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TOPOLOGICAL ASYMPTOTIC ANALYSIS OF THE KIRCHHOFF PLATE BENDING PROBLEM

Samuel Amstutz[1] and Antonio A. Novotny[2]

Abstract. The topological asymptotic analysis provides the sensitivity of a given shape functional with respect to an infinitesimal domain perturbation, like the insertion of holes, inclusions, cracks. In this work we present the calculation of the topological derivative for a class of shape functionals associated to the Kirchhoff plate bending problem, when a circular inclusion is introduced at an arbitrary point of the domain. According to the literature, the topological derivative has been fully developed for a wide range of second-order differential operators. Since we are dealing here with a forth-order operator, we perform a complete mathematical analysis of the problem.

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1. Introduction

The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation, such as the insertion of holes, inclusions, source-terms or even cracks. The topological derivative was rigorously introduced in [15]. Since then, this tool has proved extremely useful in the treatment of a wide range of problems, namely, topology optimization [1,3,14], inverse analysis [4,6,8] and image processing [5,9,10], and has become a subject of intensive research. Concerning the theoretical development of the topological asymptotic analysis, the reader may refer to [2,7,12], for instance.

In order to present this concept in more details, let us consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, which is submitted to a non-smooth perturbation confined in a small region $\omega_\varepsilon(\widehat{x}) = \widehat{x} + \varepsilon\omega$ of size ε. Here, $\widehat{x}$ is an arbitrary point of Ω and ω is a fixed domain of $\mathbb{R}^n$. We denote by Ω_ε the topologically perturbed domain which, in the case of a perforation, is defined by $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon(\widehat{x})}$. Then, we assume that a given shape functional ψ admits the following topological asymptotic expansion

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon) D_T\psi(\widehat{x}) + o(f(\varepsilon)), \tag{1.1}$$

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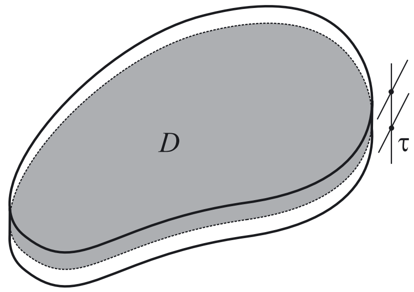

FIGURE 1. Sketch of the working domain.

where $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$. The number $D_T\psi(\widehat{x})$ is called the topological derivative of ψ at $\widehat{x}$. Therefore, this derivative can be seen as a first order correction on $\psi(\Omega)$ to estimate $\psi(\Omega_\varepsilon)$.

According to the literature, the topological derivative has been fully developed for a wide range of second-order differential operators. In [13] the formal calculation of the topological derivative for the total potential energy associated to the Kirchhoff plate bending problem, when the domain is perturbed by the introduction of an infinitesimal hole with homogeneous Neumann boundary condition, was presented. We recall that this mechanical model involves a forth-order differential operator.

In this work we provide a full mathematical justification for the formula derived in [13]. In particular, we discuss the regularity assumptions and provide precise estimates of the remainders of the topological asymptotic expansion. We also extend the result obtained in [13] by considering as topological perturbation the nucleation of an infinitesimal circular inclusion instead of a hole. Finally, we derive the closed formulas associated to a large class of shape functionals, including the total potential energy.

The paper is organized as follows. Section 2 describes the model associated to the Kirchhoff plate bending problem. The topological asymptotic analysis of the biharmonic operator is developed in Section 3, where the main result of the paper is stated, namely, a closed formula for the topological derivative. In Section 4 we provide the appropriate estimates of the remainders that come out from the topological asymptotic analysis. Finally, some examples of shape functionals are given in Section 5.

2. Problem statement

In this section we introduce a plate bending problem under Kirchhoff's kinematic assumptions. Thus, let us consider a plate represented by a two-dimensional domain $D \subset \mathbb{R}^2$ with thickness $\tau > 0$ supposed to be constant for simplicity. We assume that the plate is submitted to bending effects. In order to model this phenomenon Kirchhoff developed, in 1850, a theory based on the following *ad hoc* kinematic assumptions:

> *The normal fibers to the middle plane of the plate remain normal during deformation and do not suffer variations in their length.*

Consequently, both transversal shear and normal deformations are null. This fact limits the application of Kirchhoff's approach to plates whose deflections are small in relation to the thickness τ.

2.1. **The topology optimization problem**

Let D be a bounded domain of $\mathbb{R}^2$ as shown in Figure 1. This represents the domain in which the middle plane of the plate to be optimized must be contained. We assume that the boundary of D is a curvilinear polygon of class $\mathcal{C}^{1,1}$. Then we consider the topology optimization problem:

$$\underset{\Omega \subset D}{\text{Minimize}}\, J_\Omega(u_\Omega), \tag{2.1}$$

subject to the state equation: find $u_\Omega \in \mathcal{V}_{h,g}$, such that

$$\int_D \gamma_\Omega \mathcal{M}(u_\Omega) \cdot \nabla\nabla\varphi \, \mathrm{d}x = \int_{\Gamma_{N_q}} q\varphi \, \mathrm{d}s + \int_{\Gamma_{N_m}} m\partial_n\varphi \, \mathrm{d}s + \sum_{i=1}^{N} Q_i\varphi(x_{v_i}) \quad \forall\varphi \in \mathcal{V}_{0,0}. \tag{2.2}$$

