Var.}\{w\}\big)^{1/\alpha\beta}. \tag{1.13}$$

One conjectures that the situation is better when the patches are in "generic" position. Observe that in (1.13) the numbers α and β are determined by the order of tangency of the curves γ_1, γ_2 with the vector field g. By an arbitrarily small displacement of the curves γ_1, γ_2 we can arrange so that there is no trajectory connecting the two point of tangency A and B (Fig. 2). Moreover, we can assume that the tangency is only of first order. For generic patchy vector fields on $\mathbb{R}^2$, in Corollary 1.1 one thus expects an estimate of the form

$$\big\|y - x\big\|_{\mathbf{L}^\infty} = \mathcal{O}(1) \cdot \big(\text{Tot.Var.}\{w\}\big)^{1/2}.$$

Here the exponent 1/2 is due to the fact that first order tangencies are not removable by small perturbations. In higher space dimensions, an even lower exponent is expected. To obtain an error estimate which is linear w.r.t. the strength of the perturbation, one thus needs to remove all these tangencies. This cannot be achieved if the patches have smooth boundary, but is quite possible if we allow "polyhedral" patches (Fig. 3).

Throughout the following, we write $d(x,\, A) = \inf\big\{|x - y| \ : \ y \in A\big\}$ for the distance of a point x from the set $A \subset \mathbb{R}^n$, and denote by $\overset{\circ}{A}$ the interior of A.

Definition 1.5. Let $\Omega \subset \mathbb{R}^n$ be an open domain whose boundary is contained in a finite set of hyperplanes. Call $T_\Omega(x)$ the tangent cone to Ω at the point x, defined by

$$T_\Omega(x) \doteq \left\{ v \in \mathbb{R}^n \ : \ \lim_{t \to 0} \frac{d\big(x + tv,\ \Omega\big)}{t} = 0 \right\}. \tag{1.14}$$

We say that a smooth vector field g defined on a neighborhood of $\overline{\Omega}$ is an *inward-pointing vector field* on Ω if,

$$g(x) \in \overset{\circ}{T}_\Omega(x) \qquad \text{for all} \qquad x \in \partial\Omega. \tag{1.15}$$

The pair $(\Omega,\, g)$ will be called a *polyhedral patch*.

Clearly, at any regular point $x \in \partial\Omega$, the interior of the tangent cone $T_\Omega(x)$ is precisely the set of all vectors $v \in \mathbb{R}^n$ that satisfy

$$\langle v,\ \mathbf{n}(x)\rangle < 0$$

and hence (1.15) coincides with the inward-pointing Condition (1.7).

Replacing "patches" with "polyhedral patches" in Definition 1.2 we obtain the notion of *polyhedral patchy vector field*. For such fields, it is expected that impulsive perturbations of the form (1.1) should generically produce a perturbation on the set of trajectories which is of exactly the same order of magnitude as the strength of the impulse on the right hand side.

To avoid lengthy technicalities, we shall consider here only the planar case, *i.e. polygonal patchy vector fields*. We conjecture that the same result holds true for generic polyhedral patchy vector fields on $\mathbb{R}^n$.

Theorem 1. *For a generic polygonal patchy vector field g on $\mathbb{R}^2$, whose values are bounded away from zero, one has the the following stability property.*

Given any compact set $K \subset \mathbb{R}^2$, there exist constants $C,\ \delta > 0$, such that the following holds. For every solution $y : [0,T] \mapsto K$, of (1.1) with Tot.Var.$\{w\} < \delta$, there exists a solution $x : [0,T] \mapsto \mathbb{R}^2$ of (1.11) such that

$$\big\|x - y\big\|_{\mathbf{L}^\infty([0,T])} \leq C \cdot \textit{Tot.Var.}\{w\}\cdot \tag{1.16}$$

A precise description of the generic conditions which guarantee the estimate (1.16) will be given in Section 2. Roughly speaking, one requires that the boundary of every patch Ω_α be transversal to all fields g_β, with $\beta \leq \alpha$.

Throughout the paper, by $B(x,r)$ we denote the closed ball centered at x with radius r. The closure, the interior and the boundary of a set Ω are written as $\overline{\Omega}$, $\overset{\circ}{\Omega}$ and $\partial\Omega$, respectively.

The paper is organized as follows. In Section 2 we introduce a class of polygonal patchy vector fields for which we will establish the stability property stated in Theorem 1, and we show that we can always replace a solution of the perturbed system (1.1) with a piecewise smooth concatenation of solutions of the unperturbed system (1.11), so that their distance is of the same order of magnitude as the impulsive term $\dot{w}$. To establish this result contained in Proposition 2.4, we rely on two technical lemmas (Lems. 2.2 and 2.3) whose rather lengthy proofs are postponed to Section 4 (Appendix). In Section 3 we first show in Proposition 3.1 that, for every function $y(\cdot)$ that is a concatenation of two solutions of (1.11) (and thus admits a single jump discontinuity), there exists a solution $x(\cdot)$ of (1.11) for which the linear estimate (1.16) holds, and then we complete the proof of Theorem 1 establishing Lemma 3.2.

2. Preliminary stability estimates

Let $\mathcal{PPVF}$ denote the set of all bounded, **polygonal patchy vector fields** $\big(g,\ (\Omega_\alpha,\ g_\alpha)_{\alpha\in\mathcal{A}}\big)$ on $\mathbb{R}^2$, that are uniformly bounded away from zero. A *condition P* for a patchy vector field $\big(g,\ (\Omega_\alpha,\ g_\alpha)_{\alpha\in\mathcal{A}}\big) \in \mathcal{PPVF}$ is a logic proposition that can be expressed in terms of the fields g_α and (or) the domains Ω_α. We write $P(g)$ if $\big(g,\ (\Omega_\alpha,\ g_\alpha)_{\alpha\in\mathcal{A}}\big)$ satisfies P, and we say that P is *generic* if $\{g \in \mathcal{PPVF}\ :\ P(g)\}$ is a generic subset of $\mathcal{PPVF}$ in the sense that $\{g \in \mathcal{PPVF}\ :\ \mathcal{P}(\})\}$ is an open and dense subset of $\mathcal{PPVF}$ with respect to the $\mathbf{L}^\infty$ topology.

We state now a generic condition that yields the linear estimate (1.16) of the effect of impulsive perturbations on the solutions of the unperturbed system (1.11).

C) For any given domain Ω_α, and for any line r_γ containing an edge of the boundary $\partial\Omega_\gamma$ of some Ω_γ, $\gamma > \alpha$, the field $g_\alpha(x)$ is transversal to r_γ at every point $x \in r_\gamma \cap \overline{\big(\Omega_\alpha \setminus \bigcup_{\beta>\alpha}\Omega_\beta\big)}$.

In this section we will show that, given a polygonal patchy vector field g satisfying condition **(C)**, in order to establish the stability estimate (1.16) for an arbitrary solution $t \mapsto y(t)$ of (1.1) we can always replace $y(\cdot)$ with a piecewise smooth map $t \mapsto y^{\diamondsuit}(t)$ that is a concatenation of solutions of the unperturbed system (1.11). This result is contained in Proposition 2.4 and is based on two technical Lemmas (Lems. 2.2–2.3) whose proof is postponed to Section 4. Since we shall always consider throughout the paper solutions of (1.11) or of (1.1) that are contained in some fixed compact set K, we will assume without loss of generality that every domain Ω_α is bounded since, otherwise, one can replace Ω_α with its intersection $\Omega_\alpha \cap \Omega'$ with a polygonal domain Ω' that contains $\overline{K}$, preserving the inward-pointing condition (1.15) and the transversality condition **(C)**.

By the basic properties of a patchy vector field, for every solution $t \mapsto x(t)$ of (1.11) the corresponding map $t \mapsto \alpha^*\big(x(t)\big)$ in (1.9) is non-decreasing. Roughly speaking, a trajectory can move from a patch Ω_α to another patch Ω_β only if $\alpha < \beta$. This property no longer holds in the presence of an impulsive perturbation. However, it was shown in [2] that, for a solution $t \mapsto y(t)$ of (1.1), one can slightly modify the impulsive perturbation w, say replacing it by another perturbation $w^{\diamondsuit}$, so that the map $t \mapsto \alpha^*\big(y^{\diamondsuit}(t)\big)$ is monotone along the corresponding trajectory $t \mapsto y^{\diamondsuit}(t)$. Namely, the following holds.

Proposition 2.1. *([2], Prop. 2.2) Let g be a patchy vector field on an open domain $\Omega \subset \mathbb{R}^n$. Then, given any compact set $K \subset \Omega$, there exist constants $C' = C'(K) > 0$, $\delta' = \delta'(K) > 0$, such that the following holds.*

For every BV function $w = w(t)$ with Tot. Var.$\{w\} < \delta'$, and for every solution $y : [0,T] \mapsto K$, of the Cauchy problem (1.1)–(1.2), there is a BV function $w^{\diamondsuit} = w^{\diamondsuit}(t)$ and a left continuous solution $y^{\diamondsuit} : [0,T] \mapsto \Omega$ of

$$\dot{y}^{\diamondsuit} = g(y^{\diamondsuit}) + \dot{w}^{\diamondsuit}, \tag{2.1}$$

so that the map $t \mapsto \alpha^*(y^{\diamondsuit}(t))$ is non-decreasing, and there holds

$$\begin{aligned} \mathit{Tot.Var.}\{w^{\diamondsuit}\} &\le C' \cdot \mathit{Tot.Var.}\{w\}, \\ \big\|y^{\diamondsuit} - y\big\|_{\mathbf{L}^\infty([0,T])} &\le C' \cdot \mathit{Tot.Var.}\{w\}\cdot \end{aligned} \tag{2.2}$$

The next Lemma shows that we can replace the solution $t \mapsto y^{\diamondsuit}(t)$ of (2.1) with a piecewise smooth function $t \mapsto y^{\sharp}(t)$ so that the map $t \mapsto \alpha^*\big(y^{\sharp}(t)\big)$ is still non-decreasing and, for every interval

$$I_\alpha \doteq \big\{t \in [0,T];\ \alpha^*(y^{\sharp}(t)) = \alpha\big\}, \qquad \alpha \in \mathrm{Im}\big(\alpha^* \circ y^{\sharp}\big),$$

$y^{\sharp}\restriction_{I_\alpha}$ is a concatenation of trajectories of (1.11) whose endpoints lie on the edges of the domain

$$D_\alpha \doteq \Omega_\alpha \setminus \bigcup_{\beta > \alpha} \Omega_\beta. \tag{2.3}$$

Lemma 2.2. *Let g be a uniformly bounded away from zero polygonal patchy vector field on $\mathbb{R}^2$, associated to a family of polygonal patches $\big\{(\Omega_\alpha,\ g_\alpha);\ \alpha \in \mathcal{A}\big\}$, and assume that condition* **(C)** *is satisfied. Then, given any compact set $K \subset \mathbb{R}^2$, there exist constants $C'' = C''(K)$, $\delta'' = \delta''(K) > 0$, so that, for every BV function $w = w(t)$ with Tot. Var.$\{w\} < \delta''$, and for every solution $y : [0,T] \mapsto K$, of (1.1), there exists a left continuous, piecewise smooth function $y^{\sharp} : [0,\tau] \mapsto \mathbb{R}^2$ enjoing the properties:*

a′) the map $t \mapsto \alpha^*\big(y^{\sharp}(t)\big)$ is non-decreasing;

b′) if we let

$$\big\{\,\alpha_{i'_1}, \ldots, \alpha_{i'_{m^\sharp}}\,\big\} = \mathit{Im}\big(\alpha^* \circ y^{\sharp}\big), \tag{2.4}$$

with

$$\alpha_{i'_1} < \cdots < \alpha_{i'_{m^\sharp}}, \tag{2.5}$$

and denote $D_{\alpha_{i'_k}}$ *a polygonal domain defined as in (2.3), then, for every interval*

$$]\tau'_k, \tau'_{k+1}] \doteq \Big\{ t \in [0, T] \ : \ y^\sharp(t) \in D_{\alpha_{i'_k}} \Big\},$$

there exists a partition $\tau'_k = t_{k,1} < t_{k,2} < \cdots < t_{k,q_k} = \tau'_{k+1}$, *with* q_k *less or equal to the number of edges of the domain* $D_{\alpha_{i'_k}}$ *in (2.3), so that, on each* $]t_{k,\ell-1}, t_{k,\ell}[$, *the function* $y^\sharp(\cdot)$ *is a classical solution of*

$$\dot y = g_{\alpha_{i'_k}}(y). \tag{2.6)$_k$}$$

and the points $y^\sharp(t_{k,\ell})$, $y^\sharp({t_{k,\ell}}^+)$, $t_{k,\ell} \neq 0, T$, *lie on different edges of the domain* $D_{\alpha_{i'_k}}$;

c′)

$$\sum_{\substack{1\le k\le m^\sharp\\ 1\le \ell<q_k}} \big| y^\sharp({t_{k,\ell}}^+) - y^\sharp(t_{k,\ell}) \big| \le C'' \cdot Tot.Var.\{w\}, \tag{2.7}$$

$$\big\| y^\sharp - y \big\|_{\mathbf{L}^\infty([0,T])} \le C'' \cdot Tot.Var.\{w\}\cdot \tag{2.8}$$

A proof of the above lemma is worked out in Section 4. The next lemma shows that for every piecewise smooth function $y^\flat(\cdot)$ which is a concatenation of trajectories of (1.11) and takes values in a domain D_α as (2.3), there is a solution $x(\cdot)$ of (1.11) whose $\mathbf{L}^\infty$ distance from $y^\flat(\cdot)$ grows linearly with the total amount of jumps in $y^\flat$.

Lemma 2.3. *Given any polygonal domain* D_{α_o} *defined as in (2.3), there exist constants* $\overline{C} = \overline{C}(D_{\alpha_o})$, $\overline{\delta} = \overline{\delta}(D_{\alpha_o}) > 0$, *so that the following hold.*

Let $y^\flat :]\tau_0, \tau_1] \mapsto \mathbb{R}^2$ *be any left-continuous, piecewise smooth function having the properties:*

a″) *the function* $y^\flat(\cdot)$ *is a solution of* $\dot y = g_{\alpha_o}(y)$ *on every interval* $]t'_{\ell-1}, t'_\ell[$ *of a partition* $t'_1 = \tau_0 < t'_2 < \cdots < t'_{q_o} = \tau_1$ *of* $[\tau_0, \tau_1]$, *and one has*

$$y^\flat(t) \in D_{\alpha_o} \qquad \forall\ t \in]t'_{\ell-1}, t'_\ell[\qquad \forall\ \ell. \tag{2.9}$$

Moreover, the points $y^\flat(t'_\ell)$, $y^\flat({t'_\ell}^+)$, $1 < \ell < q_o$, *lie on different edges of the domain* D_{α_o};

b″)

$$\Delta(y^\flat) \doteq \sum_{\ell=2}^{q_o-1} \big| y^\flat({t'_\ell}^+) - y^\flat(t'_\ell) \big| < \overline{\delta}. \tag{2.10}$$

Then, there exist a point $Q_{\alpha_o} = Q_{\alpha_o}(y^\flat) \in \overline{D}_{\alpha_o}$, *and a time* $\sigma_{\alpha_o} = \sigma_{\alpha_o}(y^\flat) > 0$, *so that:*

c″)

$$x^{\alpha_o}\big(t;\ \tau_0, Q_{\alpha_o}\big) \in D_{\alpha_o} \qquad \forall\ t \in]\,\tau_0, \sigma_{\alpha_o}[; \tag{2.11}$$

d″) *if Case a″-1) occurs then* $Q_{\alpha_o}, x^{\alpha_o}\big(\sigma_{\alpha_o};\ \tau_0, Q_{\alpha_o}\big) \in \partial D_{\alpha_o}$, *if Case a″-2) occurs then* $Q_{\alpha_o} \in \partial D_{\alpha_o}$, *if Case a″-3) occurs then* $x^{\alpha_o}\big(\sigma_{\alpha_o};\ \tau_0, Q_{\alpha_o}\big) \in \partial D_{\alpha_o}$;

e″)

$$\big| \sigma_{\alpha_o} - \tau_1 \big| \le \overline{C} \cdot \Delta(y^\flat), \tag{2.12}$$

$$\big| x^{\alpha_o}\big(t;\ \tau_0, Q_{\alpha_o}\big) - y^\flat(t) \big| \le \overline{C} \cdot \Delta(y^\flat) \qquad \forall\ t \in]\,\tau_0, \min\{\tau_1, \sigma_{\alpha_o}\}]. \tag{2.13}$$

Also the proof of the above Lemma is produced in Section 4. Relying on Lemmas 2.2–2.3 we are now in the position to show that, for every solution $t \mapsto y(t)$ of the perturbed system (1.1), we can find a piecewise smooth map $t \mapsto y^\diamondsuit(t)$ that is a concatenation of solutions of the unperturbed system (1.11) and whose $\mathbf{L}^\infty$ distance from $y(\cdot)$ is of the same order of magnitude as the impulsive term $\dot w$.

