ANISOTROPIC FUNCTIONS: A GENERICITY RESULT WITH CRYSTALLOGRAPHIC IMPLICATIONS

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Abstract. In the 1950’s and 1960’s surface physicists/metallurgists such as Herring and Mullins applied ingenious thermodynamic arguments to explain a number of experimentally observed surface phenomena in crystals. These insights permitted the successful engineering of a large number of alloys, where the major mathematical novelty was that the surface response to external stress was anisotropic. By examining step/terrace (vicinal) surface defects it was discovered through lengthy and tedious experiments that the stored energy density (surface tension) along a step edge was a smooth symmetric function $\beta$ of the azimuthal angle $\theta$ to the step, and that the positive function $\beta$ attains its minimum value at $\theta = \pi/2$ and its maximum value at $\theta = 0$. The function $\beta$ provided the crucial thermodynamic parameters needed for the engineering of these materials. Moreover the minimal energy configuration of the step is determined by the values of the stiffness function $\beta'' + \beta$ which ultimately leads to the magnitude and direction of surface mass flow for these materials. In the 1990’s there was a dramatic improvement in electron microscopy which permitted real time observation of the meanderings of a step edge under Brownian heat oscillations. These observations provided much more rapid determination of the relevant thermodynamic parameters for the step edge, even for crystals at temperatures below their roughening temperature. Use of these tools led J. Hannon and his coexperimenters to discover that some crystals behave in a highly anti-intuitive manner as their temperature is varied. The present article is devoted to a model described by a class of variational problems. The main result of the paper describes the solutions of the corresponding problem for a generic integrand.

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1. INTRODUCTION

It is well-known in surface physics that when a crystalline substance is maintained at a temperature $T$ above its roughening temperature $T_R$ then the surface stored energy integrand, usually referred to as surface tension, is a smooth function $\beta$ of the azimuthal angle of orientation $\theta$. Furthermore, $\beta$ obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta), \quad 0 < \beta(\pi/2) \leq \beta(\theta) \leq \beta(0)$$

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Proof. Evidently its maximum, i.e., Remark 1.1. Note that inf$(\beta)$ for which the quantity $\beta'' + \beta$, usually referred to as the stiffness, is negative at the angle $\theta = 0$ where $\beta$ attains its maximum, i.e., $\beta''(0) + \beta(0) < 0$. That condition is present in a type of Silicon crystal which exhibited very unorthodox behavior during experimental observations involving mass transport along the surface of the crystal (cf. [3, 4]). This phenomenon led us to undertake the present investigation.

For each function $f : X \to R$ set inf$(f) = \inf\{f(x) : x \in X\}$.

Denote by $\mathcal{M}$ the set of all functions $\beta \in C^2(R)$ which satisfy the following assumption:

$\beta(t) \geq 0$ for all $t \in R$, \hspace{1cm} (1.1)  
$\beta(\pi/2) \leq \beta(t) \leq \beta(0)$ for all $t \in R$, \hspace{1cm} (1.2)  
$\beta(t) = \beta(-t)$ for all $t \in R$, \hspace{1cm} (1.3)  
$\beta(t + \pi) = \beta(t)$ for all $t \in R$, \hspace{1cm} (1.4)  
$\beta(0) + \beta''(0) \leq 0$. \hspace{1cm} (1.5)

For each $\beta_1, \beta_2 \in \mathcal{M}$ set

$$
\rho(\beta_1, \beta_2) = \sup \left\{ |\beta^{(i)}_1(t) - \beta^{(i)}_2(t)| : t \in R, i = 0, 1, 2 \right\}. \hspace{1cm} (1.6)
$$

It is not difficult to see that the metric space $(\mathcal{M}, \rho)$ is complete.

Denote by $\mathcal{M}_r$ the set of all $\beta \in \mathcal{M}$ such that

$\beta(t) > 0$ for all $t \in R$, \hspace{1cm} (1.7)  
$\beta(0) + \beta''(0) < 0$. \hspace{1cm} (1.8)

It is obvious that $\mathcal{M}_r$ is nonempty.