Above, $\mathcal{V}_{h,g}$ is the set of kinematically admissible displacements and $\mathcal{V}_{0,0}$ is the space of admissible displacements variations, which are respectively defined by

$$\mathcal{V}_{h,g} := \left\{ u \in H^2(D) : u|_{\Gamma_{D_h}} = h \text{ and } \partial_n u|_{\Gamma_{D_g}} = g \right\}, \tag{2.3}$$

$$\mathcal{V}_{0,0} := \left\{ \varphi \in H^2(D) : \varphi|_{\Gamma_{D_h}} = 0 \text{ and } \partial_n \varphi|_{\Gamma_{D_g}} = 0 \right\}. \tag{2.4}$$

Some terms in (2.2)–(2.4) require explanation. The function u_Ω is the transversal displacement (or deflection) of the plate. The Dirichlet and Neumann boundaries are respectively denoted by the pairs $(\Gamma_{D_h}, \Gamma_{D_g})$ and $(\Gamma_{N_m}, \Gamma_{N_q})$, such that $\Gamma_{D_h} \cap \Gamma_{N_q} = \varnothing$ and $\Gamma_{D_g} \cap \Gamma_{N_m} = \varnothing$ with Γ_{D_h} and Γ_{D_g} of nonzero measure. On Γ_{D_h} and Γ_{D_g} we respectively prescribe a displacement $h \in H^{3/2}(\Gamma_{D_h})$ and a rotation $g \in H^{1/2}(\Gamma_{D_g})$. The system of forces compatible with Kirchhoff's kinematic assumptions are given by $q \in H^{3/2}(\Gamma_{N_q})'$, $m \in H^{-1/2}(\Gamma_{N_m})$ and $Q_i \in \mathbb{R}$. In the right hand side of (2.2), the integrals are to be understood as duality pairings on Sobolev trace spaces. The distributions q and m stand for a transverse shear load and a moment, respectively prescribed on Γ_{N_q} and Γ_{N_m}. Finally, Q_i is a transverse shear load concentrated at the point $x_{v_i} \in \Gamma_{N_q}$ in which there is some singularity, and N is the number of such singularities. The Young modulus γ_Ω is a piecewise constant function which takes two values:

$$\gamma_\Omega = \begin{cases} \gamma_{\text{in}} & \text{in } \Omega, \\ \gamma_{\text{out}} & \text{in } D \setminus \overline{\Omega}, \end{cases} \tag{2.5}$$

where $\gamma_{\text{in}} > 0$ and $\gamma_{\text{out}} \geq 0$. If $\gamma_{\text{out}} = 0$, only the values of u_Ω restricted to Ω are to be considered in the objective functional J_Ω. The resultant moment tensor $\mathcal{M}(u_\Omega)$, normalized to a unitary Young modulus, is related to the displacement field u_Ω through the Hooke law:

$$\mathcal{M}(u) = k\mathbb{C}\nabla\nabla u, \tag{2.6}$$

where

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(I \otimes I) \tag{2.7}$$

is the elasticity tensor, and

$$k = \frac{\tau^3}{12}\cdot \tag{2.8}$$

Here, I and $\mathbb{I}$ are the second and fourth order identity tensors, respectively, and the Lamé coefficients μ and λ are given by

$$\mu = \frac{1}{2(1+\nu)} \qquad \text{and} \qquad \lambda = \frac{\nu}{1-\nu^2}, \tag{2.9}$$

where ν is the Poisson ratio. We recall that τ is the plate thickness.

In order to guarantee the existence and uniqueness of a solution to (2.2) (as a consequence of the Lax-Milgram theorem), we need to include the following additional assumptions:

- meas$(\Gamma_{D_g} \cap \Gamma_{D_h}) \neq 0$ or Γ_{D_g} is not straight (its unit normal is not constant) or Γ_{D_h} is not straight;
- if $\gamma_{\text{out}} = 0$, then in addition $\Gamma_{D_g}, \Gamma_{D_h}, \Gamma_{N_m}, \Gamma_{N_q}$ are parts of $\partial\Omega$. Note that in this case, the uniqueness holds only for the restriction of u_Ω to Ω.

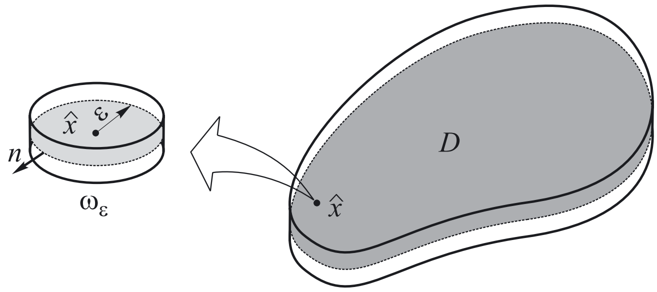

FIGURE 2. Sketch of the perturbed domain (in this example, $\Omega = D$).

TABLE 1. Coefficients $\widehat{\gamma}_0 = \gamma_0(\widehat{x})$ and $\widehat{\gamma}_1 = \gamma_1(\widehat{x})$ according to the location of $\widehat{x}$.