Proposition 2.4. *In the same setting of Lemma 2.2, given any compact set $K \subset \mathbb{R}^2$, there exist constants $C''' = C'''(K)$, $\delta''' = \delta'''(K) > 0$, so that, for every BV function $w = w(t)$ with Tot.Var.$\{w\} < \delta'''$, and for every solution $y : [0,T] \mapsto K$, of (1.1), there exists a left continuous, piecewise smooth function $y^\diamondsuit : [0,T] \mapsto \mathbb{R}^2$ with the following properties:*

a''') the map $t \mapsto \alpha^*\big(y^\diamondsuit(t)\big)$ is non-decreasing, and one has $y^\diamondsuit(0) = y^\diamondsuit(0^+)$;

b''') if we let $\alpha_{i'_1} < \cdots < \alpha_{i'_{m^\diamondsuit}}$ denote the indices defined for $y^\diamondsuit(\cdot)$ in the same way as for $y^\sharp(\cdot)$ in (2.4)–(2.5), and let $D_{\alpha_{i'_k}}$ denote a polygonal domain defined as in (2.3), then, on every interval

$$]\tau'_k,\, \tau'_{k+1}] \doteq \Big\{t \in [0,\, T] \ : \ y^\diamondsuit(t) \in D_{\alpha_{i'_k}}\Big\}, \tag{2.14}$$

the function $y^\diamondsuit(\cdot)$ is a classical solution of

$$\dot y = g_{\alpha_{i'_k}}(y). \tag{2.15$_k$}$$

Moreover, one has

$$y^\diamondsuit(\tau'_k) \in \partial D_{\alpha_{i'_{k-1}}}, \qquad y^\diamondsuit(\tau'^+_k) \in \partial D_{\alpha_{i'_k}} \qquad \forall\ 1 < k \le m^\diamondsuit; \tag{2.16}$$

c''')

$$\sum_{k=1}^{m^\diamondsuit} \big|y^\diamondsuit({\tau'_k}^+) - y^\diamondsuit(\tau'_k)\big| \le C''' \cdot Tot.Var.\{w\}, \tag{2.17}$$

$$\big\|y^\diamondsuit - y\big\|_{\mathbf{L}^\infty([0,T])} \le C''' \cdot Tot.Var.\{w\}\cdot \tag{2.18}$$

Proof. Fix a compact set K and, letting $C'' = C''(K)$, $\delta'' = \delta''(K)$ be the constants provided by Lemma 2.2, set $K'' \doteq B(K,\, C'' \cdot \delta'')$. Observe that, thanks to Lemma 2.2, in order to establish Proposition 2.4 it will be sufficient to show that there exist constants $C''' = C'''(K'')$, $\delta''' = \delta'''(K'') > 0$ so that the following holds. Given any piecewise smooth function $y^\sharp : [0,T] \mapsto K''$, enjoing properties a'), b'), c') stated in Lemma 2.2, and satisfying the condition

$$\Delta(y^\sharp) \doteq \sum_{substack} 1 \le k \le m^\sharp 1 \le \ell < q_k \big|y^\sharp({t_{k,\ell}}^+) - y^\sharp(t_{k,\ell})\big| < \delta''', \tag{2.19}$$

there exists a piecewise smooth function $y^\diamondsuit : [0,T] \mapsto \mathbb{R}^2$ having the properties a'''), b'''), and satisfying the estimate

$$\big\|y^\diamondsuit - y^\sharp\big\|_{\mathbf{L}^\infty([0,T])} \le C''' \cdot \Delta(y^\sharp). \tag{2.20}$$

To this purpose, let $\big\{\Omega_{\alpha_i} \ : \ i = 1, \ldots, N\big\}$ be the collection of polygonal domains that intersect K'', set

$$M \doteq \sup\big\{|g_{\alpha_i}(y)| \ : \ y \in \Omega_{\alpha_i}, \quad i = 1, \ldots, N\big\}, \tag{2.21}$$

and choose constants $\overline{C},\ \overline{\delta} > 0$ so that the conclusions of Lemma 2.3 hold for any piecewise smooth function $y^\sharp$ enjoing properties a''), b''), that takes values in a domain D_{α_i}, $i = 1, \ldots, N$. Now, consider a piecewise smooth function $y^\sharp : [0,T] \mapsto K''$, having the properties a'), b'), c') stated in Lemma 2.2, and satisfying (2.19) with

$$\delta''' = \overline{\delta}. \tag{2.22}$$

Let $0 = \tau'_1 < \tau'_2 < \cdots < \tau'_{m^\sharp+1} = T$, $m^\sharp \leq N$, be the partition of $[0,T]$ induced by $y^\sharp(\cdot)$ according with property b'), and observe that every restriction map $y^\sharp\lceil_{]\tau'_k,\,\tau'_{k+1}]}$, $1 \leq k \leq m^\sharp$, is a piecewise smooth function that enjoys the properties a''), b'') stated in Lemma 2.3. Let

$$\sigma_{\alpha_{i'_k}} \doteq \sigma_{\alpha_{i'_k}}\big(y^\sharp\lceil_{]\tau'_k,\,\tau'_{k+1}]}\big), \qquad Q_{\alpha_{i'_k}} \doteq Q_{\alpha_{i'_k}}\big(y^\sharp\lceil_{]\tau'_k,\,\tau'_{k+1}]}\big), \qquad 1 \leq k \leq m^\sharp,$$

be the points and times having the properties c''), d''), e'') given by Lemma 2.3. Then, consider the sequence of points $\tau''_1 \doteq 0 < \tau''_2 < \cdots < \tau''_{m''+1} \leq T$, $m'' \leq m^\sharp$, recursively defined by setting

$$\tau''_{k+1} \doteq \tau''_k - \tau'_k + \sigma_{\alpha_{i'_k}}, \tag{2.23}$$

for all $1 \leq k \leq m^\sharp$ such that $\tau''_k - \tau'_k + \sigma_{\alpha_{i'_k}} < T$, and then letting

$$\begin{aligned} m'' &\doteq \max\big\{1 < k \leq m^\sharp \;:\; \tau''_k - \tau'_k + \sigma_{\alpha_{i'_k}} < T\big\}, \\ \tau''_{m''+1} &\doteq \min\big\{T,\ \tau''_{m''} - \tau'_{m''} + \sigma_{\alpha_{i'_{m''}}}\big\}. \end{aligned} \tag{2.24}$$

Next, letting $x^g\big(t;\, t_0,\, x_0\big)$ denote a solution of (1.11) starting from x_0 at time t_0, define the map $y^\diamondsuit : [0,T] \mapsto \mathbb{R}^2$ as follows: $y^\diamondsuit(0) \doteq Q_{\alpha_{i'_1}}$, and

$$y^\diamondsuit(t) \doteq \begin{cases} x^{\alpha_{i'_k}}\big(t + \tau'_k - \tau''_k;\ \tau'_k,\, Q_{\alpha_{i'_k}}\big) & \forall\, t \in\,]\tau''_k,\, \tau''_{k+1}], \quad 1 \leq k \leq m'', \\ x^g\big(t;\, \tau''_{m''+1},\, y^\diamondsuit(\tau''_{m''+1})\big) & \forall\, t \in\,]\tau''_{m''+1},\, T]. \end{cases} \tag{2.25}$$

By construction, the properties c''), d'') of $\sigma_{\alpha_{i'_k}}$, $Q_{\alpha_{i'_k}}$ given by Lemma 2.3, together with the general properties of the solutions of a patchy system (recalled in Sect. 1), guarantee that the map $t \mapsto y^\diamondsuit(t)$ enjoys the properties a'''), b''') stated in Proposition 2.4. Moreover, observe that by property e'') of Lemma 2.3 one has

$$\left.\begin{aligned} \big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| &\leq \overline{C} \cdot \Delta(y^\sharp), \\ \big|x^{\alpha_{i'_k}}\big(t;\ \tau'_k,\, Q_{\alpha_{i'_k}}\big) - y^\sharp(t)\big| &\leq \overline{C} \cdot \Delta(y^\sharp) \qquad \forall\, t \in\,]\,\tau'_k,\ \min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\}], \end{aligned}\right. \qquad \forall\, 1 \leq k \leq m^\sharp. \tag{2.26}$$

Thanks to (2.26), and since by definition (2.23)–(2.24) one has

$$\big|\tau''_{k+1} - \tau'_{k+1}\big| \leq \big|\tau''_k - \tau'_k\big| + \big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| \qquad \forall\, 1 \leq k < m'',$$

proceeding by induction on $k \geq 1$, we derive

$$\big|\tau''_{k+1} - \tau'_{k+1}\big| \leq k \cdot \overline{C} \cdot \Delta(y^\sharp) \qquad \forall\, 1 \leq k < m''. \tag{2.27}$$

On the other hand, using (2.22) and (2.26), and relying on property b') of $y^\sharp(\cdot)$, we obtain

$$\begin{aligned}\big|y^\diamond\big({\tau''_{k+1}}^+\big) - y^\diamond(\tau''_{k+1})\big| &= \big|Q_{\alpha_{i'_{k+1}}} - x^{\alpha_{i'_k}}\big(\sigma_{\alpha_{i'_k}};\ \tau'_k,\, Q_{\alpha_{i'_k}}\big)\big| \\ &\leq \big|Q_{\alpha_{i'_{k+1}}} - y^\sharp\big({\tau'_{k+1}}^+\big)\big| + \big|y^\sharp\big({\tau'_{k+1}}^+\big) - y^\sharp(\tau'_{k+1})\big| + \\ &\quad + \big|y^\sharp(\tau'_{k+1}) - y^\sharp\big(\min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\}\big)\big| + \\ &\quad + \big|y^\sharp\big(\min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\}\big) - x^{\alpha_{i'_k}}\big(\min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\};\ \tau'_k,\, Q_{\alpha_{i'_k}}\big)\big| + \\ &\quad + \big|x^{\alpha_{i'_k}}\big(\min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\};\ \tau'_k,\, Q_{\alpha_{i'_k}}\big) - x^{\alpha_{i'_k}}\big(\sigma_{\alpha_{i'_k}};\ \tau'_k,\, Q_{\alpha_{i'_k}}\big)\big| \\ &\leq \overline{C}\cdot\Delta(y^\sharp) + \Delta(y^\sharp) + M\cdot\big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| + \overline{C}\cdot\Delta(y^\sharp) + M\cdot\big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| \\ &\leq \big(1 + 2\overline{C}(1+M)\big)\cdot\Delta(y^\sharp), \\ &\qquad\qquad \forall\ 1 \leq k < m''. \end{aligned} \tag{2.28}$$

Hence, thanks to (2.22), (2.26)–(2.28), and by definition (2.25) of $y^\diamond(\cdot)$, we derive

$$\begin{aligned}\big|y^\diamond(t) - y^\sharp(t)\big| &\leq \big|y^\diamond(t) - y^\diamond\big(t - \tau'_k + \tau''_k\big)\big| + \big|y^\diamond\big(t - \tau'_k + \tau''_k\big) - y^\sharp(t)\big| \\ &\leq \sum_{k=2}^{m''}\big|y^\diamond\big({\tau''_k}^+\big) - y^\diamond(\tau''_k)\big| + M\cdot\big|\tau''_k - \tau'_k\big| + \big|x^{\alpha_{i'_k}}\big(t;\ \tau'_k,\, Q_{\alpha_{i'_k}}\big) - y^\sharp(t)\big| \\ &\leq N\cdot\big(1 + 3\overline{C}(1+M)\big)\cdot\Delta(y^\sharp), \\ &\qquad \forall\ t \in\ ]\,\tau'_k,\ \min\{\tau'_{k+1}, \sigma_{\alpha_{i'_k}}\}], \qquad 1 \leq k \leq m'', \end{aligned} \tag{2.29}$$

while, in the case $\sigma_{\alpha_{i'_k}} < \tau'_{k+1}$, we get

$$\begin{aligned}\big|y^\diamond(t) - y^\sharp(t)\big| &\leq \big|y^\diamond(t) - y^\diamond\big(\sigma_{\alpha_{i'_k}}\big)\big| + \big|y^\diamond\big(\sigma_{\alpha_{i'_k}}\big) - y^\sharp\big(\sigma_{\alpha_{i'_k}}\big)\big| + \big|y^\sharp(t) - y^\sharp\big(\sigma_{\alpha_{i'_k}}\big)\big| \\ &\leq \sum_{k=2}^{m''}\big|y^\diamond\big({\tau''_k}^+\big) - y^\diamond(\tau''_k)\big| + M\cdot\big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| + \big|y^\diamond\big(\sigma_{\alpha_{i'_k}}\big) - y^\sharp\big(\sigma_{\alpha_{i'_k}}\big)\big| + \\ &\quad + \Delta(y^\sharp) + M\cdot\big|\sigma_{\alpha_{i'_k}} - \tau'_{k+1}\big| \\ &\leq 3N\cdot\big(1 + 3\overline{C}(1+M)\big)\cdot\Delta(y^\sharp), \\ &\qquad \forall\ t \in\ ]\,\sigma_{\alpha_{i'_k}},\ \tau'_{k+1}], \qquad 1 \leq k \leq m''. \end{aligned} \tag{2.30}$$

Thus, (2.29)–(2.30) together, yield

$$\big|y^\diamond(t) - y^\sharp(t)\big| \leq 3N\cdot\big(1 + 3\overline{C}(1+M)\big)\cdot\Delta(y^\sharp) \qquad \forall\ t \in\ [0,\ \tau'_{m''+1}]. \tag{2.31}$$

On the other hand, in the case $\tau'_{m''+1} < T$, by definition (2.24) one has $m'' = m^\sharp$, $T < \tau''_{m''} - \tau'_{m''} + \sigma_{\alpha_{i'_{m''}}}$, and hence, using (2.26)–(2.27), we get

$$\begin{aligned}\big|T - \tau'_{m''+1}\big| &\leq \big|\tau''_{m''} - \tau'_{m''} + \sigma_{\alpha_{i'_{m''}}} - \tau'_{m''+1}\big| \\ &\leq \big|\tau''_{m''} - \tau'_{m''}\big| + \big|\sigma_{\alpha_{i'_{m''}}} - \tau'_{m''+1}\big| \\ &\leq (N+1)\cdot\overline{C}\cdot\Delta(y^\sharp). \end{aligned} \tag{2.32}$$

Therefore, from (2.28), (2.31)–(2.32) we derive

$$\begin{aligned}
\big|y^{\diamondsuit}(t)-y^{\sharp}(t)\big| &\le \big|y^{\diamondsuit}(t)-y^{\diamondsuit}(\tau'_{m''+1})\big| + \big|y^{\diamondsuit}(\tau'_{m''+1})-y^{\sharp}(\tau'_{m''+1})\big| + \big|y^{\sharp}(t)-y^{\sharp}(\tau'_{m''+1})\big| \\
&\le \sum_{k=2}^{m''} \big|y^{\diamondsuit}(\tau''^{\,+}_{k})-y^{\diamondsuit}(\tau''_{k})\big| + M\cdot\big|T-\tau'_{m''+1}\big| + \\
&\quad + \big|y^{\diamondsuit}(\tau'_{m''+1})-y^{\sharp}(\tau'_{m''+1})\big| + \Delta(y^{\sharp}) + M\cdot\big|T-\tau'_{m''+1}\big| \\
&\le 4N\cdot\big(1+4\overline{C}(1+M)\big)\cdot\Delta(y^{\sharp}) \\
&\qquad\qquad \forall\, t\in\ [\tau'_{m''+1},\, T].
\end{aligned} \tag{2.33}$$

Hence, (2.32)–(2.33) together show that $y^{\diamondsuit}(\cdot)$ satisfies the estimates (2.20) choosing

$$C''' > 4N\cdot\big(1+4\overline{C}(1+M)\big), \tag{2.34}$$

which completes the proof of Proposition 2.4. □

3. Proof of Theorem 1

In view of Proposition 2.4, it is useful to introduce the following

Definition 3.1. A left-continuous, piecewise smooth function $y^{\diamondsuit} : [0,T] \mapsto \mathbb{R}^2$ that enjoys the properties $a''')$–$b''')$ stated in Proposition 2.4, is called a *concatenation of classical solutions (CCS) of* (1.11).

Notice that, in particular, any Carathéodory solution of (1.11) is always a CCS. In connection with any CCS $y : [0,T] \mapsto \mathbb{R}^2$, letting $0=\tau_1<\tau_2<\cdots<\tau_{m+1}=T$ be the partition of $[0,\, T]$ defined as in (2.4)–(2.5), we will denote the total amount of jumps in $y(\cdot)$ by

$$\Delta(y) \doteq \sum_{k=2}^{m} \big|y({\tau_k}^{+})-y(\tau_k)\big|. \tag{3.1}$$

Throughout this section, we shall work with CCS of (1.11) $y : [t_0,\, t_1] \mapsto \mathbb{R}^2$ that take values in some neighborhood

$$K_0 \doteq B(K,\delta_0), \quad \delta_0>0, \tag{3.2}$$

of a fixed compact set $K\subset\mathbb{R}^2$, and we shall adopt the following further notations. Consider the set of indices

$$\mathcal{A}_{K_0} \doteq \big\{\alpha\in\mathcal{A};\ \Omega_\alpha\cap K_0\neq\emptyset\big\}\cdot \tag{3.3}$$

Let $N=|\mathcal{A}_{K_0}|$ be the number of elements in $\mathcal{A}_{K_0}$, and set

$$M \doteq \sup\big\{|g_\alpha(y)|\ :\ y\in\Omega_\alpha, \qquad \alpha\in\mathcal{A}_{K_0}\big\}\cdot \tag{3.4}$$

Before giving the complete proof of Theorem 1 we will first show that, for every given CCS of (1.11) $y(\cdot)$ admitting a single jump discontinuity, there exists a (Carathéodory) solution $x(\cdot)$ of (1.11) for which the linear estimate (1.16) holds. Namely, we shall prove

Proposition 3.1. *Let g be a uniformly bounded away from zero polygonal patchy vector field on $\mathbb{R}^2$, associated to a family of polygonal patches $\big\{(\Omega_\alpha,\ g_\alpha);\ \alpha\in\mathcal{A}\big\}$, and assume that condition* **(C)** *is satisfied. Then, given any compact set $K\subset\mathbb{R}^2$, there exist constants $C^{\text{IV}}=C^{\text{IV}}(K)$, $\delta^{\text{IV}}=\delta^{\text{IV}}(K)>0$ so that the following hold.*

Let $y^\flat : [\tau_0, \tau_1] \mapsto K$, $y^\natural :]\tau_1, \tau_2] \mapsto K$, *be two continuous maps having the properties:*

i) the function $y^\flat(\cdot)$ *is a Carathéodory solution of (1.11) and, letting*

$$\alpha_1 \doteq \max\big\{\alpha\ ;\ \alpha \in Im(\alpha^* \circ y^\flat)\big\},$$

one has

$$y^\flat(\tau_1) \in \partial D_{\alpha_1} \tag{3.5}$$

(where D_{α_1} *denotes a polygonal domain defined as in (2.3));*

ii) the function $y^\natural(\cdot)$ *is a solution of* $\dot y = g_{\alpha_2}(y)$, *for some* $\alpha_2 > \alpha_1$, *and one has*

$$\begin{aligned} y^\natural(t) &\in D_{\alpha_2} \qquad \forall\, t \in\,]\tau_1, \tau_2], \\ y^\natural({\tau_1}^+) &\in \partial D_{\alpha_2}; \end{aligned} \tag{3.6}$$

iii)

$$\big|y^\flat(\tau_1) - y^\natural({\tau_1}^+)\big| < \delta^{iv}. \tag{3.7}$$

Then, there exists a Carathéodory solution of (1.11) $\Phi_{\flat,\natural} \doteq \Phi\big[y^\flat, y^\natural\big] : [\tau_0, \sigma_{\flat,\natural}] \longmapsto \mathbb{R}^2$, *such that*

$$\big|y^\flat(t) - \Phi_{\flat,\natural}(t)\big| \le C^{iv} \cdot \big|y^\flat(\tau_1) - y^\natural({\tau_1}^+)\big| \qquad \forall\, t \in\, [\tau_0, \min\{\tau_1, \sigma_{\flat,\natural}\}], \tag{3.8}$$

$$\big|y^\natural(t) - \Phi_{\flat,\natural}(t)\big| \le C^{iv} \cdot \big|y^\flat(\tau_1) - y^\natural({\tau_1}^+)\big| \qquad \forall\, t \in\,]\tau_1, \min\{\tau_2, \sigma_{\flat,\natural}\}], \qquad \text{if} \quad \sigma_{\flat,\natural} > \tau_1. \tag{3.9}$$

Moreover, one has

$$\begin{aligned} \alpha^*\big(\Phi_{\flat,\natural}(t)\big) \le \alpha_2 \qquad &\forall\, t \in [\tau_0, \sigma_{\flat,\natural}], \\ y^\natural(\tau_2) \in \partial D_{\alpha_2} \qquad \Longrightarrow \qquad &\Phi_{\flat,\natural}(\sigma_{\flat,\natural}) \in \partial D_{\alpha_2}, \end{aligned} \tag{3.10}$$

and there holds

$$\big|\sigma_{\flat,\natural} - \tau_2\big| \le C^{iv} \cdot \big|y^\flat(\tau_1) - y^\natural({\tau_1}^+)\big|. \tag{3.11}$$

Proof.