**Proposition 1.1.** $\mathcal{M}_r$ is an open everywhere dense subset of $(\mathcal{M}, \rho)$.

**Proof.** Evidently $\mathcal{M}_r$ is an open subset of $(\mathcal{M}, \rho)$. Let us show that $\mathcal{M}_r$ is an everywhere dense subset of $(\mathcal{M}, \rho)$.

Let $\beta \in \mathcal{M}$ and $\tilde{\beta} \in \mathcal{M}_r$. Then for each natural number $n$ the function $\beta + (n)^{-1}\tilde{\beta} \in \mathcal{M}_r$, \n
$$(\beta + (n)^{-1}\tilde{\beta})(t) \geq (n)^{-1}\tilde{\beta}(t) > 0$$

for all $t \in R$ and

$$(\beta + (n)^{-1}\tilde{\beta})(0) + (\beta + (n)^{-1}\tilde{\beta})''(0) = \beta(0) + \beta''(0) + [\tilde{\beta}(0) + \tilde{\beta}''(0)]/n < 0.$$

Thus $\beta + n^{-1}\tilde{\beta} \in \mathcal{M}_r$ for all natural numbers $n$. It is easy to see that $\beta + n^{-1}\tilde{\beta} \to \beta$ as $n \to \infty$ in $(\mathcal{M}, \rho)$. Therefore $\mathcal{M}_r$ is an everywhere dense subset of $(\mathcal{M}, \rho)$. Proposition 1.1 is proved. \hspace{1cm} $\square$

Let $\beta \in \mathcal{M}_r$. Define

$G_\beta(z) = \beta(\arctan(z))(1 + z^2)^{1/2}, z \in R$. \hspace{1cm} (1.9)

Clearly $G_\beta$ is a continuous function and

$G_\beta(z) \to \infty$ as $z \to \pm \infty$. \hspace{1cm} (1.10)

**Remark 1.1.** Note that $\inf(G_\beta) < \beta(0)$ (see [4]).
The classical model for the free energy of certain crystals is given by

\[ J(y) = \int_{0}^{s} \beta(\theta) \, ds \]

where \( s \) is arclength and \( y \) is a function defined on a fixed interval \([0, L]\) whose graph is the locus under consideration:

\[ y \in W^{1,1}(0, L), \quad \theta = \arctan(y') \in [-\pi/2, \pi/2]. \]

We can rewrite \( J \) in the form

\[ J(y) = \int_{0}^{L} G_{\beta}(y') \, dx. \]

It was shown in [4] that \( y \in W^{1,1}(0, L) \) is a minimizer of \( J \) if and only if

\[ |y'| \in \{ z \in \mathbb{R} : G_{\beta}(z) = \inf(G_{\beta}) \} \text{ a.e.} \]

In this paper we show that for a generic function \( \beta \) the set

\[ \{ z \in \mathbb{R} : G_{\beta}(z) = \inf(G_{\beta}) \} = \{ z_{\beta}, -z_{\beta} \} \]

where \( z_{\beta} \) is a unique positive number depending only on \( \beta \).

Denote by \( F \) the set of all \( \beta \in M_{r} \) which satisfy the following condition:

There is \( z_{\beta} \in R \) such that

\[ G_{\beta}(z) > G_{\beta}(z_{\beta}) \text{ for all } z \in R \setminus \{ z_{\beta}, -z_{\beta} \}. \] (1.11)

We will establish the following result.

**Theorem 1.1.** \( F \) is a countable intersection of open everywhere dense subsets of \((M, \rho)\).

2. PRELIMINARY RESULTS

**Proposition 2.1.** Let \( \beta \in M_{r} \). Then there exist \( M_{0} > 0 \) and a neighborhood \( U \) of \( \beta \) in \( M \) such that \( U \subset M_{r} \) and the following assertion holds:

if \( \phi \in U, \ z \in R \) and \( G_{\phi}(z) \leq \inf(G_{\phi}) + 1 \), then \( |z| \leq M_{0} \).