$\widehat{x}$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$
Ω	γ_{in}	γ_{out}
$D \setminus \overline{\Omega}$	γ_{out}	γ_{in}

2.2. **Topological perturbations**

Given a point $\widehat{x} \in D \setminus \partial\Omega$ and a radius $\varepsilon > 0$, we consider a circular inclusion $\omega_\varepsilon = B(\widehat{x}, \varepsilon)$, and we define the perturbed domain (see Fig. 2):

$$\Omega_\varepsilon = \begin{cases} \Omega \setminus \overline{\omega_\varepsilon} & \text{if} \quad \widehat{x} \in \Omega, \\ (\Omega \cup \omega_\varepsilon) \cap D & \text{if} \quad \widehat{x} \in D \setminus \overline{\Omega}. \end{cases} \tag{2.10}$$

We denote for simplicity $(J_{\Omega_\varepsilon}, u_{\Omega_\varepsilon}, \gamma_{\Omega_\varepsilon})$ by $(J_\varepsilon, u_\varepsilon, \gamma_\varepsilon)$ and $(J_\Omega, u_\Omega, \gamma_\Omega)$ by (J_0, u_0, γ_0). Then, for all $\varepsilon \in [0, 1]$, γ_ε can be expressed as:

$$\gamma_\varepsilon = \begin{cases} \gamma_0 & \text{in} \quad D \setminus \overline{\omega_\varepsilon}, \\ \gamma_1 & \text{in} \quad \omega_\varepsilon, \end{cases} \tag{2.11}$$

where γ_0 and γ_1 are piecewise constant functions, constant in the neighborhood of $\widehat{x}$. We will use later the notations $\widehat{\gamma}_0 := \gamma_0(\widehat{x})$ and $\widehat{\gamma}_1 := \gamma_1(\widehat{x})$. For the reader's convenience, the possible values of $\widehat{\gamma}_0$ and $\widehat{\gamma}_1$ are reported in Table 1. Of course, if $\gamma_{\text{out}} = 0$, one has to choose $\widehat{x} \in \Omega$ (one cannot create a new connected component).

For all $\varepsilon \geq 0$, the function $u_\varepsilon \in \mathcal{V}_{h,g}$ satisfies the equilibrium equation:

$$\int_D \gamma_\varepsilon \mathcal{M}(u_\varepsilon) \cdot \nabla\nabla\varphi \, \mathrm{d}x = \int_{\Gamma_{N_q}} q\varphi \, \mathrm{d}s + \int_{\Gamma_{N_m}} m\partial_n\varphi \, \mathrm{d}s + \sum_{i=1}^{N} Q_i \varphi(x_{v_i}) \quad \forall \varphi \in \mathcal{V}_{0,0}. \tag{2.12}$$

We assume that ε is small enough so that $\gamma_1 = \widehat{\gamma}_1$ in ω_ε. For $\widehat{\gamma}_1$ we have two possibilities, which depend on γ_{out}.

(1) If $\gamma_{\text{out}} > 0$, then necessarily $\widehat{\gamma}_1 > 0$. The solution of (2.12) is unique.

(2) If $\gamma_{\text{out}} = 0$, then $\widehat{\gamma}_1 = 0$ since $\widehat{x} \in \Omega$, which leads to a homogeneous Neumann boundary condition on $\partial\omega_\varepsilon$. In this case, the solution to (2.12) is not unique. To circumvent this difficulty, we introduce the following additional condition

$$\int_{\omega_\varepsilon} \mathcal{M}(u_\varepsilon) \cdot \nabla\nabla\varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in H_0^2(\omega_\varepsilon). \tag{2.13}$$

Then we clearly get uniqueness of u_ε in Ω.

In order to solve (2.1), we are looking for an asymptotic expansion of the objective functional, named as topological asymptotic expansion, of the form

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = f(\varepsilon) D_T J_\Omega(\widehat{x}) + o(f(\varepsilon)), \tag{2.14}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a function that goes to zero with ε, and $D_T J_\Omega : D \to \mathbb{R}$ is the so-called topological derivative of the functional J_Ω.

3. Topological sensitivity analysis of a class of shape functionals

In this section, the topological sensitivity analysis of the shape functional J_Ω is carried out. Possibly shifting the origin of the coordinate system, we assume henceforth for simplicity that $\widehat{x} = 0$.

3.1. A preliminary result

We start by proving an affine version of a result of [2].

Proposition 3.1. *Let $\mathcal{U}$ be a vector space, $\mathcal{V}$ be a subspace of $\mathcal{U}$, w be an element of $\mathcal{U}$, and ε_0 be a positive number. For all $\varepsilon \in [0, \varepsilon_0)$, consider a vector $u_\varepsilon \in \{w\} + \mathcal{V}$ solution of a problem of the form:*

$$a_\varepsilon(u_\varepsilon, \varphi) = \ell_\varepsilon(\varphi) \quad \forall \varphi \in \mathcal{V}, \tag{3.1}$$

where a_ε and ℓ_ε are a bilinear form on $\mathcal{U} \times \mathcal{U}$ and a linear form on $\mathcal{V}$, respectively. Consider also, for all $\varepsilon \in [0, \varepsilon_0)$, a functional $J_\varepsilon : \mathcal{U} \to \mathbb{R}$ and a linear form $L_\varepsilon(u_0) \in \mathcal{V}'$. Suppose that the following hypotheses hold.