1. Fix a compact set $K \subset \mathbb{R}^2$, let K_0 be the neighborhood of K in (3.2), and denote with $\mathcal{A}_K$, $\mathcal{A}_{K_0}$ the sets of indices defined as in (3.3) in connection with K and K_0. For each $\alpha \in \mathcal{A}_{K_0}$ denote by V_α the set of vertices of the polygonal domain D_α. Notice that, since g is a patchy vector field satisfying condition **(C)**, the Cauchy problem (1.11)–(1.2) has a unique local forward (Caratéodory) solution in the case $y_0 \in \big(\bigcup_{\alpha\in\mathcal{A}_K} \Omega_\alpha\big) \setminus \big(\bigcup_{\alpha\in\mathcal{A}_K} V_\alpha\big)$, and at most N local forward solutions if $y_0 \in \bigcup_{\alpha\in\mathcal{A}_K} V_\alpha$. On the other hand, by the properties of the solutions of a patchy system recalled in Section 1, the Cauchy problem for (1.11) has always backward uniqueness. Therefore, the set $\mathcal{T}_\alpha$ of all graphs of maximal (Caratéodory) trajectories of (1.11) that go through some vertex in $\cup_{\alpha\in\mathcal{A}_K} V_\alpha$, and are contained in $\cup_{\alpha\in\mathcal{A}_{K_0}} \overline{\Omega}_\alpha$, is finite. For convenience, with a slight abuse of notation, we will often write $\gamma \in \mathcal{T}_\alpha$ to mean $\mathrm{Im}(\gamma) \in \mathcal{T}_\alpha$. The unique backward solution of the Cauchy problem (1.11)–(1.2), whenever does exist, will be denoted by $t \mapsto x^g(t;\, t_0,\, y_0)$, $t \le t_0$. We assume that every vector field g_α is defined on a neighborhood of $\overline{\Omega}_\alpha$ and we denote, as usual, by $t \mapsto x^\alpha\big(t;\, t_0,\, x_0\big)$ the solution of the Cauchy problem

$$\dot x = g_\alpha(x), \qquad x(t_0) = x_0 \in \overline{\Omega}_\alpha. \tag{3.12}$$

By well-posedness of (3.12), there will be some constant $c_0 > 1$ so that

$$\big|x^\alpha(t;\, t_0,\, x_0) - x^\alpha(t';\, t_1,\, x_1)\big| \le c_0\big\{|t - t'| + |t_0 - t_1| + |x_0 - x_1|\big\} \qquad \forall\, x_0,\, x_1 \in \overline{\Omega}_\alpha. \tag{3.13}$$

For every $x_0 \in \partial D_\alpha$, we let $t_\alpha^+(x_0)$, $t_\alpha^-(x_0)$ denote the time that is necessary to reach the set $\overline{\Omega}_\alpha \setminus D_\alpha$ starting from x_0 and following, respectively, the forward and backward flow of the vector field g_α, *i.e.*

$$\begin{aligned} t_\alpha^+(x_0) &\doteq \inf\big\{t > 0\ ;\ x^\alpha(t;\,0,\,x_0) \in \overline{\Omega}_\alpha \setminus D_\alpha\big\},\\ t_\alpha^-(x_0) &\doteq \sup\big\{t < 0\ ;\ x^\alpha(t;\,0,\,x_0) \in \overline{\Omega}_\alpha \setminus D_\alpha\big\}. \end{aligned} \tag{3.14}$$

Using the quantities in (3.14) we define the sets of *incoming* and *outgoing boundary points*

$$\begin{aligned} \partial_{\mathcal{I}} D_\alpha &\doteq \big\{x \in \partial D_\alpha\ ;\ t_\alpha^+(x) > 0\big\},\\ \partial_{\mathcal{O}} D_\alpha &\doteq \big\{x \in \partial D_\alpha\ ;\ t_\alpha^-(x) < 0\big\}, \end{aligned} \tag{3.15}$$

that clearly consist of all the points in the boundary ∂D_α where the field g_α is pointing, respectively, towards the interior and towards the exterior of D_α. Moreover, for any pair of indices $\alpha, \beta \in \mathcal{A}_K$, $\alpha < \beta$, define the set

$$G_{\alpha,\,\beta} \doteq \partial_{\mathcal{O}} D_\alpha \cap \partial_{\mathcal{I}} D_\beta \cap \Big(V_\alpha \cup V_\beta \cup \bigcup_{\gamma\in\mathcal{T}_\beta} \mathrm{Im}(\gamma)\Big). \tag{3.16}$$

Since g_α are smooth, uniformly bounded away from zero vector fields that satisfy the inward-pointing condition (1.17) and the transversality condition **(C)**, and by the properties of the solutions of a patchy system recalled in Section 1, one can easily verify that the following properties hold.

P1) *There exist constants c_1, $\delta_1 > 0$ (depending only on K) so that, given any domain D_α, $\alpha \in \mathcal{A}_K$, one has:*

a) for any x, $y \in \partial_{\mathcal{I}} D_\alpha$ belonging to the same connected component of $\partial D_\alpha \setminus \bigcup_{\gamma\in\mathcal{T}_\alpha} Im(\gamma)$, there holds

$$|x-y| < \delta_1 \qquad \Longrightarrow \qquad \big|t_\alpha^+(x) - t_\alpha^+(y)\big| < c_1 \cdot |x-y|; \tag{3.17}$$

b) given any $x_0 \in \partial_{\mathcal{I}} D_\alpha \cap \bigcup_{\gamma\in\mathcal{T}_\alpha} Im(\gamma)$, and any $x \in \partial_{\mathcal{I}} D_\alpha \cap B(x_0,\,\delta_1)$, $x \neq x_0$, there exists $\widehat{\tau}^\alpha_{x_0,x} \in [0,\, t_\alpha^+(x_0)]$ such that

$$x^\alpha\big(\widehat{\tau}^\alpha_{x_0,x};\,0,\,x_0\big) \in \partial D_\alpha, \qquad \big|t_\alpha^+(x) - \widehat{\tau}^\alpha_{x_0,x}\big| < c_1 \cdot |x - x_0|. \tag{3.18}$$

P2) *There exist constants c_2, $\delta_2 > 0$ (depending only on K_0) so that, given any pair of domains D_α, D_β, $\alpha < \beta$, $\alpha, \beta \in \mathcal{A}_K$, with $\partial_{\mathcal{O}} D_\alpha \cap \partial_{\mathcal{I}} D_\beta \neq \emptyset$, the following holds. For every $x_0 \in G_{\alpha,\,\beta}$, and for any backward (Caratéodory) trajectory $t \mapsto x^g(t;\,0,\,x)$, $t \in [-\widehat{\tau}_x,\,0]$ of (1.11) arriving in some point $x \in \partial_{\mathcal{O}} D_\alpha \cap B(x_0,\,\delta_2)$, starting from $x^g(-\widehat{\tau}_x;\,0,\,x) \in \bigcup_{\gamma\le\alpha} \partial D_\gamma$, and contained in K, there is another backward trajectory $t \mapsto x^g(t;\,0,\,x_0)$, $t \in [-\widehat{\tau}^g_{x_0,x},\,0]$ of (1.5) arriving in x_0, contained in K_0, and such that*

$$x^g\big(-\widehat{\tau}^g_{x_0,x};\,0,\,x_0\big) \in \bigcup_{\gamma\le\alpha} \partial D_\gamma, \tag{3.19}$$

$$\big|\widehat{\tau}^g_{x_0,x} - \widehat{\tau}_x\big| < c_2 \cdot |x - x_0|, \tag{3.20}$$

$$\big|x^g\big(t - \widehat{\tau}^g_{x_0,x};\,0,\,x_0\big) - x^g\big(t - \widehat{\tau}_x;\,0,\,x\big)\big| < c_2 \cdot |x - x_0| \qquad \forall\ t \in [0,\ \min\{\widehat{\tau}^g_{x_0,x},\,\widehat{\tau}_x\}]. \tag{3.21}$$

Next, choose $\overline{\lambda} > 0$ so that, for any pair of indices $\alpha, \beta \in \mathcal{A}_K$, $\alpha < \beta$, one has

$$B(x_0,\,\overline{\lambda}\,) \cap B(y_0,\,\overline{\lambda}\,) = \emptyset \qquad \forall\ x_0,\,y_0 \in G_{\alpha,\beta}, \qquad x_0 \neq y_0. \tag{3.22}$$

Then, for any $0 \le \lambda \le \overline{\lambda}$, let $R^1_{\alpha,\beta}(\lambda), \ldots, R^{r_{\alpha,\beta}}_{\alpha,\beta}(\lambda)$ denote the connected components of

$$\big(\partial D_\alpha \cup \partial D_\beta\big) \setminus \bigcup_{P\in G_{\alpha,\,\beta}} B(P,\,\lambda),$$

and set

$$\rho_{\alpha,\beta}(\lambda) \doteq \begin{cases} \min\Big\{d\big(R^s_{\alpha,\beta}(\lambda),\ R^\ell_{\alpha,\beta}(\lambda)\big)\ \ ;\ 1 \le s,\ell \le r_{\alpha,\beta}, \quad s \ne \ell\Big\} & \text{if} \quad \partial_{\mathcal{O}} D_\alpha \cap \partial_{\mathcal{I}} D_\beta \ne \emptyset, \\ d\big(\partial_{\mathcal{O}} D_\alpha,\ \partial_{\mathcal{I}} D_\beta\big) & \text{otherwise.} \end{cases} \tag{3.23}$$

Since by construction one has

$$\inf\left\{\frac{\rho_{\alpha,\beta}(\lambda)}{\lambda}\ :\ 0 < \lambda \le \overline{\lambda}\right\} > 0 \qquad \forall\ \alpha,\beta \in \mathcal{A}_K, \quad \alpha < \beta, \qquad \text{s.t.} \quad \partial_{\mathcal{O}} D_\alpha \cap \partial_{\mathcal{I}} D_\beta \ne \emptyset, \tag{3.24}$$

there will be constants $c_3 > 1,\ 0 < \delta_3 < \overline{\lambda}/(2c_3)$, so that

$$\rho_{\alpha,\beta}(c_3 \cdot \delta) > 2\delta \qquad \forall\ 0 < \delta \le \delta_3, \qquad \forall\ \alpha,\beta \in \mathcal{A}_K, \quad \alpha < \beta. \tag{3.25}$$

2. Consider now two continuous maps $y^\flat :]\tau_0,\tau_1] \mapsto K$, $y^\natural :]\tau_1,\tau_2] \mapsto K$, having the properties *i)–iii)* with

$$\delta^{\text{IV}} \le \min\left\{\frac{\delta_1}{2c_3},\ \frac{\delta_2}{2c_3},\ \delta_3\right\}. \tag{3.26}$$

To fix the ideas, we shall assume also that

$$y^\flat(\tau_0{}^+) \in \bigcup_{\gamma \le \alpha_1} \partial D_\gamma, \qquad y^\natural(\tau_2) \in \partial D_{\alpha_2}. \tag{3.27}$$

The cases where $y^\flat(\tau_0{}^+) \in \bigcup_{\gamma\le\alpha_1} \overset{\circ}{D}_\gamma$, or $y^\natural(\tau_2) \in \overset{\circ}{D}_{\alpha_2}$, can be treated in entirely similar manner. Set

$$x^\flat \doteq y^\flat(\tau_1), \qquad x^\natural \doteq y^\natural(\tau_1{}^+), \qquad \Delta \doteq \big|x^\flat - x^\natural\big|, \tag{3.28}$$

and observe that, since the properties i)-ii) of $y^\flat(\cdot)$, $y^\natural(\cdot)$ imply

$$x^\flat \in \partial_{\mathcal{O}} D_{\alpha_1}, \qquad x^\natural \in \partial_{\mathcal{I}} D_{\alpha_2},$$

by the definition (3.23) of ρ_{α_1,α_2} and because of (3.7), (3.25) and (3.26), we deduce that, if $\partial_{\mathcal{O}} D_{\alpha_1} \cap \partial_{\mathcal{I}} D_{\alpha_2} = \emptyset$, then

$$\begin{aligned} \Delta &\ge d\big(\partial_{\mathcal{O}} D_{\alpha_1},\ \partial_{\mathcal{I}} D_{\alpha_2}\big) = \rho_{\alpha_1,\alpha_2}(c_3 \cdot \Delta) \\ &> 2\Delta \end{aligned}$$

which yields a contradiction. Therefore, it must be $\partial_{\mathcal{O}} D_{\alpha_1} \cap \partial_{\mathcal{I}} D_{\alpha_2} \ne \emptyset$ which, in turn, by definition (3.16) implies $G_{\alpha_1,\alpha_2} \ne \emptyset$. In order to construct the Carathéodory solution of (1.11) $\Phi_{\flat,\natural}$ satisfying (3.8)–(3.11), we will handle separately the case in which the endpoints $x^\flat$, $x^\natural$ lie on the same connected component of

$$\big(\partial D_{\alpha_1} \cup \partial D_{\alpha_2}\big) \setminus \bigcup_{x_0 \in G_{\alpha_1,\alpha_2}} B(x_0,\ c_3 \cdot \Delta), \tag{3.29}$$

and the case where $x^\flat$, $x^\natural$ belong to the ball $B(x_0,\ c_3 \cdot \Delta)$ centered at some point $x_0 \in G_{\alpha_1,\alpha_2}$.

3. Case 1. Assume that

$$x^\flat \in \partial D_{\alpha_1} \setminus \bigcup_{x_0 \in G_{\alpha_1,\alpha_2}} B(x_0,\, c_3 \cdot \Delta),$$
$$x^\natural \in \partial D_{\alpha_2} \setminus \bigcup_{x_0 \in G_{\alpha_1,\alpha_2}} B(x_0,\, c_3 \cdot \Delta), \tag{3.30}$$

and let $R^s_{\alpha_1,\alpha_2}(c_3 \cdot \Delta)$, $R^\ell_{\alpha_1,\alpha_2}(c_3 \cdot \Delta)$ be the connected components of the set in (3.29) that contain, respectively, $x^\flat$ and $x^\natural$. Observe that, if $s \neq \ell$, then, by the definition (3.23) of ρ_{α_1,α_2} and because of (3.7), (3.25) and (3.26), we deduce

$$\begin{aligned}\Delta &\geq d\big(R^s_{\alpha_1,\alpha_2}(c_3 \cdot \Delta),\ R^\ell_{\alpha_1,\alpha_2}(c_3 \cdot \Delta)\big)\\ &\geq \rho_{\alpha_1,\alpha_2}(c_3 \cdot \Delta)\\ &> 2\Delta\end{aligned}$$

which yields a contradiction. Therefore it must be $s = \ell$, *i.e.* $x^\flat$, $x^\natural$ lie on the same connected component of the set in (3.29) and hence one has

$$x^\flat,\, x^\natural \in R^s_{\alpha_1,\alpha_2}(c_3 \cdot \Delta) \subset \partial_{\mathcal{O}} D_{\alpha_1} \cap \partial_{\mathcal{I}} D_{\alpha_2}.$$

But then, since (3.7) and (3.26) together imply $\Delta < \delta_1$, applying property **P1**-*a)* we derive

$$\big|t^+_{\alpha_2}(x^\flat) - t^+_{\alpha_2}(x^\natural)\big| < c_1 \cdot \Delta. \tag{3.31}$$

On the other hand, since $y^\natural(\cdot)$ satisfies property *ii*), and because (3.30) implies

$$x^\natural \in \partial_{\mathcal{I}} D_{\alpha_2} \setminus \bigcup_{\gamma \in \mathcal{T}_{\alpha_\in}} \mathrm{Im}(\gamma), \tag{3.32}$$

using also $(3.27)_2$ we deduce

$$t^+_{\alpha_2}(x^\natural) = \tau_2 - \tau_1, \tag{3.33}$$

$$y^\natural(t) = x^{\alpha_2}\big(t;\, \tau_1,\, x^\natural\big) \qquad \forall\, t \in\,]\tau_1,\, \tau_2]. \tag{3.34}$$

Thus, setting

$$\sigma_{\flat,\natural} \doteq \tau_1 + t^+_{\alpha_2}(x^\flat), \tag{3.35}$$

from (3.31), (3.33) we derive

$$\begin{aligned}&x^{\alpha_2}\big(\sigma_{\flat,\natural};\, \tau_1,\, x^\flat\big) \in \partial D_{\alpha_2},\\ &\big|\sigma_{\flat,\natural} - \tau_2\big| < c_1 \cdot \Delta,\end{aligned} \tag{3.36}$$

and, thanks to (3.13), we obtain

$$\big|y^\natural(t) - x^{\alpha_2}\big(t;\, \tau_1,\, x^\flat\big)\big| < c_0 \cdot \Delta \qquad \forall\, t \in\,]\tau_1, \min\{\tau_2, \sigma_{\flat,\natural}\}]. \tag{3.37}$$

Then, define the map

$$\Phi_{\flat,\natural}(t) \doteq \begin{cases} y^\flat(t) & \forall\, t \in\ [\tau_0,\, \tau_1],\\ x^{\alpha_2}\big(t;\, \tau_1,\, x^\flat\big) & \forall\, t \in\,]\,\tau_1, \sigma_{\flat,\natural}],\end{cases} \tag{3.38}$$

and observe that, by construction, $\Phi_{\flat,\natural}(\cdot)$ is a solution of (1.11) verifying (3.10). Moreover, (3.8) trivially holds, while from (3.36)–(3.37) we recover the estimates (3.9) and (3.11) taking the constant $C^{\text{iv}} > \max\{c_0, c_1\}$. Thus, (3.38) provides the desired map whenever (3.30) is verified.