**Proof.** There is \( c_{0} > 0 \) such that

\[ \beta(t) \geq c_{0} \text{ for all } t \in R. \] (2.1)

There is a neighborhood \( U \) of \( \beta \) in \((M, \rho)\) such that

\[ U \subset M_{r} \] (2.2)

and for each \( \phi \in U \)

\[ \phi(t) \geq c_{0}/2 \text{ for all } t \in R, \] (2.3)

\[ \phi(0) \leq 2\beta(0). \] (2.4)

Assume that \( \phi \in U, \ z \in R, \)

\[ G_{\phi}(z) \leq \inf(G_{\phi}) + 1. \] (2.5)

Then (2.3), (2.4) hold. By (1.9), (2.3), (2.5), (2.4)

\[ (1 + z^{2})^{1/2}c_{0}/2 \leq \phi(\arctan(z))(1 + z^{2})^{1/2} = G_{\phi}(z) \leq G_{\phi}(0) + 1 = \phi(0) + 1 \leq 2\beta(0) + 1 \]

and

\[ |z| \leq 2(2\beta(0) + 1)c_{0}^{-1}. \]
Thus the assertion of Proposition 2.1 holds with $M_0 = 2(2\beta(0) + 1)c_0^{-1}$.

It is not difficult to see that the next proposition is true.

**Proposition 2.2.** Let $\beta \in \mathcal{M}_r$, $\epsilon, M > 0$. Then there exists a neighborhood $U$ of $\beta$ in $\mathcal{M}$ such that $U \subset \mathcal{M}_r$ and the following assertion holds:

if $\phi \in U$, $z \in R$, $|z| \leq M$, then

$$|G_\phi(z) - G_\beta(z)| \leq \epsilon.$$ 

**Proposition 2.3.** Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists a neighborhood $U$ of $\beta$ in $\mathcal{M}$ such that $U \subset \mathcal{M}_r$ and the following assertion holds:

if $\phi_1, \phi_2 \in U$, $z \in R$,

$$G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1, \quad (2.6)$$
then

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon.$$ 

**Proof.** By Proposition 2.1 there exist $M > 0$ and a neighborhood $U_1$ of $\beta$ in $(\mathcal{M}, \rho)$ such that

$$U_1 \subset \mathcal{M}_r \quad (2.7)$$

and the following property holds:

(P1) if $\phi \in U_1$, $z \in R$, $G_\phi(z) \leq \inf(G_\phi) + 1$, then $|z| \leq M$.

By Proposition 2.2 there is a neighborhood $U$ of $\beta$ in $(\mathcal{M}, \rho)$ such that $U \subset U_1$ and the following property holds:

(P2) If $\phi_1, \phi_2 \in U$, $z \in R$, $|z| \leq M$, then

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon. \quad (2.8)$$

Now assume that $\phi_1, \phi_2 \in U$, $z \in R$ and (2.6) holds. By (2.6) and the property (P1),

$$|z| \leq M. \quad (2.9)$$

By (2.9) and property (P2), the inequality (2.8) is true. Proposition 2.3 is proved. 

**Proposition 2.4.** Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists a neighborhood $U$ of $\beta$ in $\mathcal{M}$ such that $U \subset \mathcal{M}_r$ and for each $\phi \in U$,

$$|\inf(G_\phi) - \inf(G_\beta)| \leq \epsilon.$$ 

**Proof.** Let a neighborhood $U$ of $\beta$ in $\mathcal{M}$ be as guaranteed by Proposition 2.3. Let $\phi_1, \phi_2 \in U$. It is enough to show that

$$\inf(G_{\phi_2}) \leq \inf(G_{\phi_1}) + \epsilon.$$ 

By the choice of $U$ and Proposition 2.3 the inequality (2.8) holds for any $z \in R$ satisfying (2.6). This implies that

$$\inf(G_{\phi_2}) \leq \inf\{G_{\phi_2}(z) : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\}$$
$$\leq \inf\{G_{\phi_1}(z) + \epsilon : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\}$$
$$= \epsilon + \inf\{G_{\phi_1}(z) : z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\}$$
$$= \epsilon + \inf(G_{\phi_1})$$
\[ \inf(G_{\phi_2}) \leq \inf(G_{\phi_1}) + \epsilon. \]

This completes the proof of Proposition 2.4.

\[ \square \]