(1) *For all $\varepsilon \in [0, \varepsilon_0)$, there exists $v_\varepsilon \in \mathcal{V}$ solution of*

$$a_\varepsilon(\varphi, v_\varepsilon) = -\langle L_\varepsilon(u_0), \varphi \rangle \quad \forall \varphi \in \mathcal{V}. \tag{3.2}$$

(2) *There exist two numbers δa and $\delta \ell$ and a function $\varepsilon \in [0, \varepsilon_0) \mapsto f(\varepsilon) \in \mathbb{R}$ such that, when ε goes to zero,*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \tag{3.3}$$

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = f(\varepsilon)\delta \ell + o(f(\varepsilon)). \tag{3.4}$$

(3) *There exist two numbers δJ_1 and δJ_2 such that*

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + \langle L_\varepsilon(u_0), u_\varepsilon - u_0 \rangle + f(\varepsilon)\delta J_1 + o(f(\varepsilon)), \tag{3.5}$$

$$J_\varepsilon(u_0) = J_0(u_0) + f(\varepsilon)\delta J_2 + o(f(\varepsilon)). \tag{3.6}$$

Then we have

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = f(\varepsilon)(\delta a - \delta \ell + \delta J_1 + \delta J_2) + o(f(\varepsilon)).$$

Proof. From (3.5) and (3.6), we obtain

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \langle L_\varepsilon(u_0), u_\varepsilon - u_0 \rangle + f(\varepsilon)(\delta J_1 + \delta J_2) + o(f(\varepsilon)).$$

Taking into account (3.2) and the fact that $u_\varepsilon - u_0 \in \mathcal{V}$, we get

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J_0(u_0) &= -a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + f(\varepsilon)(\delta J_1 + \delta J_2) + o(f(\varepsilon)) \\ &= -a_\varepsilon(u_\varepsilon, v_\varepsilon) + (a_\varepsilon - a_0)(u_0, v_\varepsilon) + a_0(u_0, v_\varepsilon) + f(\varepsilon)(\delta J_1 + \delta J_2) + o(f(\varepsilon)). \end{aligned}$$

The state equation (3.1) yields

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = -\ell_\varepsilon(v_\varepsilon) + (a_\varepsilon - a_0)(u_0, v_\varepsilon) + \ell_0(v_\varepsilon) + f(\varepsilon)(\delta J_1 + \delta J_2) + o(f(\varepsilon)).$$

Finally, from the hypotheses (3.3) and (3.4), it comes

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = -f(\varepsilon)\delta\ell + f(\varepsilon)\delta a + f(\varepsilon)(\delta J_1 + \delta J_2) + o(f(\varepsilon)). \qquad \square$$

For readability, we focus in Sections 3.2 through 3.4 on the case where $\gamma_{\text{out}} > 0$. The case $\gamma_{\text{out}} = 0$ is discussed in Section 3.5.

3.2. Some notations

By defining the spaces

$$\mathcal{U} = H^2(D), \tag{3.7}$$

$$\mathcal{V} = \left\{\varphi \in H^2(D) : \varphi|_{\Gamma_{D_h}} = 0 \text{ and } \partial_n\varphi|_{\Gamma_{D_g}} = 0\right\}, \tag{3.8}$$

and the function w as an arbitrary lifting of the Dirichlet boundary condition (h, g) in $\mathcal{U}$, problem (2.12) can be written in the form (3.1) with the help of the bilinear and linear forms:

$$a_\varepsilon(u,\varphi) = \int_D \gamma_\varepsilon \mathcal{M}(u)\cdot\nabla\nabla\varphi \, \mathrm{d}x \qquad \forall u,\varphi \in \mathcal{U}, \tag{3.9}$$

$$\ell_\varepsilon(\varphi) = \int_{\Gamma_{N_q}} q\varphi \, \mathrm{d}s + \int_{\Gamma_{N_m}} m\partial_n\varphi \, \mathrm{d}s + \sum_{i=1}^{N} Q_i\varphi(x_{v_i}) \qquad \forall \varphi \in \mathcal{V}. \tag{3.10}$$

We consider an objective functional satisfying the hypotheses (3.5)–(3.6) for a function $f(\varepsilon)$ which will be specified later. Then the perturbed adjoint state $v_\varepsilon \in \mathcal{V}$ has to solve the following problem:

$$\int_D \gamma_\varepsilon \mathcal{M}(v_\varepsilon)\cdot\nabla\nabla\varphi \, \mathrm{d}x = -\langle L_\varepsilon(u_0), \varphi\rangle \quad \forall\varphi \in \mathcal{V}. \tag{3.11}$$

By the Lax-Milgram theorem, this problem admits a unique solution.

3.3. Variation of the bilinear form

In order to apply Proposition 3.1, we need to obtain a closed form for the leading term of the quantity:

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \int_{\omega_\varepsilon} (\gamma_1 - \gamma_0)\mathcal{M}(u_0)\cdot\nabla\nabla v_\varepsilon \, \mathrm{d}x. \tag{3.12}$$

In the course of the analysis, the remainders detached from this expression will be denoted by $\mathcal{E}_i(\varepsilon)$, $i = 1, 2, ...$

By setting $\tilde{v}_\varepsilon = v_\varepsilon - v_0$ and assuming that ε is sufficiently small so that γ_0 and γ_1 are constant in ω_ε, we obtain:

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = (\widehat{\gamma}_1 - \widehat{\gamma}_0)\left(\int_{\omega_\varepsilon} \mathcal{M}(u_0)\cdot\nabla\nabla v_0 \, \mathrm{d}x + \int_{\omega_\varepsilon} \mathcal{M}(u_0)\cdot\nabla\nabla\tilde{v}_\varepsilon \, \mathrm{d}x\right).$$

Since u_0 and v_0 are smooth in the vicinity of $\widehat{x}$ (at least $\mathcal{C}^{4,\alpha}$ under the assumptions of Thm. 3.1), we approximate $\mathcal{M}(u_0)$ and $\nabla\nabla v_0$ in the first integral by their values at the point $\widehat{x}$, and write:

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = (\widehat{\gamma}_1 - \widehat{\gamma}_0)\left(\pi\varepsilon^2\mathcal{M}(u_0)(\widehat{x})\cdot\nabla\nabla v_0(\widehat{x}) + \int_{\omega_\varepsilon} \mathcal{M}(u_0)\cdot\nabla\nabla\tilde{v}_\varepsilon \, \mathrm{d}x + \mathcal{E}_1(\varepsilon)\right). \tag{3.13}$$