4. Case 2. Assume that

$$x^{\flat} \in \partial D_{\alpha_1} \cap B(x_0, c_3 \cdot \Delta) \qquad \text{or} \qquad x^{\natural} \in \partial D_{\alpha_2} \cap B(x_0, c_3 \cdot \Delta), \tag{3.39}$$

for some

$$x_0 \in G_{\alpha_1,\alpha_2} \cap \bigcup_{\gamma \in \mathcal{T}_{\alpha_2}} \text{Im}(\gamma). \tag{3.40}$$

Then, by (3.7), (3.26) and (3.39), one has

$$\left|x^{\flat} - x_0\right| < (1 + c_3) \cdot \Delta < \delta_2, \tag{3.41}$$

$$\left|x^{\natural} - x_0\right| < (1 + c_3) \cdot \Delta < \min\{\delta_1, \overline{\lambda}\}\cdot \tag{3.42}$$

Thus, observing that by property *i)* one has

$$y^{\flat}(t) = x^{g}(t - \tau_1;\, 0,\, x^{\flat}) \qquad \forall\, t \in [\tau_0,\, \tau_1], \tag{3.43}$$

and because of $(3.27)_1$, (3.41), applying property **P2** we deduce that there is another backward trajectory $t \mapsto x^{g}(t;\, 0,\, x_0)$, $t \in [-\widehat{\tau}^{g}_{x_0,x^{\flat}},\, 0]$, of (1.11) arriving in x_0, such that

$$\left|\widehat{\tau}^{g}_{x_0,x^{\flat}} - \tau_1 + \tau_0\right| < c_2(1 + c_3) \cdot \Delta, \tag{3.44}$$

$$\left|y^{\flat}(t) - x^{g}\big(t - \tau_0 - \widehat{\tau}^{g}_{x_0,x^{\flat}};\, 0,\, x_0\big)\right| < c_2(1 + c_3) \cdot \Delta \qquad \forall\, t \in\ [\tau_0,\ \min\{\tau_0 + \widehat{\tau}^{g}_{x_0,x^{\flat}},\, \tau_1\}]. \tag{3.45}$$

To fix the ideas assume that

$$\tau_1 - \tau_0 < \widehat{\tau}^{g}_{x_0,x^{\flat}} < \tau_2 - \tau_0. \tag{3.46}$$

Then, observing that by property *ii)* there holds (3.34), using (3.4), (3.44)–(3.45), we obtain

$$\begin{aligned}
\left|y^{\natural}(t) - x^{g}\big(t - \tau_0 - \widehat{\tau}^{g}_{x_0,x^{\flat}};\, 0,\, x_0\big)\right| &\le \left|y^{\natural}(t) - x^{\natural}\right| + \left|x^{\natural} - x^{\flat}\right| + \left|x^{\flat} - x^{g}(\tau_1 - \tau_0 - \widehat{\tau}^{g}_{x_0,x^{\flat}};\, 0,\, x_0)\right| + \\
&\quad + \left|x^{g}(\tau_1 - \tau_0 - \widehat{\tau}^{g}_{x_0,x^{\flat}};\, 0,\, x_0) - x^{g}(t - \tau_0 - \widehat{\tau}^{g}_{x_0,x^{\flat}};\, 0,\, x_0)\right| \\
&\le 2M \cdot |t - \tau_1| + \Delta + c_2(1 + c_3) \cdot \Delta \\
&< 2(M + 1)c_2(1 + c_3) \cdot \Delta \\
&\qquad\qquad \forall\, t \in\ ]\tau_1,\, \tau_0 + \widehat{\tau}^{g}_{x_0,x^{\flat}}].
\end{aligned} \tag{3.47}$$

On the other hand, if $x^{\natural} \neq x_0$, since (3.42) together with (3.22), imply (3.32), using $(3.27)_2$ we deduce as in Case 1 that (3.33) holds. Hence, thanks to (3.42), by property **P1**-*b)* it follows that there exists $\widehat{\tau}^{\alpha_2}_{x_0,x^{\natural}} \in [0,\, t^{+}_{\alpha_2}(x_0)]$ such that

$$x^{\alpha_2}\big(\widehat{\tau}^{\alpha_2}_{x_0,x^{\natural}};\, 0,\, x_0\big) \in \partial D_{\alpha_2}, \tag{3.48}$$

$$\left|\tau_2 - \tau_1 - \widehat{\tau}^{\alpha_2}_{x_0,x^{\natural}}\right| < c_1(1 + c_3) \cdot \Delta. \tag{3.49}$$

Then, setting

$$\sigma_{\flat,\natural} \doteq \tau_0 + \widehat{\tau}^{g}_{x_0,x^{\flat}} + \widehat{\tau}^{\alpha_2}_{x_0,x^{\natural}}, \tag{3.50}$$

and relying on (3.44) and (3.49), we derive

$$\left|\sigma_{\flat,\natural}-\tau_2\right| \le \left|\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat}-\tau_1\right|+\left|\tau_1-\tau_2+\widehat{\tau}^{\alpha_2}_{x_0,x^\natural}\right| < (c_1+c_2)(1+c_3)\cdot\Delta, \tag{3.51}$$

while, using (3.13), (3.42) and (3.44), and because of (3.34), we get

$$\begin{aligned}\left|y^\natural(t)-x^{\alpha_2}\big(t;\,\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat},\,x_0\big)\right| &\le c_0\cdot\big(\left|x^\natural-x_0\right|+\left|\tau_1-\tau_0-\widehat{\tau}^{g}_{x_0,x^\flat}\right|\big)\\ &\le c_0\cdot(1+c_2)(1+c_3)\cdot\Delta\\ &\qquad \forall\, t\in\,]\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat},\ \min\{\tau_2,\,\sigma_{\flat,\natural}\}].\end{aligned} \tag{3.52}$$

Thus, define

$$\Phi_{\flat,\natural}(t) \doteq \begin{cases} x^g\big(t-\tau_0-\widehat{\tau}^{g}_{x_0,x^\flat};\,0,\,x_0\big) & \forall\, t\in\ [\tau_0,\,\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat}],\\ x^{\alpha_2}\big(t;\,\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat},\,x_0\big) & \forall\, t\in\,]\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat},\,\sigma_{\flat,\natural}], \end{cases} \tag{3.53}$$

and observe that, by construction and because of (3.48), $\Phi_{\flat,\natural}(\cdot)$ is a solution of (1.11) verifying (3.10). Moreover, from (3.45), (3.47), (3.49) and (3.52), it follows that $\Phi_{\flat,\natural}(\cdot)$ satisfies the estimates (3.8), (3.9) and (3.11) with the constant $C^{\mathrm{iv}} > 2(M+c_0)(1+c_2)(1+c_3)$, which shows that (3.53) provides the desired map whenever (3.39) holds.

5. Case 3. Assume that

$$x^\flat\in\partial D_{\alpha_1}\cap B(x_0,\,c_3\cdot\Delta)\qquad\text{for}\quad\text{some}\qquad x_0\in G_{\alpha_1,\alpha_2}\setminus\bigcup_{\gamma\in\mathcal{T}_{\alpha_2}}\mathrm{Im}(\gamma), \tag{3.54}$$

and that

$$x^\natural\in\partial D_{\alpha_2}\cap B(y_0,\,c_3\cdot\Delta)\qquad\text{for}\quad\text{some}\qquad y_0\in G_{\alpha_1,\alpha_2}\setminus\bigcup_{\gamma\in\mathcal{T}_{\alpha_2}}\mathrm{Im}(\gamma). \tag{3.55}$$

Observe that, by (3.7), (3.26) and (3.54)–(3.55), one has

$$\left|x^\flat-x_0\right| < c_3\cdot\Delta < \delta_2, \tag{3.56}$$

$$\left|x^\natural-y_0\right| < c_3\cdot\Delta < \min\{\delta_1,\,\overline{\lambda}\}, \tag{3.57}$$

$$\left|x^\natural-x_0\right| < (1+c_3)\cdot\Delta < \overline{\lambda}. \tag{3.58}$$

But then from (3.57)–(3.58), because of (3.22), we deduce that $x_0=y_0$. Hence, by (3.22), (3.54) and (3.57), it follows that x_0, $x^\natural$ belong to the same connected component of $\partial D_{\alpha_2}\setminus\bigcup_{\gamma\in\mathcal{T}_{\alpha_2}}\mathrm{Im}(\gamma)$. Moreover, since $y^\natural(\cdot)$ satisfies property *ii*), we deduce also that (3.32)–(3.34) hold. Therefore, relying on (3.57), and applying property **P1**-*a)*, we derive

$$\left|\tau_2-\tau_1-t^+_{\alpha_2}(x_0)\right| < c_1\,c_3\cdot\Delta. \tag{3.59}$$

On the other hand, since by property *i*) there holds (3.43), and because of $(3.27)_1$ and (3.56), applying property **P2** as in Case 2 we deduce that there is another backward trajectory $t\mapsto x^g(t;\,0,\,x_0)$, $t\in[-\widehat{\tau}^{g}_{x_0,x^\flat},\,0]$, of (1.11) arriving in x_0, for which the estimates (3.44), (3.45) and (3.47) are verified. Thus, setting

$$\sigma_{\flat,\natural} \doteq \tau_0+\widehat{\tau}^{g}_{x_0,x^\flat}+t^+_{\alpha_2}(x_0), \tag{3.60}$$

and relying on (3.44) and (3.59), we derive

$$\left|\sigma_{\flat,\natural}-\tau_2\right| \le \left|\tau_0+\widehat{\tau}^{g}_{x_0,x^\flat}-\tau_1\right|+\left|\tau_1-\tau_2+\widehat{\tau}^{\alpha_2}_{x_0,x^\natural}\right| < (c_1+c_2)(1+c_3)\cdot\Delta, \tag{3.61}$$

while, using (3.13), (3.44) and (3.57), and because of (3.34), we get

$$\begin{aligned}\left|y^\natural(t) - x^{\alpha_2}\big(t;\, \tau_0 + \widehat{\tau}^g_{x_0,x^\flat},\, x_0\big)\right| &\le c_0 \cdot \big(\left|x^\natural - x_0\right| + \left|\tau_1 - \tau_0 - \widehat{\tau}^g_{x_0,x^\flat}\right|\big)\\ &\le c_0 \cdot (1+c_2)(1+c_3)\cdot \Delta\\ &\qquad \forall\, t \in \,]\tau_0 + \widehat{\tau}^g_{x_0,x^\flat},\ \min\{\tau_2,\, \sigma_{\flat,\natural}\}]. \end{aligned} \tag{3.62}$$

Observe now that the map $\Phi_{\flat,\natural}(\cdot)$ defined in (3.53) is a solution of (1.11) verifying (3.10) since, by the definition (3.14) of the quantity $t^+_{\alpha_2}$, one has

$$\Phi_{\flat,\natural}(\sigma_{\flat,\natural}) = x^{\alpha_2}\big(t^+_{\alpha_2}(x_0);\, 0,\, x_0\big) \in \partial D_{\alpha_2}. \tag{3.63}$$

Moreover, from (3.45), (3.47), (3.59) and (3.62), it follows that $\Phi_{\flat,\natural}(\cdot)$ satisfies the estimates (3.8), (3.9) and (3.11) with the constant $C^{\mathrm{iv}} > 2(M + c_0)(1 + c_2)(1 + c_3)$, thus showing that (3.53) provides the desired map even in the case where (3.54)–(3.55) hold. This completes the proof of Proposition 3.1 since, because of (3.5)–(3.6) one has $x^\flat \in \partial D_{\alpha_1}$, $x^\natural \in \partial D_{\alpha_2}$, and hence the above three Cases 1–3 cover all the possibilities. □

Completion of the proof of Theorem 1

Let g be a uniformly bounded away from zero, polygonal patchy vector field on $\mathbb{R}^2$, satisfying condition **(C)**. Fix a compact set $K \subset \mathbb{R}^2$. Observe that, thanks to Proposition 2.4, in order to establish Theorem 1 it will be sufficient to take in consideration only perturbed solutions of (1.1) with values in the compact set $K''' = B(K, C''' \cdot \delta''')$ that are CCS of (1.11), and derive for any such solution $y(\cdot)$ a linear estimate of the distance from some solution $x(\cdot)$ of (1.11), of the type

$$\left\|x - y\right\|_{\mathbf{L}^\infty} \le C \cdot \Delta(y), \tag{3.64}$$

where $\Delta(y)$ denotes the total amount of jumps in $y(\cdot)$ as defined in (3.1). To this end we will establish the following

Lemma 3.2. *In the same setting of Proposition 3.1, given any compact set $K \subset \mathbb{R}^2$, there exist constants $C^{\mathrm{v}} = C^{\mathrm{v}}(K)$, $\delta^{\mathrm{v}} = \delta^{\mathrm{v}}(K) > 0$, for which the following hold.*

For every CCS of (1.11) $y : [0,T] \mapsto K$, such that $\Delta(y) < \delta^{\mathrm{v}}$, letting $0 = \tau_1 < \tau_2 < \cdots < \tau_{m+1} = T$ be the partition of $[0,\, T]$ induced by $y(\cdot)$ according with (2.4)–(2.5), there exist a sequence of points $\widehat{\tau}_2, \widehat{\tau}_3, \cdots, \widehat{\tau}_{m'} = T$, together with a sequence of CCS $y^k : [0,T] \mapsto \mathbb{R}^2$, $k = 1, \cdots, m'-1$, having the following properties.

I) $y^k\restriction_{[0,\,\widehat{\tau}_{k+1}]}$ *is a Carathéodory solution of (1.11) and, letting*

$$\alpha_k \doteq \max\big\{\alpha\ ;\ \alpha \in Im\big((\alpha^* \circ y^k)\restriction_{[0,\,\widehat{\tau}_{k+1}]}\big)\big\}, \tag{3.65}$$

one has

$$\begin{aligned} \alpha_k > \alpha_{k-1} \qquad & if \qquad k > 1,\\ y^k(\widehat{\tau}_{k+1}) \in \partial D_{\alpha_k} \qquad & if \qquad k < m'-1, \end{aligned} \tag{3.66}$$

(where D_{α_k} denotes a polygonal domain defined as in (2.3)).