**Proposition 2.5.** Let \( \beta \in M_r, \bar{z} \in R, \)
\[ G_{\beta}(z) > G_{\beta}(\bar{z}) \text{ for all } z \in R \setminus \{ \bar{z}, -\bar{z} \}. \] (2.10)

Let \( \epsilon > 0. \) Then there exist a neighborhood \( U \) of \( \beta \) in \( M \) and \( \delta > 0 \) such that \( U \subset M_r \) and that for each \( \phi \in U \) and each \( z \in R \) satisfying
\[ G_{\phi}(z) \leq \inf(G_{\phi}) + \delta \] (2.11)

the inequality
\[ \min\{|z - \bar{z}|, |z + \bar{z}|\} \leq \epsilon \]
is true.

**Proof.** Let us assume the converse. Then for each natural number \( n \) there exist \( \phi_n \in M_r \) and \( z_n \in R \) such that
\[ \rho(\beta, \phi_n) \leq 1/n, \] (2.12)
\[ G_{\phi_n}(z_n) \leq \inf(G_{\phi_n}) + 1/n \] (2.13)
and
\[ \min\{|z_n - \bar{z}|, |z_n + \bar{z}|\} > \epsilon. \] (2.14)

By (2.12), (2.13) and Proposition 2.1 the sequence \( \{z_n\}_{n=1}^\infty \) is bounded. Extracting a subsequence and reindexing, if necessary, we may assume without loss of generality that there exists
\[ z_* = \lim_{n \to \infty} z_n. \] (2.15)

By (2.12) and Proposition 2.4
\[ \lim_{n \to \infty} \inf(G_{\phi_n}) = \inf(G_{\beta}). \] (2.16)

Since the sequence \( \{z_n\}_{n=1}^\infty \) is bounded it follows from (2.12) and Proposition 2.2 that
\[ \lim_{n \to \infty} [G_{\phi_n}(z_n) - G_{\beta}(z_n)] = 0. \] (2.17)

It follows from (2.15), (2.17), (2.13), (2.16) that
\[ G_{\beta}(z_*) = \lim_{n \to \infty} G_{\beta}(z_n) = \lim_{n \to \infty} G_{\phi_n}(z_n) = \lim_{n \to \infty} \inf(G_{\phi_n}) = \inf(G_{\beta}). \]

Thus
\[ G_{\beta}(z^*) = \inf(G_{\beta}). \]

By (2.10) either \( z_* = \bar{z} \) or \( z_* = -\bar{z} \) and

either \( z_n \to \bar{z} \) or \( z_n \to -\bar{z} \) as \( n \to \infty. \)

This contradicts (2.14). The contradiction we have reached proves Proposition 2.5.

\[ \square \]
For $\psi \in C^2(R)$ set

$$||\psi||_{C^2(R)} = \sup \left\{ |\psi^{(i)}(t)| : t \in R, \ i = 0, 1, 2 \right\}.$$ 

**Lemma 3.1.** Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$,

$$\beta(t) > \beta(\pi/2) \text{ for all } t \in [0, \pi/2).$$

Then there exist $\phi \in \mathcal{M}_r$, $\bar{z} \in R$ such that

$$\rho(\phi, \beta) \leq \epsilon,$$

$$G_\phi(z) > G_\phi(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}.$$

**Proof.** There is $\bar{z} \in R$ such that

$$G_\beta(\bar{z}) = \inf(G_\beta).$$

By Remark 1.1 $\bar{z} \neq 0$. We may assume that

$$\bar{z} > 0.$$ 

Set

$$\bar{\theta} = \arctan(\bar{z}) \in (0, \pi/2).$$

There exists $\psi \in C^\infty(R)$ such that

$$0 \leq \psi(t) \leq 1 \text{ for all } t \in R, \ \psi(t) = 0 \text{ if } |t| \geq 1, \ \psi(t) = 1 \text{ if } |t| \leq 1/2.$$ 