We assume that the linear functional $L_\varepsilon(u_0)$ is of the form:

$$\langle L_\varepsilon(u_0),\varphi\rangle = \int_D \gamma_\varepsilon(b(u_0)\varphi + \mathcal{B}(u_0)\cdot\nabla\nabla\varphi)\,\mathrm{d}x \;+\langle L,\varphi\rangle \qquad \forall\varphi\in\mathcal{V}, \tag{3.14}$$

where $L\in\mathcal{V}'$, $b(u_0)\in L^2(D)$ is a scalar field and $\mathcal{B}(u_0)\in L^2(D)$ is a second order tensor field. We assume further that $\langle L,\varphi\rangle$ does not depend on the value of φ in a neighborhood B of $\widehat{x}$, *i.e.*, $\langle L,\varphi\rangle = \langle\tilde{L},\varphi|_{D\setminus B}\rangle$. As v_ε is solution of (3.2), then, by difference, we find that the function $\tilde{v}_\varepsilon\in\mathcal{V}$ solves

$$\int_D \gamma_\varepsilon\mathcal{M}(\widetilde{v}_\varepsilon)\cdot\nabla\nabla\varphi\,\mathrm{d}x = -(\widehat{\gamma}_1-\widehat{\gamma}_0)\left(\int_{\partial\omega_\varepsilon} M^{nn}\partial_n\varphi\,\mathrm{d}s - \int_{\partial\omega_\varepsilon}(\partial_t M^{tn} + \mathrm{div}M\cdot n)\varphi\,\mathrm{d}s\right)\quad\forall\varphi\in\mathcal{V}. \tag{3.15}$$

The tensor field M introduced above is defined by

$$M = M_1 + M_2,$$

with

$$M_1 = \mathcal{M}(v_0) \qquad\text{and}\qquad M_2 = \mathcal{B}(u_0).$$

The notation ∂_t stands for the tangential derivative, and $M^{tn} = t\cdot Mn$, where t and n are the unit tangent and outward unit normal to $\partial\omega_\varepsilon$. The corresponding strong formulation for $\tilde{v}_\varepsilon$ reads:

$$\left\{\begin{array}{lll}
\mathrm{div}[\mathrm{div}(\gamma_0\mathcal{M}(\widetilde{v}_\varepsilon))] = 0 & \text{in} & D\setminus\overline{\omega_\varepsilon},\\
\mathrm{div}[\mathrm{div}(\mathcal{M}(\widetilde{v}_\varepsilon))] = 0 & \text{in} & \omega_\varepsilon\\
\widetilde{v}_\varepsilon = 0 & \text{on} & \Gamma_{D_h},\\
\partial_n\widetilde{v}_\varepsilon = 0 & \text{on} & \Gamma_{D_g},\\
\gamma_0\mathcal{M}^{nn}(\widetilde{v}_\varepsilon) = 0 & \text{on} & \partial D\setminus\Gamma_{D_g},\\
\partial_t[\gamma_0\mathcal{M}^{tn}(\widetilde{v}_\varepsilon)] + \mathrm{div}[\gamma_0\mathcal{M}(\widetilde{v}_\varepsilon)]\cdot n = 0 & \text{on} & \partial D\setminus\Gamma_{D_h},\\
\left.\begin{array}{l}[\![\gamma_\varepsilon\mathcal{M}^{nn}(\widetilde{v}_\varepsilon)]\!] = -(\gamma_1-\gamma_0)M^{nn}\\ [\![\gamma_\varepsilon(\partial_t\mathcal{M}^{tn}(\widetilde{v}_\varepsilon) + \mathrm{div}\mathcal{M}(\widetilde{v}_\varepsilon)\cdot n)]\!] = -(\gamma_1-\gamma_0)(\partial_t M^{tn} + \mathrm{div}M\cdot n)\end{array}\right\} & \text{on} & \partial\omega_\varepsilon,
\end{array}\right. \tag{3.16}$$

where $[\![\gamma_\varepsilon\mathcal{M}^{nn}(\widetilde{v}_\varepsilon)]\!]$ and $[\![\gamma_\varepsilon(\partial_t\mathcal{M}^{tn}(\widetilde{v}_\varepsilon) + \mathrm{div}\mathcal{M}(\widetilde{v}_\varepsilon)\cdot n)]\!]$ are the jumps of the normal moment and transversal shear through the interface $\partial\omega_\varepsilon$. We use the convention that in the jump $[\![(\cdot)]\!]$ the quantity $(\cdot)$ is taken positively on the inclusion side. As we will see later, due to the fast decrease of the fundamental solution associated with this problem, it is important to have a sufficiently accurate approximation of $\tilde{v}_\varepsilon$ near the inclusion, but the external boundary conditions can be rejected at infinity. Thus we approximate $\mathcal{M}(\tilde{v}_\varepsilon)$ by $\mathcal{M}(h_\varepsilon^M)$, solution of the auxiliary exterior problem:

$$\left\{\begin{array}{lll}
\mathrm{div}[\mathrm{div}(\mathcal{M}(h_\varepsilon^M))] = 0 & \text{in} & \mathbb{R}^2\setminus\overline{\omega_\varepsilon},\\
\mathrm{div}[\mathrm{div}(\mathcal{M}(h_\varepsilon^M))] = 0 & \text{in} & \omega_\varepsilon\\
\mathcal{M}(h_\varepsilon^M)\to 0 & \text{at} & \infty,\\
\left.\begin{array}{l}[\![\gamma_\varepsilon\mathcal{M}^{nn}(h_\varepsilon^M)]\!] = -(\widehat{\gamma}_1-\widehat{\gamma}_0)M^{nn}(\widehat{x})\\ [\![\gamma_\varepsilon(\partial_t\mathcal{M}^{tn}(h_\varepsilon^M) + \mathrm{div}\mathcal{M}(h_\varepsilon^M)\cdot n)]\!] = -(\widehat{\gamma}_1-\widehat{\gamma}_0)(\partial_t M^{tn}(\widehat{x}) + \mathrm{div}M(\widehat{x})\cdot n)\end{array}\right\} & \text{on} & \partial\omega_\varepsilon.
\end{array}\right. \tag{3.17}$$

In the present case of a circular inclusion, the tensor $\mathcal{M}(h_\varepsilon^M)$ admits the following expression in a polar coordinate system (r,θ) centered in $\widehat{x}$ (the general solution associated to the biharmonic operator can be found in [11], for instance):

- for $r \geq \varepsilon$

$$\mathcal{M}_r(r,\theta) = -(\alpha_1+\alpha_2)\frac{1-\gamma}{1+\xi\gamma}\frac{\varepsilon^2}{r^2} - \frac{1-\gamma}{1+\eta\gamma}\left(\frac{4\nu}{3+\nu}\frac{\varepsilon^2}{r^2}+3\eta\frac{\varepsilon^4}{r^4}\right)(\beta_1\cos 2\theta+\beta_2\cos 2(\theta+\phi)), \tag{3.18}$$

$$\mathcal{M}_\theta(r,\theta) = (\alpha_1+\alpha_2)\frac{1-\gamma}{1+\xi\gamma}\frac{\varepsilon^2}{r^2} - \frac{1-\gamma}{1+\eta\gamma}\left(\frac{4}{3+\nu}\frac{\varepsilon^2}{r^2}-3\eta\frac{\varepsilon^4}{r^4}\right)(\beta_1\cos 2\theta+\beta_2\cos 2(\theta+\phi)), \tag{3.19}$$

$$\mathcal{M}_{r\theta}(r,\theta) = \eta\frac{1-\gamma}{1+\eta\gamma}\left(2\frac{\varepsilon^2}{r^2}-3\frac{\varepsilon^4}{r^4}\right)(\beta_1\sin 2\theta+\beta_2\sin 2(\theta+\phi)); \tag{3.20}$$

- for $0 < r < \varepsilon$

$$\mathcal{M}_r(r,\theta) = (\alpha_1+\alpha_2)\,\xi\frac{1-\gamma}{1+\xi\gamma} + \eta\frac{1-\gamma}{1+\eta\gamma}(\beta_1\cos 2\theta+\beta_2\cos 2(\theta+\phi)), \tag{3.21}$$

$$\mathcal{M}_\theta(r,\theta) = (\alpha_1+\alpha_2)\,\xi\frac{1-\gamma}{1+\xi\gamma} - \eta\frac{1-\gamma}{1+\eta\gamma}(\beta_1\cos 2\theta+\beta_2\cos 2(\theta+\phi)), \tag{3.22}$$

$$\mathcal{M}_{r\theta}(r,\theta) = -\eta\frac{1-\gamma}{1+\eta\gamma}(\beta_1\sin 2\theta+\beta_2\sin 2(\theta+\phi)). \tag{3.23}$$

The notations used above are the following. The parameter ϕ denotes the angle between the eigenvectors of the tensors $M_1(\widehat{x})$ and $M_2(\widehat{x})$,

$$\alpha_i = \frac{1}{2}(\mu_I^i+\mu_{II}^i) \quad \text{and} \quad \beta_i = \frac{1}{2}(\mu_I^i-\mu_{II}^i), \quad i=1,2;$$

where μ_I^i and μ_{II}^i are the eigenvalues of the tensors M_i for $i=1,2$. In addition, the constants ξ and η are respectively given by

$$\xi = \frac{1+\nu}{1-\nu} \qquad \text{and} \qquad \eta = \frac{1-\nu}{3+\nu}, \tag{3.24}$$

and γ is the contrast, that is,

$$\gamma = \frac{\widehat{\gamma}_1}{\widehat{\gamma}_0}\cdot \tag{3.25}$$

From these elements, we obtain successively:

$$\int_{\omega_\varepsilon}\mathcal{M}(u_0)\cdot\nabla\nabla\tilde{v}_\varepsilon\,\mathrm{d}x = \int_{\omega_\varepsilon}\mathcal{M}(\tilde{v}_\varepsilon)\cdot\nabla\nabla u_0\,\mathrm{d}x = \int_{\omega_\varepsilon}\mathcal{M}(h_\varepsilon^M)\cdot\nabla\nabla u_0\,\mathrm{d}x + \mathcal{E}_2(\varepsilon). \tag{3.26}$$