II) *If $k < m' - 1$, one has*

$$y^k(\widehat{\tau}_{k+1}{}^+) \in \bigcup_{\gamma > \alpha_k} D_\gamma, \tag{3.67}$$

$$\alpha^*\big(y^k(t)\big) > \alpha_k \qquad \forall\, t \in \,]\widehat{\tau}_{k+1},\, T]. \tag{3.68}$$

III) *If* $k>1$, *there holds*

$$\big|y^k(t)-y^{k-1}(t)\big|\leq C^{\text{v}}\cdot\Delta(y^{k-1})\qquad\forall\ t\in[0,\,T],\tag{3.69$_k$}$$

$$\Delta(y^k)\leq\big(1+2C^{\text{v}}\big)^{k-1}\cdot\Delta(y).\tag{3.70$_k$}$$

Proof of Lemma 3.2. Fix a compact set $K\subset\mathbb{R}^2$. Let K_0 and $\mathcal{A}_{K_0}$ be, respectively, the neighborhood of K (3.2) and the set of indices in (3.3), and choose the constants $C^{\text{IV}}=C^{\text{IV}}(K_0)$, $\delta^{\text{IV}}=\delta^{\text{IV}}(K_0)>0$ according with Proposition 3.1. By the properties of a CCS and because of the regularity of the vector fields g_α, $\alpha\in\mathcal{A}_{K_0}$, there will be some constant $c_4>0$ so that, for any CCS of (1.11) $y:[0,T]\mapsto K_0$, one has

$$\big|y(t)-y(t')\big|\leq c_4\cdot\big(|t-t'|+\Delta(y)\big)\qquad\forall\ t,t'\in[0,\,T].\tag{3.71}$$

Then, set

$$\delta^{\text{v}}\doteq\min\left\{\frac{\delta^{\text{IV}}}{\big(1+2C^{\text{v}}\big)^N},\ \frac{\delta_4}{\big(1+2C^{\text{v}}\big)^N},\ \frac{\delta_0}{N\big(1+2C^{\text{v}}\big)^N},\ \right\},\qquad C^{\text{v}}\doteq 3c_4(1+2C^{\text{IV}}),\tag{3.72}$$

and consider a CCS of (1.11) $y:[0,T]\mapsto K$, with

$$\Delta(y)<\delta^{\text{v}}.\tag{3.73}$$

We shall construct the sequence of of CCS $y^k:[0,T]\mapsto K_0$ and of points $\widehat{\tau}_{k+1}$, enjoing the properties I–III, applying Proposition 3.1 and proceeding by induction on $k\geq 1$. Set

$$\widehat{\tau}_2\doteq\tau_2,\qquad\qquad y^1(y)\doteq y(t)\qquad\forall\ t\in[0,T]\tag{3.74}$$

and, if $m=1$, *i.e.* $\tau_2=T$, set $m'\doteq 2$, otherwise let $m'>2$. Observe that, by the properties $a''')$–$b''')$ of a CCS stated in Proposition 2.4, the point $\widehat{\tau}_2$ and the map $y^1(\cdot)$ in (3.74) clearly verify the conditions I–II of Lemma 3.2. Next, assume to have constructed, for some $1<k\leq N$, a sequence of CCS $y^1,\ldots,y^{k-1}$, together with a sequence of points $0<\widehat{\tau}_2,\cdots,\widehat{\tau}_k<T$, enjoing the properties I–III, with C^{v}, as in (3.72). Set

$$\widehat{\tau}'_{k+1}\doteq\sup\big\{t\in\,]\,\widehat{\tau}_k,\,T]\ ;\ \alpha^*\big(y^{k-1}(t)\big)=\alpha^*\big(y^{k-1}(\widehat{\tau}_k{}^+)\big)\big\},\tag{3.75}$$

and observe that, because of I–III, and by $(3.70)_{k-1}$, (3.72)–(3.73), the maps

$$y^{k,\flat}\doteq y^{k-1}\restriction_{[0,\,\widehat{\tau}_k]},\qquad\qquad y^{k,\natural}\doteq y^{k-1}\restriction_{]\,\widehat{\tau}_k,\,\widehat{\tau}'_{k+1}]},$$

have the properties $i)-iii)$ stated in Proposition 3.1. Moreover, since the estimates $(3.69)_h$, $(3.70)_h$, $h=2,\ldots,k-1$, together with (3.72)–(3.73), imply

$$\begin{aligned}\big|y^{k-1}(t)-y(t)\big|&\leq\sum_{h=2}^{k-1}\big|y^h(t)-y^{h-1}(t)\big|\\&\leq\sum_{h=2}^{k-1}\big(1+2C^{\text{v}}\big)^{h-1}\cdot\Delta(y)\\&\leq N\big(1+2C^{\text{v}}\big)^N\cdot\Delta(y)<\delta_0\qquad\forall\ t\in[0,\,T],\end{aligned}\tag{3.76}$$

by the above assumptions it follows that $y^{k-1}(\cdot)$, and hence $y^{k,\flat}(\cdot)$, $y^{k,\natural}(\cdot)$, take values in the set K_0. Then, letting

$$\Phi^k_{\flat,\natural} \doteq \Phi\big[y^{k,\flat},\, y^{k,\natural}\big] :\ [0,\, \sigma^k_{\flat,\natural}] \longmapsto \mathbb{R}^2$$

be the Carathéodory solution of (1.11) provided by Proposition 3.1, and denoting by $x^g\big(t;\, t_0,\, x_0\big)$ a Carathéodory solution of (1.11) starting from x_0 at time t_0, set

$$\widehat{\tau}_{k+1} \doteq \min\big\{\sigma^k_{\flat,\natural},\ T\big\}, \qquad \widehat{\tau}''_{k+1} \doteq \min\big\{T,\ T + \widehat{\tau}_{k+1} - \widehat{\tau}'_{k+1}\big\}, \tag{3.77}$$

$$y^k(t) \doteq \begin{cases} \Phi^k_{\flat,\natural}(t) & \text{if} \quad t \in [0,\, \widehat{\tau}_{k+1}], \\ y^{k-1}\big(t + \widehat{\tau}'_{k+1} - \widehat{\tau}_{k+1}\big) & \text{if} \quad t \in \,]\,\widehat{\tau}_{k+1},\, \widehat{\tau}''_{k+1}], \\ x^g\big(t;\, \widehat{\tau}''_{k+1},\, y^{k-1}(T)\big) & \text{if} \quad t \in \,]\,\widehat{\tau}''_{k+1},\, T]. \end{cases} \tag{3.78}$$

Next, if $\widehat{\tau}_{k+1} = T$, set $m' \doteq k+1$, otherwise let $m' > k+1$. By construction, and because $\Phi^k_{\flat,\natural}$ satisfies condition (3.10) of Proposition 3.1, the map in (3.78) defines a CCS of (1.11) that enjoys the properties I–II. Moreover, by (3.8), (3.9) and (3.11), and because of the above definition (3.77) of $\widehat{\tau}_{k+1}$, we derive

$$\big|\widehat{\tau}_{k+1} - \widehat{\tau}'_{k+1}\big| \leq C^{\text{IV}} \cdot \Delta(y^{k-1}), \tag{3.79}$$

$$\big|y^k(t) - y^{k-1}(t)\big| \leq C^{\text{IV}} \cdot \Delta(y^{k-1}) \qquad \forall\, t \in [0,\ \min\{\widehat{\tau}_{k+1},\, \widehat{\tau}'_{k+1}\}]. \tag{3.80}$$

On the other hand, using (3.71) and (3.79)–(3.80), if $\widehat{\tau}'_{k+1} < \widehat{\tau}_{k+1}$ we obtain

$$\begin{aligned} \big|y^k(t) - y^{k-1}(t)\big| &\leq \big|\Phi^k_{\flat,\natural}(t) - \Phi^k_{\flat,\natural}(\widehat{\tau}'_{k+1})\big| + \big|y^k(\widehat{\tau}'_{k+1}) - y^{k-1}(\widehat{\tau}'_{k+1})\big| + \big|y^{k-1}(t) - y^{k-1}(\widehat{\tau}'_{k+1})\big| \\ &\leq 2c_4 \cdot \big|t - \widehat{\tau}'_{k+1}\big| + c_4 \cdot \big(\Delta(\Phi^k_{\flat,\natural}) + \Delta(y^{k-1})\big) + C^{\text{IV}} \cdot \Delta(y^{k-1}) \\ &\leq 3c_4(1 + 2C^{\text{IV}}) \cdot \Delta(y^{k-1}) \qquad \forall\, t \in \,]\,\widehat{\tau}'_{k+1},\, \widehat{\tau}_{k+1}], \end{aligned} \tag{3.81}$$

while, in the case $\widehat{\tau}_{k+1} < T$, we get

$$\begin{aligned} \big|y^k(t) - y^{k-1}(t)\big| &\leq c_4 \cdot \big(\big|\widehat{\tau}_{k+1} - \widehat{\tau}'_{k+1}\big| + \Delta(y^{k-1})\big) \\ &\leq c_4(1 + C^{\text{IV}}) \cdot \Delta(y^{k-1}) \qquad \forall\, t \in \,]\,\widehat{\tau}_{k+1},\, T]. \end{aligned} \tag{3.82}$$

From (3.80)–(3.82) we recover the estimate $(3.69)_k$, with C^{V} as in (3.72), while $(3.70)_{k-1}$, together with $(3.69)_k$, immediately yields $(3.70)_k$, showing that the map in (3.78) enjoys also the property III.

To complete the proof of Lemma 3.2, observe that proceeding by induction on $k \geq 2$, either we find some $m' \leq N$ such that $\widehat{\tau}_{m'} = T$, or else we construct a sequence of CCS of (1.11) $y^1, \ldots, y^{N-1}$, together with a sequence of points $0 < \widehat{\tau}_2, \cdots, \widehat{\tau}_N < T$, enjoing the properties I–III. But then, if we define y^N and $\widehat{\tau}_{N+1}$ according with (3.77)–(3.78), we certainly find $\widehat{\tau}_{N+1} = T$, since otherwise, relying on properties I–III, one deduces

$$y^N(t) \in \mathbb{R}^2 \setminus \bigcup_{\alpha \in \mathcal{A}_{K_0}} \Omega_\alpha \qquad \forall\, t \in \,]\widehat{\tau}_{N+1},\, T], \tag{3.83}$$

while, performing a computation as in (3.76), one deduces

$$\big|y^N(t) - y(t)\big| < \delta_0 \qquad \forall\, t \in\ [0,\, T],$$

which, by (3.3), implies

$$y^N(t) \in B(K_0, \delta_0) \subset \bigcup_{\alpha \in \mathcal{A}_{K_0}} \Omega_\alpha \qquad \forall\, t \in\ [0,\, T],$$

yielding a contradiction with (3.83). This concludes the proof of the lemma. □

We are in the position now to complete the proof of Theorem 1, relying on Lemma 3.2. Let $C^{\mathrm{v}} = C^{\mathrm{v}}(K''')$, $\delta^{\mathrm{v}} = \delta^{\mathrm{v}}(K''') > 0$ be constants chosen according with Lemma 3.2 and consider a CCS of (1.11) $y : [0,T] \mapsto K'''$, such that $\Delta(y) < \delta^{\mathrm{v}}$. Then, letting $y^1 = y,\, y^2, \dots y^{m'-1}$ be the sequence of CCS provided by Lemma 3.2 that enjoy the properties I–III, and using $(3.69)_k$–$(3.70)_k$, we derive

$$\begin{aligned}\big|y^{m'-1}(t) - y(t)\big| &\le \sum_{k=2}^{m'-1} \big|y^k(t) - y^{k-1}(t)\big| \\ &\le N\big(1 + 2C^{\mathrm{v}}\big)^N \cdot \Delta(y) \qquad \forall\, t \in\ [0,\, T].\end{aligned} \tag{3.84}$$

By property I, $y^{m'-1} : [0,T] \to \mathbb{R}^2$ is a Carathéodory solution of (1.11), and hence (3.84) yields the estimate in (3.63), taking $C = N\big(1 + 2C^{\mathrm{v}}\big)^N$, which concludes the proof of the theorem.

4. Appendix

We provide here the proofs of the Lemmas 2.2–2.3 stated in Section 2. To this end we shall first establish the following

Lemma 4.1. *Let g be a uniformly bounded away from zero polygonal patchy vector field on $\mathbb{R}^2$, associated to a family of polygonal patches $\big\{(\Omega_\alpha,\ g_\alpha); \alpha \in \mathcal{A}\big\}$. Assume that condition* **(C)** *(stated in Sect. 2) is satisfied. Then, given $C > 0$ and any compact set $K \subset \mathbb{R}^2$, there exist constants $C^{\mathrm{vi}} = C^{\mathrm{vi}}(K, C) \ge C$, $C^{\mathrm{vii}} = C^{\mathrm{vii}}(K, C)$, $\delta^{\mathrm{vi}} = \delta^{\mathrm{vi}}(K, C) > 0$, so that the following property holds.*

For every BV perturbation $w = w(t)$ with Tot.Var.$\{w\} < \delta^{\mathrm{vi}}$, and for every left continuous solution $y : [0,T] \mapsto K$ of (1.1), for which the map $t \mapsto \alpha^*\big(y(t)\big)$ is non-decreasing, letting

$$\big\{\, \alpha_{i_1}, \dots, \alpha_{i_m} \,\big\} = Im\big(\alpha^* \circ y\big), \tag{4.1}$$

with

$$\alpha_{i_1} < \cdots < \alpha_{i_m}, \tag{4.2}$$

and setting

$$D_{\alpha_{i_j}} \doteq \Omega_{\alpha_{i_j}} \setminus \bigcup_{\beta > \alpha_{i_j}} \Omega_\beta \qquad j = 1, \dots, m, \tag{4.3}$$

$$]\tau_j,\, \tau_{j+1}] \doteq \Big\{ t \in [0,\, T] \ :\ y(t) \in D_{\alpha_{i_j}} \Big\} \qquad j = 1, \dots, m, \tag{4.4}$$

there holds

$$\mathrm{meas}\left(\bigcup_j \Big\{ t \in]\tau_j,\, \tau_{j+1}] \ :\ d(y(t),\, \partial D_{\alpha_{i_j}}) < C^{\mathrm{vi}} \cdot \mathrm{Tot.Var.}\{w\} \Big\} \right) < C^{\mathrm{vii}} \cdot \mathrm{Tot.Var.}\{w\} \cdot \tag{4.5}$$

Proof. **1.** Fix $C > 0$ and a compact set $K \subset \mathbb{R}^2$. Letting $\mathcal{A}_K = \{\alpha_1, \cdots, \alpha_N\}$ be the set of indices defined as in (3.3), for each $\alpha \in \mathcal{A}_K$ call $E^1_\alpha, \dots, E^{p_\alpha}_\alpha$, and $r^1_\alpha, \dots, r^{p_\alpha}_\alpha$, respectively, the edges of the domain D_α (defined as in (4.3)) that form the boundary ∂D_α, and the corresponding lines in which the edges are contained. By construction, every edge E_α^ℓ is a part of the boundary of some Ω_β, $\beta \geq \alpha$. Call $\mathbf{n}_{\boldsymbol{\alpha}}^{\boldsymbol{\ell}}$ the normal to E_α^ℓ pointing towards the interior of D_α, and let $\varphi_\alpha^\ell(x)$ denote the signed distance of the point x from r_α^ℓ, *i.e.*

$$\varphi_\alpha^\ell(x) \doteq \begin{cases} d\big(x,\ r_\alpha^\ell\big) & \text{if} \quad x \in r_\alpha^\ell + \big\{\lambda\, \mathbf{n}_{\boldsymbol{\alpha}}^{\boldsymbol{\ell}}\ :\ \lambda \geq 0\big\}, \\ -d\big(x,\ r_\alpha^\ell\big) & \text{if} \quad x \in r_\alpha^\ell + \big\{\lambda\, \mathbf{n}_{\boldsymbol{\alpha}}^{\boldsymbol{\ell}}\ :\ \lambda \leq 0\big\}\cdot \end{cases} \tag{4.6}$$

Given any BV perturbation $w\ :\ [t_0,\, t_1] \mapsto D_\alpha$, and any solution $y\ :\ [t_0,\, t_1] \mapsto D_\alpha$ of the Cauchy problem (1.1)–(1.2), consider the map $\varphi_\alpha^\ell \circ y\ :\ [t_0,\, t_1] \mapsto \mathbb{R}$. One can easily verify that, for every Borel set $E \subset [t_0,\, t_1]$, the Radon measure $\mu \doteq D\,(\varphi_\alpha^\ell \circ y)$ satisfies

$$\left|\mu(E) - \int_E \big\langle \nabla\varphi_\alpha^\ell(y(t)),\ g_\alpha(y(t))\big\rangle\ dt\right| \leq c_5 \cdot \text{Tot.Var.}\{w\}, \tag{4.7}$$

for some constant $c_5 > 0$ depending only on the compact set K. Then, fix

$$C^{\text{VI}} > \max\{C,\, c_5\}, \tag{4.8}$$

and take $\delta_5 > 0$ so that one has

$$\Big\{x \in D_{\alpha_i}\ :\ d(x,\, \partial D_{\alpha_i}) > 2C^{\text{VI}} \cdot \delta_5\Big\} \neq \emptyset \qquad \forall\ i. \tag{4.9}$$

Observe now that, since g_α are smooth, bounded away from zero vector fields, the transversality condition **(C)** and the inward-pointing condition (1.15) guarantee that there exists some constant $c_6 > 0$ such that, for every α_i, $i = 1, \ldots, N$, and for every $\ell = 1, \ldots, p_{\alpha_i}$, one of the following two conditions holds

$$\big\langle g_{\alpha_i}(x),\ \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\big\rangle \geq c_6 \qquad\qquad \forall\ x \in \overline{D}_{\alpha_i} \cap r_{\alpha_i}^\ell \cap K, \tag{4.10}$$

$$\big\langle g_{\alpha_i}(x),\ \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\big\rangle \leq -c_6 \qquad\qquad \forall\ x \in \overline{D}_{\alpha_i} \cap r_{\alpha_i}^\ell \cap K. \tag{4.11}$$

For each α_i, define the sets $\mathcal{I}_{\alpha_i}$ and $\mathcal{O}_{\alpha_i}$ of (incoming and outgoing) indices

$$\begin{aligned} \mathcal{I}_{\alpha_i} &\doteq \big\{1 \leq \ell \leq p_{\alpha_i}\ :\ (4.10) \quad \text{holds}\big\}, \\ \mathcal{O}_{\alpha_i} &\doteq \big\{1 \leq \ell \leq p_{\alpha_i}\ :\ (4.11) \quad \text{holds}\big\}\cdot \end{aligned} \tag{4.12}$$

Because of the regularity assumptions on the fields g_α, there will be some constants $0 < \delta_6 \leq \delta_5$, $c_7 > 0$, so that

$$\begin{aligned} &\sup\Big\{\big\langle g_{\alpha_i}(x),\ \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\big\rangle\ :\ \ x \in D_{\alpha_i} \cap \Big(r_{\alpha_i}^\ell + \big\{\lambda\, \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\ :\ |\lambda| \leq 2C^{\text{VI}} \cdot \delta_6\big\}\Big) \cap K, \\ &\qquad\qquad\qquad\qquad i = 1, \ldots, N, \qquad \ell \in \mathcal{I}_{\alpha_i}\Big\} \geq c_7, \\ &\sup\Big\{\big\langle g_{\alpha_i}(x),\ \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\big\rangle\ :\ \ x \in D_{\alpha_i} \cap \Big(r_{\alpha_i}^\ell + \big\{\lambda\, \mathbf{n}_{\boldsymbol{\alpha_i}}^{\boldsymbol{\ell}}\ :\ |\lambda| \leq 2C^{\text{VI}} \cdot \delta_6\big\}\Big) \cap K, \\ &\qquad\qquad\qquad\qquad i = 1, \ldots, N, \qquad \ell \in \mathcal{O}_{\alpha_i}\Big\} \leq -c_7. \end{aligned} \tag{4.13}$$

2. Consider now a BV perturbation $w = w(t)$ with Tot.Var.$\{w\} < \delta_6$, and let $y : [0, T] \mapsto K$ be a solution of (1.1), for which there is a partition $\tau_1 < \tau_2 < \cdots < \tau_{m+1}$ of $[0,\, T]$, such that

$$\alpha^*(y(t)) = \alpha_{i_j} \qquad \forall\ t \in\ ]\tau_j, \tau_{j+1}], \qquad j = 1, \ldots, m, \tag{4.14}$$

with

$$\alpha_{i_1} < \cdots < \alpha_{i_m}. \tag{4.15}$$

For any $\alpha \in \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$, $\ell \in \{1, \dots, p_\alpha\}$, call S_α^ℓ the connected component of the set

$$\Big\{x \in D_\alpha \ :\ d(x, \partial D_\alpha) < C^{\text{VI}} \cdot \text{Tot.Var.}\{w\}\Big\} \bigcap \Big(r_\alpha^\ell + \big\{\lambda\, \mathbf{n}_{\boldsymbol{\alpha}}^{\boldsymbol{\ell}} \ :\ 0 \le \lambda < C^{\text{VI}} \cdot \text{Tot.Var.}\{w\}\big\}\Big) \tag{4.16}$$

whose boundary contains the edge E_α^ℓ. Then, by similar computations as those used in [2] to establish ([2], Prop. 2.2), relying on (4.7), (4.8) and (4.13), one can establish the following claims.