Set

$$\psi_1(t) = \psi(t)(1 - t^2), \ t \in R.$$ 

Clearly $\psi_1 \in C^\infty(R),$

$$0 \leq \psi_1(t) \leq 1 \text{ for all } t \in R, \ \psi_1(t) = 0 \text{ if } |t| \geq 1, \ \psi_1(t) = 1 - t^2 \text{ if } |t| \leq 1/2,$$

$$1 = \psi_1(0) > \psi_1(t) \text{ for all } t \in R \setminus \{0\}.$$ 

By (3.4), (3.1), and the relation $\beta \in \mathcal{M}_r$, we can choose positive constants $c_0, c_1$ such that $c_0 > 1$, $c_1 < 1$,

$$[\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}] \subset (0, \pi/2),$$

$$\inf \left\{ \beta(t) : t \in [\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}] \right\} - \beta(\pi/2) > 4c_1, \ c_1 < -|\beta(0) + \beta''(0)|,$$

$$||\psi_1||_{C^2(R)} < c.$$ 

Consider the function

$$\psi_2(t) = c_1 - c_1 \psi_1(c_0(t - \bar{\theta})), \ t \in R.$$ 

Clearly $\psi_2 \in C^\infty(R)$. By (3.12), (3.7)

$$0 \leq \psi_2(t) \leq c_1, \ t \in R,$$

$$\psi_2(t) = c_1 \text{ for each } t \in R \text{ satisfying } |t - \bar{\theta}| \geq c_0^{-1}.$$ 

By (3.12), (3.8)

$$\psi_2(\bar{\theta}) = 0,$$

$$\psi_2(t) > 0 \text{ for each } t \in R \setminus \{\bar{\theta}\}.$$ 

It is not difficult to see that there exists a function $\psi_3 : R \to R$ such that

$$\psi_3(t) = \psi_2(t), \ t \in [0, \pi/2],$$

$$\psi_3(-t) = \psi_3(t), \ t \in R,$$

$$\psi_3(t + \pi) = \psi_3(t), \ t \in R.$$
It is not difficult to see that $\psi_3 \in C^\infty(R)$,
\begin{align*}
0 & \leq \psi_3(t) \leq c_1, \ t \in R, \quad (3.18) \\
\psi_3(\bar{\theta}) & = 0, \\
\psi_3(t) & > 0 \text{ for all } t \in [0, \pi/2] \setminus \{\bar{\theta}\}, \\
\psi_3(t) & > 0 \text{ for all } t \in [-\pi/2, 0] \setminus \{-\bar{\theta}\}. \quad (3.19)
\end{align*}
Define
\begin{equation*}
\phi(t) = \beta(t) + \psi_3(t), \ t \in R. \quad (3.20)
\end{equation*}
Clearly $\phi \in C^2(R)$. By (3.20), (1.1), (3.18), (1.3), (3.17), (1.4) we have
\begin{equation*}
\phi(t) \geq 0 \text{ for all } t \in R \text{ and } \phi(t) = \phi(-t) = \phi(t + \pi) \text{ for all } t \in R. \quad (3.21)
\end{equation*}
By (3.20), the relation $\beta \in \mathcal{M}_r$, (3.18)
\begin{equation*}
\phi(t) > 0 \text{ for all } t \in R. \quad (3.22)
\end{equation*}
We show that
\begin{equation*}
\phi(0) + \phi''(0) < 0. \quad (3.23)
\end{equation*}
By (3.20), (3.17)
\begin{align*}
\phi(0) + \phi''(0) & = \beta(0) + \beta''(0) + \psi_3(0) + \psi''_3(0) \\
& = \beta(0) + \beta''(0) + \psi_2(0) + \psi''_2(0). \quad (3.24)
\end{align*}
By (3.14), (3.9),
\begin{equation*}
\psi_2(0) = c_1, \ \psi''_2(0) = 0. \quad (3.25)
\end{equation*}
Combined with (3.24), (3.10) this implies that
\begin{equation*}
\phi(0) + \phi''(0) = \beta(0) + \beta''(0) + c_1 < 0.
\end{equation*}
Thus (3.23) is true. We show that
\begin{equation*}
\phi(\pi/2) \leq \phi(t) \leq \phi(0) \quad (3.26)
\end{equation*}
for all $t \in R$. Clearly it is enough to show that this inequality holds for all $t \in [0, \pi/2]$.
Let $t \in [0, \pi/2]$. Then by (3.20), (3.17)
\begin{equation*}
\phi(t) = \beta(t) + \psi_2(t). \quad (3.27)
\end{equation*}
By (3.27), (3.13), (3.25), (1.2), (3.17), (3.20)
\begin{align*}
\phi(t) & \leq \beta(t) + c_1 = \beta(t) + \psi_2(0) \leq \beta(0) + \psi_2(0) \\
& = \beta(0) + \psi_3(0) = \phi(0).
\end{align*}
Thus
\begin{equation*}
\phi(t) \leq \phi(0). \quad (3.28)
\end{equation*}
By (3.20), (3.17), (3.14), (3.9)
\begin{equation*}
\phi(\pi/2) = \beta(\pi/2) + \psi_2(\pi/2) = \beta(\pi/2) + c_1. \quad (3.29)
\end{equation*}
There are two cases:
(a) $|t - \bar{\theta}| \geq c_0^{-1}$; (b) $|t - \bar{\theta}| < c_0^{-1}$.
Consider the case (a). Then it follows from (3.17), (3.14) that
\[ \psi_3(t) = \psi_2(t) = c_1. \]
Combined with (3.20), (1.2), (3.29) this implies that
\[ \phi(t) = \beta(t) + c_1 \geq \beta(\pi/2) + c_1 = \phi(\pi/2) \]
and
\[ \phi(t) \geq \phi(\pi/2). \]  
(3.30)