Then approximating $\nabla\nabla u_0$ in ω_ε by its value at $\widehat{x}$ and calculating the resulting integral with the help of the expressions (3.21)–(3.23) yields:

$$\begin{aligned}\int_{\omega_\varepsilon}\mathcal{M}(u_0)\cdot\nabla\nabla\tilde{v}_\varepsilon\,\mathrm{d}x &= \int_{\omega_\varepsilon}\mathcal{M}(h_\varepsilon^M)\cdot\nabla\nabla u_0(\widehat{x})\,\mathrm{d}x + \mathcal{E}_2(\varepsilon)+\mathcal{E}_3(\varepsilon)\\ &= -\pi\varepsilon^2\rho\,(\mathbb{T}M\cdot\nabla\nabla u_0)\,(\widehat{x}) + \mathcal{E}_2(\varepsilon)+\mathcal{E}_3(\varepsilon),\end{aligned} \tag{3.27}$$

with

$$\rho = \frac{\gamma - 1}{1 + \gamma\eta} \quad \text{and} \quad \mathbb{T} = \eta \mathbb{I} + \frac{1}{2}\frac{\xi - \eta}{1 + \gamma\xi} I \otimes I. \tag{3.28}$$

Finally, the variation of the bilinear form can be written as:

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \pi\varepsilon^2(\widehat{\gamma}_1 - \widehat{\gamma}_0)\left[(\mathbb{I} - \rho\mathbb{T})\mathcal{M}(u_0) \cdot \nabla\nabla v_0 - \rho\mathbb{T}\mathcal{B}(u_0) \cdot \nabla\nabla u_0\right](\widehat{x}) + (\widehat{\gamma}_1 - \widehat{\gamma}_0)\sum_{i=1}^{3} \mathcal{E}_i(\varepsilon). \tag{3.29}$$

3.4. Variation of the linear form

Since here ℓ_ε is independent of ε, it follows trivially that

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = 0. \tag{3.30}$$

3.5. Study of the limit case $\gamma_{\text{out}} = 0$

Let us now examine what changes in the preceding derivations when $\gamma_{\text{out}} = 0$. The variational formulation is still given by the bilinear and linear forms (3.9) and (3.10) with the spaces (3.7) and (3.8). The additional condition (2.13) is assumed. The perturbed adjoint state is defined as solution of (3.11) complemented with the condition inside the hole

$$\int_{\omega_\varepsilon} \gamma_0 \mathcal{M}(v_\varepsilon) \cdot \nabla\nabla\varphi \, \mathrm{d}x = -\langle L_0(u_0), \varphi\rangle \quad \forall \varphi \in H_0^2(\omega_\varepsilon), \tag{3.31}$$

which yields uniqueness of v_ε in Ω. The functional L_ε is assumed to satisfy (3.14), where $b(u_0)$ and $\mathcal{B}(u_0)$ are as in the previous case, and $L \in \mathcal{V}'$ is of the form $\langle L, \varphi\rangle = \langle \tilde{L}, \varphi|_{\Omega\setminus B}\rangle$ for some neighborhood B of $\widehat{x}$. Then we find that the variation $\tilde{v}_\varepsilon$ solves (3.15) together with

$$\int_{\omega_\varepsilon} \mathcal{M}(\tilde{v}_\varepsilon) \cdot \nabla\nabla\varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in H_0^2(\omega_\varepsilon). \tag{3.32}$$

This corresponds to

$$\begin{cases} \operatorname{div}[\operatorname{div}(\gamma_0 \mathcal{M}(\widetilde{v}_\varepsilon))] = 0 & \text{in} \quad D \setminus \overline{\omega_\varepsilon}, \\ \operatorname{div}[\operatorname{div}(\mathcal{M}(\widetilde{v}_\varepsilon))] = 0 & \text{in} \quad \omega_\varepsilon \\ \widetilde{v}_\varepsilon = 0 & \text{on} \quad \Gamma_{D_h}, \\ \partial_n \widetilde{v}_\varepsilon = 0 & \text{on} \quad \Gamma_{D_g}, \\ \gamma_0 \mathcal{M}^{nn}(\widetilde{v}_\varepsilon) = 0 & \text{on} \quad \partial\Omega \setminus \Gamma_{D_g}, \\ \partial_t[\gamma_0 \mathcal{M}^{tn}(\widetilde{v}_\varepsilon)] + \operatorname{div}[\gamma_0 \mathcal{M}(\widetilde{v}_\varepsilon)] \cdot n = 0 & \text{on} \quad \partial\Omega \setminus \Gamma_{D_h}, \\ \left.\begin{array}{l} [\![\gamma_\varepsilon \mathcal{M}^{nn}(\widetilde{v}_\varepsilon)]\!] = -(\gamma_1 - \gamma_0) M^{nn} \\ [\![\gamma_\varepsilon(\partial_t \mathcal{M}^{tn}(\widetilde{v}_\varepsilon) + \operatorname{div}\mathcal{M}(\widetilde{v}_\varepsilon) \cdot n)]\!] = -(\gamma_1 - \gamma_0)(\partial_t M^{tn} + \operatorname{div} M \cdot n) \end{array}\right\} & \text{on} \quad \partial\omega_\varepsilon. \end{cases}$$

Compared with (3.16), only the external boundary condition is modified. Thus the approximation (3.17) remains valid. All the subsequent derivations leading to (3.29) are unchanged (the contrast is now $\gamma = 0$). Of course, (3.30) still holds true.

3.6. Topological derivative

In Section 4 we prove that the remainders $\mathcal{E}_i(\varepsilon)$, $i = 1, 2, 3$, behave like a $o(\varepsilon^2)$ in both situations $\gamma_{\text{out}} > 0$ and $\gamma_{\text{out}} = 0$. Therefore, after summation of the different terms according to Proposition 3.1 and a few simplifications, we arrive at the following result.