Claim 1. If, for some α_{i_j}, $j = 1, \dots, m$, $\ell \in \mathcal{I}_{\alpha_{i_j}}$, and for some constant $0 \le c \le 2C^{\text{VI}}$, there exists $t' \in]\tau_j, \tau_{j+1}]$, such that

$$\varphi_{\alpha_{i_j}}^\ell(y(t')) \ge c \cdot \text{Tot.Var.}\{w\}, \tag{4.17}$$

then, there holds

$$\varphi_{\alpha_{i_j}}^\ell(y(t)) > c - C^{\text{VI}} \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [t', \tau_j]. \tag{4.18}$$

Claim 2. If, for some α_{i_j}, $j = 1, \dots, m$, $\ell \in \mathcal{O}_{\alpha_{i_j}}$, and for some constant $|c| \le C^{\text{VI}}$, there exists $t' \in]\tau_j, \tau_{j+1}]$, such that

$$\varphi_{\alpha_{i_j}}^\ell(y(t')) \le c \cdot \text{Tot.Var.}\{w\}, \tag{4.19}$$

then, there holds

$$\varphi_{\alpha_{i_j}}^\ell(y(t)) < c + C^{\text{VI}} \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [t', \tau_j]. \tag{4.20}$$

Claim 3. If, for some interval $[t_1, t_2] \subset]\tau_j, \tau_{j+1}]$, $j = 1, \dots, m$, and for some $\ell \in \mathcal{I}_{\alpha_{i_j}}$, the following two conditions hold

$$\varphi_{\alpha_{i_j}}^\ell(y(t)) < 2C^{\text{VI}} \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [t_1, t_2], \tag{4.21}$$

$$\text{meas}\big\{t \in [t_1, t_2] \ :\ y(t) \in S_{\alpha_{i_j}}^\ell\big\} > c_8 \cdot \text{Tot.Var.}\{w\}, \tag{4.22}$$

with

$$c_8 \doteq \frac{2C^{\text{VI}}}{c_7}, \tag{4.23}$$

then, one has

$$\varphi_{\alpha_{i_j}}^\ell(y(t_2)) > C^{\text{VI}} \cdot \text{Tot.Var.}\{w\}\cdot \tag{4.24}$$

Claim 4. If, for some interval $[t_1, t_2] \subset]\tau_j, \tau_{j+1}]$, $j = 1, \dots, m$, and for some $\ell \in \mathcal{O}_{\alpha_{i_j}}$, there holds

$$\varphi_{\alpha_{i_j}}^\ell(y(t)) > -C^{\text{VI}} \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [t_1, t_2], \tag{4.25}$$

together with the condition (4.22), then, one has

$$\varphi_{\alpha_{i_j}}^\ell(y(t_2)) < 0. \tag{4.26}$$

From Claim 1 (taking the constant $c = 2C^{\text{VI}}$), Claim 2 (taking the constant $c = -C^{\text{VI}}$), and Claims 3–4, it clearly follows that, for every fixed $j = 1, \dots, m$, and every $\ell = 1, \dots, p_{\alpha_{i_j}}$, there holds

$$\text{meas}\big\{t \in]\tau_j, \tau_{j+1}] \ :\ y(t) \in S_{\alpha_{i_j}}^\ell\big\} \le c_8 \cdot \text{Tot.Var.}\{w\}\cdot \tag{4.27}$$

Thus, observing that by construction we have

$$\bigcup_j \Big\{ t \in]\tau_j,\, \tau_{j+1}] \ :\ d(y(t),\, \partial D_{\alpha_{i_j}}) < C^{\text{VI}} \cdot \text{Tot.Var.}\{w\} \Big\} = \bigcup_{j,\,\ell} S^\ell_{\alpha_{i_j}}, \tag{4.28}$$

from (4.27) we derive the estimate (4.5) with

$$C^{\text{VII}} \doteq \Big(\sum_{i=1}^{N} p_{\alpha_i} \Big) \cdot c_8, \tag{4.29}$$

where p_{α_i} denotes the number of edges of the domain D_{α_i} defined as in (4.3), while c_8 is the constant defined in (4.23). This completes the proof of Lemma 4.1, taking $\delta^{\text{VI}} = \delta_6$ and C^{VI} as in (4.8). □

Proof of Lemma 2.2.

1. Fix a compact set $K \subset \mathbb{R}^2$, let $C' = C'(K)$, $\delta' = \delta'(K) > 0$ be the constants provided by Proposition 2.1, and set $K' \doteq B(K, C' \cdot \delta')$. Observe that, thanks to Proposition 2.1, in order to establish Lemma 2.2 it will be sufficient to show that, there exist constants $C'' = C''(K')$, $\delta'' = \delta''(K') > 0$, so that the following holds. Given any BV function $w(\cdot)$ with Tot.Var.$\{w\} < \delta''$, for every left continuous solutions $y : [0,T] \mapsto K'$ of (1.1) for which the map $t \mapsto \alpha^*\big(y(t)\big)$ is non-decreasing, there exists a piecewise smooth function $y^\sharp(\cdot)$ enjoing the properties $a')$–$c')$. Letting $\mathcal{A}_{K'} = \{\alpha_1, \cdots, \alpha_N\}$ be the set of indices defined as in (3.3), we shall assume that every vector field g_{α_i} is defined on a neighborhood $B(\Omega_{\alpha_i}, \rho)$, $\rho > 0$, of the domain Ω_{α_i} and, for any fixed $t_0 > 0$, $x_0 \in B(\Omega_{\alpha_i}, \rho)$, we will denote by $t \mapsto x^{\alpha_i}(t;\, t_0,\, x_0)$ the solution of the Cauchy problem

$$\dot y = g_{\alpha_i}(y), \qquad y(t_0) = x_0, \tag{4.30}$$

and set

$$M \doteq \sup \big\{ |g_{\alpha_i}(y)| \ :\ y \in B(\Omega_{\alpha_i},\, \rho), \quad i = 1, \ldots, N \big\}. \tag{4.31}$$

Similarly, for every given $w \in$ BV, we denote by $t \mapsto z^{\alpha_i}(t;\, w,\, t_0,\, x_0)$ the left-continuous solution of

$$\dot z = g_{\alpha_i}(z) + \dot w, \qquad z(t_0) = x_0. \tag{4.32}$$

For every $x_0 \in B(\Omega_{\alpha_i}, \rho)$, $t_0 > 0$, we let $t^{\alpha_i,+}(t_0, x_0)$, $t^{\alpha_i,-}(t_0, x_0)$ denote the time that is necessary to reach the set $B(\Omega_{\alpha_i}, \rho) \setminus D_{\alpha_i}$, starting from x_0 at time t_0, and following, respectively, the forward and backward flow of g_{α_i}, *i.e.*

$$\begin{aligned} t^+_{\alpha_i}(t_0, x_0) &\doteq \inf \big\{ t > t_0 \ :\ x^{\alpha_i}(t;\, t_0,\, x_0) \in B(\Omega_{\alpha_i},\, \rho) \setminus D_{\alpha_i} \big\}, \\ t^-_{\alpha_i}(t_0, x_0) &\doteq \sup \big\{ t < t_0 \ :\ x^{\alpha_i}(t;\, t_0,\, x_0) \in B(\Omega_{\alpha_i},\, \rho) \setminus D_{\alpha_i} \big\}. \end{aligned} \tag{4.33}$$

Since g_{α_i} are smooth vector fields, and because of the linear estimate (1.6), the Cauchy Problems (4.30) and (4.32) are well posed. Hence, there will be some constant $c_9 > 0$ so that there holds

$$\Big| x^{\alpha_i}(t;\, t_0,\, x_0) - z^{\alpha_i}(t';\, w,\, t_1,\, x_1) \Big| \le c_9 \Big\{ |t - t'| + |t_0 - t_1| + |x_0 - x_1| + \text{Tot.Var.}\{w\} \Big\} \tag{4.34}$$

for any t, t', t_0, t_1, x_0, x_1, w, and for every α_i. Moreover, recalling that g_{α_i} are uniformly bounded away from zero vector fields that satisfy the inward-pointing condition (1.15) and the transversality condition **(C)**, we deduce that there will be constants $c_{10} > 1$, $\delta_7 > 0$, so that, if

$$d\Big(x_0,\ B(\Omega_{\alpha_i},\, \rho) \setminus D_{\alpha_i} \Big) < \delta_7, \tag{4.35}$$

then

$$\left|t^{\pm}_{\alpha_i}(t_0,x_0)-t_0\right| \le c_{10}\cdot d\Big(x_0,\ B(\Omega_{\alpha_i},\,\rho)\setminus D_{\alpha_i}\Big). \tag{4.36}$$

Set

$$c_{11} \doteq 4(1+c_9)^2\cdot(1+2Mc_{10})^{N+1}, \tag{4.37}$$

and let $C^{\text{VI}} = C^{\text{VI}}(K',2c_{11}) \ge 2c_{11}$, $C^{\text{VII}} = C^{\text{VII}}(K',2c_{11})$, $\delta^{\text{VI}} = \delta^{\text{VI}}(K',2c_{11}) > 0$, be constants chosen according with Lemma 4.1. Then, fix any $w \in \mathrm{BV}$ with

$$\text{Tot.Var.}\{w\} < \delta'' \doteq \min\left\{\delta^{\text{VI}},\ \frac{\delta_7}{c_{11}},\ \frac{\rho}{c_{11}}\right\}, \tag{4.38}$$

and consider a left continuous solution $y:[0,T]\mapsto K'$ of (1.1), for which the map $t\mapsto\alpha^*\big(y(t)\big)$ is non-decreasing. As an intermediate step towards the construction of the map $y^\sharp:[0,T]\mapsto\mathbb{R}^2$ enjoing properties $a')$–$c')$, we shall first produce a piecewise smooth function $\widetilde{y}:[0,T]\mapsto\mathbb{R}^2$ whose $\mathbf{L}^\infty$ distance from $y(\cdot)$ is bounded by $c_{11}\cdot\text{Tot.Var.}\{w\}$, and for which there is a partition $0=\tau'_1<\tau'_2<\cdots<\tau'_{m'+1}=T$ of $[0,T]$, together with an increasing sequence of indices $\alpha_{i'_1}<\cdots<\alpha_{i'_{m'}}$, so that $\widetilde{y}(\cdot)$ is a classical solution of $\dot y = g_{\alpha_{i'_k}}(y)$ on every interval $]\tau'_k,\,\tau'_{k+1}[$, $1\le k\le m'$, but does not satisfy the condition $\widetilde{y}(t)\in D_{\alpha_{i'_k}}$ for all $t\in\,]\tau'_k,\,\tau'_{k+1}[$.

2. In order to define the map $\widetilde{y}(\cdot)$, in connection with the partition $0=\tau_1<\tau_2<\cdots<\tau_{m+1}=T$ of $[0,T]$ induced by $y(\cdot)$ according with (4.14)–(4.15), consider the sequence of points $\tau'_1\doteq 0<\tau'_2<\cdots<\tau'_{m'+1}=T$, $m'\le m$, and of sub-indices $j(1)=1,\,j(2),\,\ldots,\,j(m')\in\{1,\ldots,m\}$, recursively defined as follows. If $\tau_2+t^+_{\alpha_{i_1}}\big(0,\,x^{\alpha_{i_1}}\big(\tau_2;\,0,\,y(0^+)\big)\big)\ge T$, set $\tau'_2\doteq T$, otherwise set

$$\tau'_2\doteq\tau_2+t^+_{\alpha_{i_1}}\big(0,\,x^{\alpha_{i_1}}\big(\tau_2;\,0,\,y(0^+)\big)\big). \tag{4.39}$$

Next, for all $1<k\le m$ such that $\tau'_k<T$, let $j(k)$ be the subindex of α_i for which there holds $\alpha_{i_{j(k)}}=\alpha^*\big(y(\tau'^{\,+}_k)\big)$, so that

$$\tau'_k\in[\tau_{j(k)},\,\tau_{j(k)+1}], \tag{4.40}$$

and set

$$\tau'_{k+1}\doteq\min\Big\{T,\ \tau_{j(k)+1}-t^-_{\alpha_{i_{j(k)}}}\big(0,\,y(\tau'^{\,+}_k)\big)+t^+_{\alpha_{i_{j(k)}}}\big(0,\,x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,\tau'_k,\,y(\tau'^{\,+}_k)\big)\big)\Big\}. \tag{4.41}$$

Then, set

$$m'\doteq\max\big\{1\le k\le m\ :\ \tau'_k<T\big\},\qquad\qquad \tau'_{m'+1}\doteq T.$$

Observe that, using (4.34), and because of (4.37)–(4.38), one finds, for every $k\ge1$,

$$\begin{aligned}
d\Big(&x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,\tau'_k,\,y(\tau'^{\,+}_k)\big),\ B(\Omega_{\alpha_{i_{j(k)}}},\,\rho)\setminus D_{\alpha_{i_{j(k)}}}\Big)\\
&\le\Big(\big|x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,\tau'_k,\,y(\tau'^{\,+}_k)\big)-z^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,w,\,\tau'_k,\,y(\tau'^{\,+}_k)\big)\big|\\
&\quad+\big|z^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,w,\,\tau'_k,\,y(\tau'^{\,+}_k)\big)-y(\tau_{j(k)+1}{}^+)\big|\Big)\\
&\le c_9\cdot\text{Tot.Var.}\{w\}+\big|y\big(\tau_{j(k)+1}\big)-y\big(\tau_{j(k)+1}{}^+\big)\big|\\
&\le(1+c_9)\cdot\text{Tot.Var.}\{w\}<\delta_7.
\end{aligned}\tag{4.42}$$

Thanks to (4.42), and because of (4.35), we can apply (4.36) obtaining

$$\begin{aligned}
\big|t^+_{\alpha_{i_{j(k)}}}\big(0,\,x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,\tau'_k,\,y(\tau'^{\,+}_k)\big)\big)\big|&\le c_{10}\cdot d\Big(x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\,\tau'_k,\,y(\tau'^{\,+}_k)\big),\ B(\Omega_{\alpha_{i_{j(k)}}},\,\rho)\setminus D_{\alpha_{i_{j(k)}}}\Big)\\
&\le c_{10}\cdot\Big(c_9\cdot\text{Tot.Var.}\{w\}+\big|y\big(\tau_{j(k)+1}\big)-y\big(\tau_{j(k)+1}{}^+\big)\big|\Big)\qquad\forall\,k\ge1.
\end{aligned}\tag{4.43}$$

Thus, (4.39)–(4.41) and (2.46) together, imply

$$\begin{aligned}\tau_2' - \tau_{j(2)} &\le \tau_2' - \tau_2 \\ &\le \big|t^+_{\alpha_{i_1}}\big(0,\, x^{\alpha_{i_1}}(\tau_2;\, 0,\, y(0^+))\big)\big| \\ &\le c_{10}\cdot\Big(c_9\cdot \text{Tot.Var.}\{w\} + \big|y(\tau_2) - y({\tau_2}^+)\big|\Big),\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\tau_k' - \tau_{j(k)} &\le \tau_k' - \tau_{j(k-1)+1} \\ &\le \big|t^+_{\alpha_{i_{j(k)}}}\big(0,\, x^{\alpha_{i_{j(k)}}}\big(\tau_{j(k)+1};\, \tau_k',\, y({\tau_k'}^+)\big)\big)\big| + \big|t^-_{\alpha_{i_{j(k-1)}}}\big(0,\, y({\tau_{k-1}'}^+)\big)\big| \\ &\le c_{10}\cdot\Big(c_9\cdot \text{Tot.Var.}\{w\} + \big|y\big(\tau_{j(k-1)+1}\big) - y\big({\tau_{j(k-1)+1}}^+\big)\big|\Big) + \big|t^-_{\alpha_{i_{j(k-1)}}}\big(0,\, y({\tau_{k-1}'}^+)\big)\big| \qquad \forall\ k > 2.\end{aligned} \tag{4.45}$$

On the other hand, since $y(\cdot)$ satisfies (1.3), using (4.31) we find

$$\begin{aligned} d\Big(y({\tau_k'}^+),\ B(\Omega_{\alpha_{i_{j(k)}}}, \rho)\setminus D_{\alpha_{i_{j(k)}}}\Big) &\le \big|y({\tau_k'}^+) - y\big({\tau_{j(k)}}^+\big)\big| + \big|y\big({\tau_{j(k)}}^+\big) - y\big(\tau_{j(k)}\big)\big| \\ &\le M\big|\tau_k' - \tau_{j(k)}\big| + \text{Tot.Var.}\{w\} + \big|y\big({\tau_{j(k)}}^+\big) - y\big(\tau_{j(k)}\big)\big| \qquad \forall\ k.\end{aligned} \tag{4.46}$$