Consider the case (b) with
\[ |t - \bar{\theta}| < c_0^{-1}. \]  
(3.31)
By (3.20), (3.17), (3.31), (3.13), (3.10), (3.29)
\[ \phi(t) = \beta(t) + \psi_2(t) \geq \beta(t) > \beta(\pi/2) + 4c_1 > \phi(\pi/2) \]
and (3.30) is true. Thus (3.30) is true in both cases. (3.30), (3.28) imply (3.26).

We have shown that \( \phi \in \mathcal{M}_r \). By (1.6), (3.20), (3.17), (3.13), (3.11), (3.12)
\[ \rho(\beta, \phi) = \sup \{ \psi_3(t), |\psi'_3(t)|, |\psi''_3(t)| : t \in [-\pi/2, \pi/2] \} \]
\[ = \sup \{ \psi_2(t), |\psi'_2(t)|, |\psi''_2(t)| : t \in [0, \pi/2] \} \]
\[ \leq \max \{ c_1, ||\psi_1||_{C^2(R)}c_1c_0, ||\psi_1||_{C^2(R)}c_1c_0 \} = c_1c_0^2 ||\psi_1||_{C^2} < \epsilon. \]
Thus
\[ \rho(\phi, \beta) < \epsilon. \]

We have
\[ \phi(t) \geq \beta(t) \text{ for all } t \in R. \]  
(3.33)
This implies that
\[ G_\phi(t) \geq G_\beta(t) \text{ for all } t \in R. \]  
(3.34)
By (1.9), (3.4), (3.20), (3.17), (3.15), (3.2)
\[ G_\phi(\bar{\delta}) = \phi(\arctan(\bar{\delta}))(1 + \bar{\delta}^2)^{1/2} = \phi(\bar{\theta})(1 + \bar{\delta}^2)^{1/2} \]
\[ = (\beta + \psi_3)(\bar{\theta})(1 + \bar{\delta}^2)^{1/2} = \beta(\bar{\theta})(1 + \bar{\delta}^2)^{1/2} = G_\beta(\bar{\delta}) = \inf(G_\beta). \]  
(3.35)
(3.35), (3.34) imply that
\[ \inf(G_\phi) = \inf(G_\beta) = G_\phi(\bar{\delta}) = G_\beta(\bar{\delta}). \]  
(3.36)

Let
\[ z \in R \setminus \{ \bar{\delta}, -\bar{\delta} \}. \]  
(3.37)
It follows from (1.9), (3.20) that
\[ G_\phi(z) = \phi(\arctan(z))(1 + z^2)^{1/2} = \beta(\arctan(z))(1 + z^2)^{1/2} \]
\[ + \psi_3(\arctan(z))(1 + z^2)^{1/2} = G_\beta(z) + \psi(\arctan(z))(1 + z^2)^{1/2}. \]  
(3.38)
By (3.37), (3.19)
\[ \psi_3(\arctan(z)) > 0. \]
Combined with (3.38), (3.36) this inequality implies that

\[ G_\phi(z) > G_\beta(z) \geq G_\beta(\bar{z}) = G_\phi(\bar{z}). \]

Lemma 3.1 is proved.