Therefore, proceeding by induction on k, using (4.36) (thanks to (4.35) and (4.38)), and relying on (4.37) and (4.43)–(4.46), we obtain for every $k > 1$ the estimates

$$\begin{aligned}\tau_k' - \tau_{j(k)} \le \tau_k' - \tau_{j(k-1)+1} &\le (2 + c_9)c_{10}\cdot(1 + Mc_{10})^{k-2}\cdot \text{Tot.Var.}\{w\} \\ &\le \frac{c_{11}}{2(1+c_9)(1+2Mc_{10})}\cdot \text{Tot.Var.}\{w\},\end{aligned} \tag{4.47}$$

$$\begin{aligned} d\Big(y({\tau_k'}^+),\ B(\Omega_{\alpha_{i_{j(k)}}}, \rho)\setminus D_{\alpha_{i_{j(k)}}}\Big) &\le (2 + c_9)\cdot(1 + Mc_{10})^{k-1}\cdot \text{Tot.Var.}\{w\} \\ &\le c_{11}\cdot \text{Tot.Var.}\{w\} < \delta_7\,,\end{aligned} \tag{4.48}$$

$$\begin{aligned}\big|t^-_{\alpha_{i_{j(k)}}}\big(0,\, y({\tau_k'}^+)\big)\big| &\le c_{10}\cdot d\Big(y({\tau_k'}^+),\ B(\Omega_{\alpha_{i_{j(k)}}}, \rho)\setminus D_{\alpha_{i_{j(k)}}}\Big) \\ &\le (2+c_9)c_{10}\cdot(1 + Mc_{10})^{k-1}\cdot \text{Tot.Var.}\{w\} \\ &\le \frac{c_{11}}{2c_9(1+2Mc_{10})}\cdot \text{Tot.Var.}\{w\}\cdot\end{aligned} \tag{4.49}$$

3. Consider now the piecewise smooth map $\widetilde{y} : [0, T] \mapsto \mathbb{R}^2$ defined by setting

$$\widetilde{y}(0) \doteq y(0), \qquad \widetilde{y}(t) \doteq x^{\alpha_{i_1}}\big(t;\ 0,\, y(0^+)\big) \qquad t \in]0, \tau_2'], \tag{4.50}$$

and

$$\widetilde{y}(t) \doteq x^{\alpha_{i_{j(k)}}}\Big(t;\ \tau_k',\, x^{\alpha_{i_{j(k)}}}\big(t^-_{\alpha_{i_{j(k)}}}\big(0,\, y({\tau_k'}^+)\big);\ 0,\, y({\tau_k'}^+)\big)\Big) \qquad t \in\,]\tau_k', \tau_{k+1}'], \quad 1 < k \le m'. \tag{4.51}$$

By construction one has

$$
\begin{aligned}
&\widetilde{y}(\tau_k'^{\,+}) \in \partial D_{\alpha_{i_{j(k)}}} && \forall\ 1 < k \le m',\\
&\widetilde{y}(\tau_{k+1}') \in \Omega \setminus \overset{\circ}{D}_{\alpha_{i_{j(k)}}} && \forall\ 1 \le k < m'.
\end{aligned}
\tag{4.52}
$$

Moreover, since $\widetilde{y}(\cdot)$ is a classical solution of $\dot{y} = g_{\alpha_{i_{j(k)}}}(y)$ on $]\tau_k',\, \tau_{k+1}'[$, and because $y(\cdot)$ satisfies (1.3), setting $t^-_{\alpha_{i_1}}(0,\, y(0^+)) \doteq 0$, and using (4.31), (4.34), (4.47) and (4.49), we derive, for all k, the estimates

$$
\begin{aligned}
\big|\widetilde{y}(t) - y(t)\big| &\le \big|\widetilde{y}(t) - y(\tau_k'^{\,+})\big| + \big|y(\tau_k'^{\,+}) - y(t)\big|\\
&= \big|\widetilde{y}(t) - \widetilde{y}(\tau_k' - t^-_{\alpha_{i_{j(k)}}}(0,\, y(\tau_k'^{\,+})))\big| + \big|y(\tau_k'^{\,+}) - y(t)\big|\\
&\le 2M \cdot \big|t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big)\big| + \mathrm{Tot.Var.}\{w\}\\
&\le c_{11} \cdot \mathrm{Tot.Var.}\{w\}\\
&\qquad\qquad \forall\ t \in \big]\tau_k',\, \tau_k' - t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big)\big],
\end{aligned}
\tag{4.53}
$$

$$
\begin{aligned}
\big|\widetilde{y}(t) - y(t)\big| &= \Big|x^{\alpha_{i_{j(k)}}}\Big(t;\ \tau_k' - t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big),\, y(\tau_k'^{\,+})\Big) - z^{\alpha_{i_{j(k)}}}\big(t;\ w,\, \tau_k',\, y(\tau_k'^{\,+})\big)\Big|\\
&\le c_9 \cdot \Big(\big|t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big)\big| + \mathrm{Tot.Var.}\{w\}\Big)\\
&\le c_{11} \cdot \mathrm{Tot.Var.}\{w\}\\
&\qquad\qquad \forall\ t \in \big]\tau_k' - t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big),\, \tau_{j(k)+1}\big],
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
\big|\widetilde{y}(t) - y(t)\big| &\le \big|\widetilde{y}(t) - \widetilde{y}(\tau_{j(k)+1})\big| + \big|\widetilde{y}(\tau_{j(k)+1}) - y(\tau_{j(k)+1})\big| + \big|y(\tau_{j(k)+1}^{\ +}) - y(\tau_{j(k)+1})\big| + \big|y(t) - y(\tau_{j(k)+1}^{\ +})\big|\\
&\le 2M \cdot \big|\tau_{k+1}' - \tau_{j(k)+1}\big| + c_9 \cdot \big|t^-_{\alpha_{i_{j(k)}}}\big(0,\, y(\tau_k'^{\,+})\big)\big| + (2 + c_9) \cdot \mathrm{Tot.Var.}\{w\}\\
&\le (2M + c_9) \cdot \big|\tau_{k+1}' - \tau_{j(k)+1}\big| + (2 + c_9) \cdot \mathrm{Tot.Var.}\{w\}\\
&\le c_{11} \cdot \mathrm{Tot.Var.}\{w\}\\
&\qquad \forall\ t \in \big]\tau_{j(k)+1},\, \tau_{k+1}'\big],
\end{aligned}
\tag{4.55}
$$

which, together, and thanks to (4.38), yield

$$
\big\|\widetilde{y} - y\big\|_{\mathbf{L}^\infty([0,T])} \le c_{11} \cdot \mathrm{Tot.Var.}\{w\} < \rho. \tag{4.56}
$$

Notice that (4.56), in particular, implies $\widetilde{y}(t) \in \bigcup_i B(\Omega_{\alpha_i},\, \rho)$ for all $t \in\,]\tau_k',\, \tau_{k+1}']$, $1 < k \le m'$, and hence guarantees that $\widetilde{y}(\cdot)$ is well defined by (4.51).

4. Because of (4.56), the choice of the constants $C^{VI} \ge 2c_{11}$, C^{VII}, δ^{VI} according with Lemma 4.1 guarantees that

$$
\mathrm{meas}\left(\bigcup_k \Big\{t \in]\tau_k',\, \tau_{k+1}']\ :\ \widetilde{y}(t) \notin D_{\alpha_{i_{j(k)}}}\Big\}\right) < C^{VII} \cdot \mathrm{Tot.Var.}\{w\}\cdot \tag{4.57}
$$

Therefore, relying on (4.52) and (4.57), the inward-pointing condition (1.15) together with the transversality condition **(C)** imply that, for every $k = 1, \dots, m'$, there exists a partition $\tau_k' = \widetilde{t}_{k,1} < \widetilde{t}_{k,1} < \cdots < \widetilde{t}_{k,\widetilde{q}_k} = \tau_{k+1}'$

of $[\tau'_k, \tau'_{k+1}]$, so that

$$\widetilde{y}(t) \in D_{\alpha_{i_{j(k)}}} \qquad \forall\, t \in]\,\widetilde{t}_{k,\ell},\, \widetilde{t}_{k,\ell+1}\,[\qquad \text{for all odd} \quad \ell,$$
$$\sum_{\ell \text{ even}} (\,\widetilde{t}_{k,\ell+1} - \widetilde{t}_{k,\ell}) \le C^{\text{VII}} \cdot \text{Tot.Var.}\{w\}, \tag{4.58}$$

and with the property that the points

$$\widetilde{y}(\,\widetilde{t}_{k,1}{}^{+}), \quad t_{k,1} \neq 0, \qquad\qquad \widetilde{y}(\,\widetilde{t}_{k,\ell}), \quad 1 < \ell \le 2\lfloor \widetilde{q}_k/2 \rfloor,$$

lie on different edges of the domain $D_{\alpha_{i_{j(k)}}}$ ($\lfloor a \rfloor$ denoting the integer part of a). For each $k = 1, \dots, m'$, consider the sequence of points $t_{k,1} < t_{k,2} < \cdots < t_{k,q_k}$, $q_k \doteq \lfloor \widetilde{q}_k/2 \rfloor + 1$, recursively defined by setting $t_{1,1} \doteq 0$, and

$$t_{k,1} \doteq t_{k-1,\,q_{k-1}} \qquad 1 < k \le m',$$
$$t_{k,\ell+1} \doteq t_{k,\ell} + \widetilde{t}_{k,2\ell} - \widetilde{t}_{k,2\ell-1} \qquad 1 \le \ell < q_k, \qquad 1 \le k \le m'. \tag{4.59}$$

Then, letting $x^g\big(t;\, t_0,\, x_0\big)$ denote a solution of (1.11) starting from x_0 at time t_0, we define the map $y^\sharp : [0, T] \mapsto \mathbb{R}^2$ as follows: $y^\sharp(0) \doteq \widetilde{y}(0)$, and

$$y^\sharp(t) \doteq \begin{cases} \widetilde{y}\big(\,t + \widetilde{t}_{k,2\ell-1} - t_{k,\ell}\,\big) & \forall\, t \in]\,t_{k,\ell},\, t_{k,\ell+1}\,], \qquad 1 \le \ell < q_k, \qquad 1 \le k \le m', \\ x^g\big(t;\, t_{m',q_{m'}},\, y^\sharp(t_{m',q_{m'}})\big) & \forall\, t \in]\,t_{m',q_{m'}},\, T\,]. \end{cases} \tag{4.60}$$

Notice that, by construction, and by the properties of the solutions of a patchy system (recalled in Sect. 1), the map $t \mapsto y^\sharp(t)$ enjoys the properties $a')$–$b')$ stated in Lemma 2.2. Moreover, since one has

$$\widetilde{t}_{k,2\ell-1} - t_{k,\ell} \le \sum_{p \text{ even}} \widetilde{t}_{k,p+1} - \widetilde{t}_{k,p} = T - t_{m',q_{m'}} \qquad \forall\, l, \quad \forall\, k,$$

and because $\widetilde{y}(\cdot)$ is a solution of $\dot{y} = g_{\alpha_{i_{j(k)}}}(y)$ on $]\tau'_k,\, \tau'_{k+1}[$, using (4.31), (4.56) and (4.58), we derive

$$\begin{aligned} \big|y^\sharp(t) - \widetilde{y}(t)\big| &\le \sum_{1<k\le m'} \big|\widetilde{y}(\tau'_k{}^{+}) - \widetilde{y}(\tau'_k)\big| + M \cdot \max_{k,\ell} \big|\,\widetilde{t}_{k,2\ell-1} - t_{k,\ell}\big| \\ &\le \sum_{1<k\le m'} \Big(\big|\widetilde{y}(\tau'_k{}^{+}) - y(\tau'_k{}^{+})\big| + \big|y(\tau'_k{}^{+}) - y(\tau'_k)\big| + \big|\widetilde{y}(\tau'_k) - y(\tau'_k)\big|\Big) + MC^{\text{VII}} \cdot \text{Tot.Var.}\{w\} \\ &\le \big(1 + 2c_{11} + MC^{\text{VII}}\big) \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [0,\, t_{m',q_{m'}}], \end{aligned} \tag{4.61}$$

and

$$\begin{aligned} \big|y^\sharp(t) - \widetilde{y}(t)\big| &\le \big|y^\sharp(t) - y^\sharp(t_{m',q_{m'}})\big| + \big|y^\sharp(t_{m',q_{m'}}) - \widetilde{y}(t_{m',q_{m'}})\big| + \big|\widetilde{y}(t) - \widetilde{y}(t_{m',q_{m'}})\big| \\ &\le \sum_{1<k\le m'} \big|\widetilde{y}(\tau'_k{}^{+}) - \widetilde{y}(\tau'_k)\big| + 2M \cdot \big|T - t_{m',q_{m'}}\big| + \big|y^\sharp(t_{m',q_{m'}}) - \widetilde{y}(t_{m',q_{m'}})\big| \\ &\le \big(2 + 4c_{11} + 3MC^{\text{VII}}\big) \cdot \text{Tot.Var.}\{w\} \qquad \forall\, t \in [t_{m',q_{m'}},\, T]. \end{aligned} \tag{4.62}$$

Thus, (4.61)–(4.62) together with (4.56), show that $y^\sharp(\cdot)$ satisfies the estimates (2.7)–(2.8) of property c'), taking

$$C'' > \big(5 + 10c_{11} + 6MC^{\text{VII}}\big), \tag{4.63}$$

which completes the proof of Lemma 2.2. □

Proof of Lemma 2.3.
1. As in the proof of Lemma 4.1, call $E^1, \dots, E^{p_{\alpha_o}}$, and $r^1, \dots, r^{p_{\alpha_o}}$, respectively, the edges of the polygonal domain D_{α_o} that form the boundary ∂D_{α_o}, and the corresponding lines in which the edges are contained. Let $\mathbf{n}^{\boldsymbol{\ell}}$ be the normal to E^ℓ pointing towards the interior of D_{α_o}, and call $\mathcal{I}_{\alpha_o}$ and $\mathcal{O}_{\alpha_o}$, respectively, the set of incoming and of outgoing indices defined as in (4.12), so that

$$\begin{aligned} &\big\langle g_{\alpha_o}(x),\ \mathbf{n}^{\boldsymbol{\ell}}\big\rangle \ > 0 \qquad \forall\, x \in \overline{D}_{\alpha_o} \cap r^\ell, \qquad \forall\, \ell \in \mathcal{I}_{\alpha_o},\\ &\big\langle g_{\alpha_o}(x),\ \mathbf{n}^{\boldsymbol{\ell}}\big\rangle \ < 0 \qquad \forall\, x \in \overline{D}_{\alpha_o} \cap r^\ell, \qquad \forall\, \ell \in \mathcal{O}_{\alpha_o}. \end{aligned} \tag{4.64}$$

Let V_{α_o} be the set of vertices of the domain D_{α_o}, and denote

$$\mathcal{T} \doteq \Big\{\gamma_j : [0,\, \widehat{\tau}^j\,] \mapsto \mathbb{R}^2\ ;\ j = 1, \dots, m\Big\} \tag{4.65}$$

the set of maximal trajectories of $\dot{y} = g_{\alpha_o}(y)$ that go through some vertex in V_{α_o}, and whose graph is contained in $\overline{D}_{\alpha_o}$. Thus, we have

$$\begin{aligned} \mathrm{Im}(\gamma_j) \cap V_{\alpha_o} \neq \emptyset, \qquad &\gamma_j(0),\ \gamma_j(\widehat{\tau}^j) \in \partial D_{\alpha_o},\\ \gamma_j(t) = x^{\alpha_o}\big(t;\ 0,\ \gamma_j(0)\big) \in \overline{D}_{\alpha_o} \qquad &\forall\, t \in [0,\, \widehat{\tau}^j\,]. \end{aligned} \tag{4.66}$$

In connection with every trajectory γ_j, there will be a partition $\widehat{t}_{j,1} = 0 < \widehat{t}_{j,2} < \cdots < \widehat{t}_{j,k_j} = \widehat{\tau}^j$ of $[0,\, \widehat{\tau}^j\,]$ such that

$$\begin{aligned} \gamma_j(\widehat{t}_{j,h}) \in \partial D_{\alpha_o} \qquad &\forall\, 1 \le h \le k_j,\\ \gamma_j(t) \in \overset{\circ}{D}_{\alpha_o} \qquad \forall\, t \in]\,\widehat{t}_{j,h},\, \widehat{t}_{j,h+1}[\,, \qquad &1 \le h < k_j. \end{aligned} \tag{4.67}$$

Then, set

$$\gamma_{j,h} \doteq \gamma_j\, \restriction_{[\widehat{t}_{j,h},\, \widehat{t}_{j,k_j}]},$$

and let

$$\begin{aligned} \mathcal{L}^{\mathcal{I}}_{j,h} &\doteq \big\{\ell \in 1, \dots, p_{\alpha_o}\ ;\ \ \gamma_j(\widehat{t}_{j,h}) \in E^\ell \quad \text{and} \quad \ell \in \mathcal{I}_{\alpha_o}\big\},\\ \mathcal{L}^{\mathcal{O}}_{j,h} &\doteq \big\{\ell \in 1, \dots, p_{\alpha_o}\ ;\ \ \gamma_j(\widehat{t}_{j,h}) \in E^\ell \quad \text{and} \quad \ell \in \mathcal{O}_{\alpha_o}\big\}, \end{aligned} \tag{4.68}$$

denote the incoming and outgoing indices of the edges of D_{α_o} that pass through $\gamma_j(\widehat{t}_{j,h})$. Moreover, set

$$\mathcal{L}_{j,h} \doteq \mathcal{L}^{\mathcal{I}}_{j,h} \cup \mathcal{L}^{\mathcal{O}}_{j,h}. \tag{4.69}$$

Since we are assuming that g_{α_o} is a smooth vector field defined on a neighborhood of $\overline{D}_{\alpha_o}$ that satisfies the transversality condition **(C)** and the inward-pointing condition (1.15), there will be some constant $\delta_8 > 0$ so that one has

$$\forall\, x_0 \in \overline{D}_{\alpha_o} \cap B\big(\gamma_j(\widehat{t}_{j,h}),\, \delta_8\big), \quad \forall\, h' > h, \quad \forall\, \ell \in \mathcal{L}^{\mathcal{O}}_{j,h'} \qquad \exists! \quad t > 0 \quad : \quad x^{\alpha_o}\big(t;\ 0,\ x_0\big) \in r^\ell. \tag{4.70}$$

Thus, for such x_0, ℓ, define

$$t^\ell(x_0) \doteq t > 0 \quad \text{s.t.} \quad x^{\alpha_o}(t;\ 0,\, x_0) \in r^\ell. \tag{4.71}$$

The regularity properties of the flow map of g_{α_o} together with the transversality condition **(C)** and the inward-pointing condition (1.15) guarantee that there exists some constant $c_{12} > 1$ such that

$$\left|t^\ell(x_0) - t^\ell(y_0)\right| \leq c_{12} \cdot \left|x_0 - y_0\right| \qquad \forall\, x_0,\, y_0. \tag{4.72}$$

2. We shall construct now an increasing tube $\Gamma(\gamma_j,\,\lambda)$ of size λ around each trajectory $\gamma_j \in \mathcal{T}$, which is positively invariant w.r.t. left-continuous, piecewise smooth function having the properties $a'')$–$b'')$. Take $0 < \lambda < \delta_8$, and let

$$\begin{aligned} F(\gamma_{_{j,1}},\,\lambda) &\doteq B\Big(\gamma_j(\widehat{t}_{j,1}),\,\lambda\Big) \cap \Bigg(\bigcup_{\ell\in\mathcal{L}_{j,1}} E^\ell\Bigg), \\ G(\gamma_{_{j,1}},\,\lambda) &\doteq \Big\{x^{\alpha_o}(s;\ 0,\, x_0) \quad ; \quad x_0 \in F(\gamma_{_{j,1}},\,\lambda), \quad 0 \leq s \leq t^{\ell'}(x_0), \quad \ell' \in \mathcal{L}^{\mathcal{O}}_{j,k_j}\Big\} \cap D_{\alpha_o}. \end{aligned} \tag{4.73}$$

Then, proceeding by induction on $h > 1$, and relying on (4.34), (4.70) and (4.72), one can show that there exists constants $c_{13} > 1$, $\overline{\lambda} < \delta_8/c_{13}$, such that, letting

$$\left.\begin{aligned} F(\gamma_{_{j,h}},\,\lambda) &\doteq B\Big(G(\gamma_{_{j,h-1}},\,\lambda),\,\lambda\Big) \cap \Bigg(\textstyle\bigcup_{\ell\in\mathcal{L}_{j,h}} E^\ell\Bigg), \\ G(\gamma_{_{j,h}},\,\lambda) &\doteq \Big\{x^{\alpha_o}(s;0,x_0) \ ;\ x_0 \in F(\gamma_{_{j,h}},\,\lambda), \quad 0 \leq s \leq t^{\ell'}(x_0), \quad \ell' \in \mathcal{L}^{\mathcal{O}}_{j,k_j}\Big\} \cap D_{\alpha_o}, \end{aligned}\right. \qquad \lambda \leq \overline{\lambda}, \tag{4.74}$$

for $1 < h < k_j$, one has

$$\left|\gamma_j(\widehat{t}_{j,h}) - x_0\right| \leq c_{13} \cdot \lambda \qquad \forall\, x_0 \in F(\gamma_{_{j,h}},\,\lambda), \quad 1 \leq h < k_j, \quad \lambda \leq \overline{\lambda}. \tag{4.75}$$

Moreover, we may choose $\overline{\lambda}$ so that, setting

$$\Gamma(\gamma_j,\,\lambda) \doteq \bigcup_{h=1}^{k_j-1} G(\gamma_{_{j,h}},\,\lambda), \qquad 1 \leq j \leq m, \tag{4.76}$$

there holds

$$\Gamma(\gamma_j,\,\lambda) \cap \Gamma(\gamma_i,\,\lambda) = \emptyset, \qquad \forall\, 1 \leq i,j \leq m, \quad i \neq j, \qquad \forall\, \lambda \leq \overline{\lambda}. \tag{4.77}$$

Let $R^1(\lambda), \dots, R^r(\lambda)$, denote the connected components of $\overline{D}_{\alpha_o} \setminus \cup_{j=1}^m \Gamma(\gamma_j,\,\lambda)$, and set

$$\rho(\lambda) \doteq \min\Big\{d\big(R^s(\lambda),\, E^\ell\big) \ \ ;\ \ 1 \leq s \leq r, \quad 1 \leq \ell \leq p_{\alpha_o}, \quad R^s(\lambda) \cap E^\ell = \emptyset\Big\}. \tag{4.78}$$

Observe that, by the transversality condition **(C)** and because of the inward-pointing condition (1.15), one has

$$\inf\left\{\frac{\rho(\lambda)}{\lambda} \ :\ 0 < \lambda \leq \overline{\lambda}\right\} > 0. \tag{4.79}$$

Therefore, there exist constants $c_{14} > 1$, $0 < \delta_9 < \overline{\lambda}/c_{14}$, so that

$$\rho(c_{14} \cdot \delta) > 2\delta \qquad \forall\, 0 < \delta \leq \delta_9. \tag{4.80}$$

3. Consider now a left-continuous, piecewise smooth function $y^\flat :]\tau_0, \tau_1] \mapsto \mathbb{R}^2$ having the properties $a'')$–$b'')$ with

$$\overline{\delta} \leq \delta_9. \tag{4.81}$$

Observe that, since $y^\flat$ is left-continuous, $a'')$ in particular implies

$$y^\flat(\tau_1) \in \partial D_{\alpha_o}. \tag{4.82}$$

The other two cases can be treated in an entirely similar manner. Since, by construction, the tubes $G(\gamma_{j,h}, \lambda)$ defined in (4.73)–(4.74) are invariant subsets of $\overline{D}_{\alpha_o}$ for the flow map of g_{α_o}, if

$$y^\flat(\tau_0{}^+) \in D_{\alpha_o} \setminus \bigcup_{j=1}^{m} \Gamma(\gamma_j, c_{14} \cdot \Delta(y^\flat)), \tag{4.83}$$

then, by property $a'')$ it follows that

$$y^\flat(t) \in \overline{D}_{\alpha_o} \setminus \bigcup_{j=1}^{m} \Gamma(\gamma_j, c_{14} \cdot \Delta(y^\flat)) \qquad \forall\ t \in]t_1', t_2']. \tag{4.84}$$

We claim that (4.84) implies $q_o = 2$. Indeed, if $R^s(c_{14} \cdot \Delta(y^\flat))$ denotes the connected component of $\overline{D}_{\alpha_o} \setminus \big(\cup_{j=1}^m \Gamma(\gamma_j, c_{14} \cdot \Delta(y^\flat))\big)$ that contains $\mathrm{Im}(y^\flat \restriction_{]t_1', t_2']})$, and we assume by contradiction that $q_o > 2$, relying on (4.77) and on property a''-*3*) we deduce that $y^\flat(t_2'{}^+)$ lies on some edge E^ℓ such that $R^s(c_{14} \cdot \Delta(y^\flat)) \cap E^\ell = \emptyset$. But then, using the inequality (4.80), and because of the definition (4.78) of the quantity ρ, we would obtain

$$\begin{aligned} \big|y^\flat(t_2') - y^\flat(t_2'{}^+)\big| &\geq d\big(R^s(c_{14} \cdot \Delta(y^\flat)),\, E^\ell\,\big) \\ &\geq \rho\big(c_{14} \cdot \Delta(y^\flat)\big) \\ &> 2\Delta(y^\flat) \end{aligned}$$

which yields a contradiction by the definition of $\Delta(y^\flat)$ at (2.10). Therefore, it must be $q_0 = 2$, and hence we have

$$y^\flat(t) = x^{\alpha_o}\big(t;\, \tau_0,\, y^\flat(\tau_0{}^+)\big) \qquad \forall\ t \in]\tau_0,\, \tau_1],$$

which, together with (2.9), (4.82), clearly shows that $Q_{\alpha_o} \doteq y^\flat(\tau_0{}^+)$, $\sigma_{\alpha_o} \doteq \tau_1$ enjoy pro-perties $c'')$–$e'')$ whenever (4.83) holds.

4. Assume now that (4.83) is not verified *i.e.* that, for some $1 \leq j \leq m$, $1 \leq h' \leq k_j$, there holds

$$y^\flat(\tau_0{}^+) \in G\big(\gamma_{_{j,h'}},\, c_{14} \cdot \Delta(y^\flat)\big). \tag{4.85}$$

Proceeding by induction on $h' \leq h < k_j$, and relying on (4.70) and (4.77) and on property $a'')$, one then easily derives that

$$y^\flat(t) \in \bigcup_{h=h'}^{k_j - 1} G\big(\gamma_{_{j,h}},\, c_{14} \cdot \Delta(y^\flat)\big) \subset \Gamma\big(\gamma_j,\, c_{14} \cdot \Delta(y^\flat)\big) \qquad \forall\ t \in]\tau_0, \tau_1]. \tag{4.86}$$

In connection with the partition $t_1' = \tau_0 < t_2' < \cdots < t_{q_o}' = \tau_1$ of $[\tau_0, \tau_1]$, induced by $y^\flat(\cdot)$ according with property $a'')$, define the triplet of indices $h(\ell)$, $p^-(\ell)$, $p^+(\ell)$, $1 < \ell \leq q_o$ so that $E^{p^-(\ell)}$, $E^{p^+(\ell)}$ denote the edges of D_{α_o} which cross the trajectory γ_j in $\gamma_j(\widehat{t}_{j,h(\ell)})$, $h(\ell) > h'$, and contain, respectively, the point $y^\flat(t'_\ell)$, and the point $y^\flat({t'_\ell}^+)$, *i.e.* such that

$$\begin{aligned} y^\flat(t'_\ell) \quad &\in E^{p^-(\ell)}, \qquad p^-(\ell) \quad \in \mathcal{L}^{\mathcal{O}}_{j,h(l)}, \qquad\quad \forall\ 1 < \ell \le q_0, \\ y^\flat({t'_\ell}^+) \quad &\in E^{p^+(\ell)}, \qquad p^+(\ell) \quad \in \mathcal{L}^{\mathcal{I}}_{j,h(l)} \qquad\quad \forall\ 1 < \ell < q_0. \end{aligned} \tag{4.87}$$

Then, set

$$Q_{\alpha_o} \doteq x^{\alpha_o}\big(t'_1 - t'_2;\ 0,\ \gamma_j(\widehat{t}_{j,h(2)})\big), \tag{4.88}$$

and observe that, by property $a'')$, one has

$$y^\flat(\tau_0^+) = x^{\alpha_o}\big(t'_1 - t'_2;\ 0,\ y^\flat(t'_2)\big). \tag{4.89}$$

Thus, using (4.34), (4.88)–(4.89), we obtain

$$\big|Q_{\alpha_o} - y^\flat(\tau_0^+)\big| \le c_9 \cdot \big|\gamma_j(\widehat{t}_{j,h(2)}) - y^\flat(t'_2)\big|. \tag{4.90}$$

On the other hand, since one can show with an inductive argument that (4.86)–(4.87) imply

$$\begin{aligned} y^\flat(t'_\ell) &\in F\big(\gamma_{j,h},\ c_{14} \cdot \Delta(y^\flat)\big), \qquad \forall\ 1 < \ell \le q_0, \\ y^\flat({t'_\ell}^+) &\in F\big(\gamma_{j,h},\ c_{14} \cdot \Delta(y^\flat)\big), \qquad \forall\ 1 < \ell < q_0, \end{aligned}$$

relying on (4.75) we derive

$$\begin{aligned} \big|y^\flat(t'_\ell) - \gamma_j(\widehat{t}_{j,h(\ell)})\big| &\le c_{13} \cdot c_{14} \cdot \Delta(y^\flat), \qquad \forall\ 1 < \ell \le q_0, \\ \big|y^\flat({t'_\ell}^+) - \gamma_j(\widehat{t}_{j,h(\ell)})\big| &\le c_{13} \cdot c_{14} \cdot \Delta(y^\flat), \qquad \forall\ 1 < \ell < q_0. \end{aligned} \tag{4.91}$$

In turn, the first estimate in (4.91) for $\ell = 2$, together with (4.90), yields

$$\big|Q_{\alpha_o} - y^\flat(\tau_0^+)\big| \le c_9 \cdot c_{13} \cdot c_{14} \cdot \Delta(y^\flat). \tag{4.92}$$

Moreover, observing that by the definitions (4.71), (4.87), one has

$$\left.\begin{aligned} t'_{\ell+1} - t'_\ell &= t^{p^-(\ell+1)}\big(y^\flat(t'_\ell)\big), \\ \widehat{t}_{j,h(\ell+1)} - \widehat{t}_{j,h(\ell)} &= t^{p^-(\ell+1)}\big(\gamma_j(\widehat{t}_{j,h(\ell)})\big), \end{aligned}\right. \qquad \forall\ 1 < \ell < q_0,$$

thanks to (4.72) we get

$$\big|(t'_{\ell+1} - t'_\ell) - (\widehat{t}_{j,h(\ell+1)} - \widehat{t}_{j,h(\ell)})\big| \le c_{12} \cdot c_{13} \cdot c_{14} \cdot \Delta(y^\flat), \qquad \forall\ 1 < \ell < q_0. \tag{4.93}$$

Therefore, since by the definition (4.88) we have

$$Q_{\alpha_o} = \gamma_j\big(\widehat{t}_{j,h(2)} + t'_1 - t'_2\big),$$

which, in particular, implies

$$\begin{aligned} x^{\alpha_o}\big(t;\, \tau_0,\, Q_{\alpha_o}\big) &= \gamma_j\big(\widehat{t}_{j,h(2)} - t'_2 + t\big), \\ x^{\alpha_o}\big(\widehat{t}_{j,h(\ell)} - \widehat{t}_{j,h(2)} + t'_2;\ \tau_0,\, Q_{\alpha_o}\big) &= \gamma_j\big(\widehat{t}_{j,h(\ell)}\big) \qquad \forall\ \ell \ge 2, \end{aligned} \tag{4.94}$$

and because by property $a'')$ one has

$$y^\flat(t) = x^{\alpha_o}\big(t;\ t'_\ell,\, y^\flat({t'_\ell}^+)\big) \qquad \forall\, t \in\,]t'_\ell,\, t'_{\ell+1}], \qquad 1 \le \ell < q_0, \tag{4.95}$$

relying on (4.91)–(4.93), and using (4.34) and (4.94)–(4.95), we derive

$$\begin{aligned} \big|y^\flat(t) - x^{\alpha_o}\big(t;\, \tau_0,\, Q_{\alpha_o}\big)\big| &\le \big|x^{\alpha_o}\big(t;\ t'_\ell,\, y^\flat({t'_\ell}^+)\big) - x^{\alpha_o}\big(t;\ t'_\ell,\, \gamma_j\big(\widehat{t}_{j,h(\ell)}\big)\big)\big| \\ &\quad + \big|x^{\alpha_o}\big(t + (\widehat{t}_{j,h(\ell)} - \widehat{t}_{j,h(2)}) - (t'_\ell - t'_2);\ t'_1,\, Q_{\alpha_o}\big) - x^{\alpha_o}\big(t;\, t'_1,\, Q_{\alpha_o}\big)\big| \\ &\le c_9 \cdot \Big(\big|y^\flat(t'_\ell) - \gamma_j\big(\widehat{t}_{j,h(\ell)}\big)\big| + \big|\big(\widehat{t}_{j,h(\ell)} - \widehat{t}_{j,h(2)}\big) - (t'_\ell - t'_2)\big|\Big) \\ &\le c_9 \cdot c_{13} \cdot c_{14} \cdot \big(1 + q_0 \cdot c_{12}\big) \cdot \Delta(y^\flat), \\ &\qquad \forall\, t \in\,]t'_\ell,\ \min\{t'_{\ell+1},\ \widehat{t}_{j,h(q_0)} - \widehat{t}_{j,h(2)} + t'_2\}], \qquad 1 \le \ell < q_0, \end{aligned} \tag{4.96}$$

and

$$\begin{aligned} \big|\big(\widehat{t}_{j,h(q_0)} - \widehat{t}_{j,h(2)} + t'_2\big) - \tau_1\big| &= \big|\big(\widehat{t}_{j,h(q_0)} - \widehat{t}_{j,h(2)}\big) - \big(t'_{q_0} - t'_2\big)\big| \\ &\le q_0 \cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot \Delta(y^\flat). \end{aligned} \tag{4.97}$$

Moreover, if we set $\sigma_{\alpha_o} \doteq \widehat{t}_{j,h(q_0)} - \widehat{t}_{j,h(2)} + t'_2$, thanks to (4.67) and (4.94) we obtain

$$x^{\alpha_o}\big(\sigma_{\alpha_o};\, \tau_0,\, Q_{\alpha_o}\big) = \gamma_j\big(\widehat{t}_{j,h(q_0)}\big) \in \partial D_{\alpha_o}. \tag{4.98}$$

Hence, (4.96)–(4.98) together, show that Q_{α_o} defined as in (4.88) and $\sigma_{\alpha_o} = \widehat{t}_{j,h(q_0)} - \widehat{t}_{j,h(2)} + t'_2$ enjoy properties $c'')$–$e'')$, taking $\overline{C} > c_9 \cdot c_{13} \cdot c_{14} \cdot \big(1 + q_0 \cdot c_{12}\big)$, in the case where (4.85) is verified, which completes the proof of Lemma 2.3. □

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