Lemma 3.2. Let \( \beta \in \mathcal{M}_r, \, \epsilon > 0. \) Then there exists \( \tilde{\beta} \in \mathcal{M}_r \) such that \( \rho(\beta, \tilde{\beta}) < \epsilon, \, \tilde{\beta}(t) > \tilde{\beta}(\pi/2) \) for all \( t \in [0, \pi/2). \)

Proof. Consider the function \( \beta_0(t) = \cos(2t) + 3/2, \, t \in \mathbb{R}. \)

Clearly \( \beta_0 \in \mathcal{M}_r. \) For all \( t \in [0, \pi/2) \)

\[ \beta_0(t) = \cos(2t) + 3/2 > (\cos \pi) + 3/2 = \beta_0(\pi/2). \]

For each natural number \( n \) set \( \beta_n(t) = \beta(t) + n^{-1}\beta_0(t), \, t \in \mathbb{R}. \)

Clearly for all natural \( n, \beta_n \in \mathcal{M}_r, \beta_n(t) > \beta_n(\pi/2) \) for all \( t \in [0, \pi/2), \)

\( \beta_n \to \beta \) as \( n \to \infty \) in \( (\mathcal{M}, \rho). \)

Lemma 3.2 is proved.

Lemmas 3.1 and 3.2 imply

Lemma 3.3 (basic lemma). Let \( \beta \in \mathcal{M}_r, \, \epsilon > 0. \) Then there exists \( \phi \in \mathcal{M}_r, \, \bar{z} \in \mathbb{R} \) such that \( \rho(\phi, \beta) < \epsilon, \)

\[ G_\phi(z) > G_\phi(\bar{z}) \] for all \( z \in \mathbb{R} \setminus \{\bar{z}, -\bar{z}\}. \)

4. Proof of Theorem 1.1

By Proposition 1.1 and Lemma 3.3, \( \mathcal{F} \) is an everywhere dense subset of \( (\mathcal{M}, \rho). \) Let \( \beta \in \mathcal{F}, \, z_\beta > 0 \) satisfy (1.11), \( n \) be a natural number.

By Proposition 2.5 there are an open neighborhood \( U(\beta, n) \) of \( \beta \) in \( (\mathcal{M}, \rho) \) and a number \( \delta(\beta, n) > 0 \) such that

\[ U(\beta, n) \subset \mathcal{M}_r \]

and the following property holds:

(P3) if \( \phi \in U(\beta, n), \, z \in \mathbb{R}, G_\phi(z) \leq \inf(G_\phi) + \delta(\beta, n), \) then

\[ \min\{|z - z_\beta|, |z + z_\beta|\} \leq 1/n. \]

Set

\[ \mathcal{F}_0 = \cap_{n=1}^{\infty} \cup \{U(\beta, n) : \beta \in \mathcal{F}, \, n \text{ is a natural number}\}. \]

Clearly \( \mathcal{F} \subset \mathcal{F}_0 \) and \( \mathcal{F}_0 \) is a countable intersection of open everywhere dense subsets of \( (\mathcal{M}, \rho). \) Let \( \phi \in \mathcal{F}_0, \, z_1, z_2 \in [0, \infty), \)

\[ G_\phi(z_1) = G_\phi(z_2) = \inf(G_\beta). \]

Let \( n \geq 1 \) be an integer. By (4.1) there is \( \beta \in \mathcal{F} \) such that

\[ \phi \in U(\beta, n). \]
By (4.3), (P3), (4.2),
\[
|z_i - z_{\beta}| \leq 1/n, \ i = 1, 2,
\]
\[
|z_1 - z_2| \leq 2/n.
\]
Since \(n\) is any natural number we conclude that \(z_1 = z_2, \ \phi \in \mathcal{F}\) and \(\mathcal{F}_0 = \mathcal{F}\). This completes the proof of Theorem 1.1.

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