Theorem 3.1. *Let $J_\varepsilon(u_\varepsilon)$ be an objective functional satisfying the hypotheses (3.5) and (3.6) with $f(\varepsilon) = \pi\varepsilon^2$ and $L_\varepsilon(u_0)$ such that (3.14) holds true. We assume that $b(u_0)$ and $\mathcal{B}(u_0)$ are respectively of class $\mathcal{C}^{0,\alpha}$ and $\mathcal{C}^{2,\alpha}$ in a neighborhood of $\widehat{x}$, $0 < \alpha < 1$, and that ∂D ($\partial\Omega$ if $\gamma_{\rm out} = 0$) is Lipschitz. Then, $J_\varepsilon(u_\varepsilon)$ admits the topological asymptotic expansion*

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = \pi\varepsilon^2 D_T J_\Omega(\widehat{x}) + o(\varepsilon^2),$$

with the topological derivative given by

$$D_T J_\Omega = (\gamma_1 - \gamma_0)\left[(\mathbb{I} - \rho\mathbb{T})\mathcal{M}(u_0)\cdot\nabla\nabla v_0 - \rho\mathbb{T}\mathcal{B}(u_0)\cdot\nabla\nabla u_0\right] + \delta J_1 + \delta J_2. \tag{3.33}$$

We recall that ρ and $\mathbb{T}$ are given by (3.28), and that the coefficients γ_0 and γ_1 are given by Table 1. Moreover, $u_0 = u_\Omega$ is the solution of the state equation (2.2) and $v_0 = v_\Omega$ is the solution of the adjoint equation (3.2) for $\varepsilon = 0$, i.e., *$v_0 \in \mathcal{V}$ and*

$$\int_D \gamma_0 \mathcal{M}(v_0)\cdot\nabla\nabla\varphi \,{\rm d}x = -\langle L_0(u_0), \varphi\rangle \quad \forall\varphi\in\mathcal{V}. \tag{3.34}$$

Formula (3.33) is valid for all $\widehat{x} \in D\setminus\partial\Omega$ ($\widehat{x}\in\Omega$ if $\gamma_{\rm out} = 0$).

4. Estimation of the remainders

In this section, we proceed to the estimation of the remainders $\mathcal{E}_i(\varepsilon)$, $i = 1,2,3$. We use the letter c to denote any constant independent of ε. In order to be able to treat simultaneously the cases $\gamma_{\rm out} > 0$ and $\gamma_{\rm out} = 0$, we introduce the set

$$\Xi = \begin{cases} D & \text{if} \quad \gamma_{\rm out} > 0, \\ \Omega & \text{if} \quad \gamma_{\rm out} = 0. \end{cases} \tag{4.1}$$

We start by two preliminary lemmas.

Lemma 4.1. *Let B be a bounded Lipschitz domain, $\mathcal{H}$ be a closed subspace of $H^2(B)$ and $\|.\|$ be a norm on $\mathcal{H}$ verifying*

$$\exists c_1, c_2 > 0 \ \textit{s.t.} \qquad c_1\|\nabla\nabla u\|_{L^2(B)} \le \|u\| \le c_2\|u\|_{H^2(B)} \qquad \forall u\in\mathcal{H}.$$

Then the norm $\|.\|$ is equivalent on $\mathcal{H}$ to the norm $\|.\|_{H^2(B)}$.

Proof. We assume by contradiction that there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that

$$\forall n\in\mathbb{N}, \qquad \|v_n\|_{H^2(B)} = 1 \text{ and } \|v_n\| < \frac{1}{n}\cdot$$

Thanks to the compact imbedding of $H^2(B)$ into $H^1(B)$, we can extract a subsequence, still denoted by v_n, such that $v_n \to v \in H^1(B)$ for the H^1 norm. In addition, the fact that $\|v_n\| \to 0$ implies that $\|\nabla\nabla v_n\|_{L^2(B)} \to 0$. We deduce that (v_n) is a Cauchy sequence in $H^2(B)$, hence $v_n \to v \in H^2(B)$ for the H^2 norm. As $\mathcal{H}$ is closed, we have $v \in \mathcal{H}$, and $v_n \to v$ for the norm $\|.\|$. In particular, $\|v_n\| \to \|v\|$, thus $v = 0$. This contradicts the assumption that $\|v_n\|_{H^2(B)} = 1$ for all $n \in \mathbb{N}$. □

Lemma 4.2. *For any tensor field M of class $\mathcal{C}^{1,\alpha}$ in a neighborhood of the point $\widehat{x}$, $0 < \alpha < 1$, let $w_\varepsilon \in \mathcal{V}$ be solution of:*

$$\int_\Xi \gamma_\varepsilon \mathcal{M}(w_\varepsilon)\cdot\nabla\nabla\varphi\,{\rm d}x = -(\widehat{\gamma}_1 - \widehat{\gamma}_0)\left(\int_{\partial\omega_\varepsilon} M^{nn}\partial_n\varphi\,{\rm d}s - \int_{\partial\omega_\varepsilon}(\partial_t M^{tn} + {\rm div}M\cdot n)\varphi\,{\rm d}s\right) \quad \forall\varphi\in\mathcal{V}.$$

In the case of a hole, it is assumed the additional condition

$$\int_{\omega_\varepsilon} \mathcal{M}(w_\varepsilon)\cdot\nabla\nabla\varphi\,{\rm d}x = 0 \quad \forall\varphi\in H_0^2(\omega_\varepsilon).$$

Then, there exists $\delta > 0$ such that

$$\left\|\mathcal{M}(w_\varepsilon - h_\varepsilon^M)\right\|_{L^2(\Xi)} = O(\varepsilon^{1+\delta}). \tag{4.2}$$

Proof. We take an arbitrary test function $\varphi \in \mathcal{V}$. By integration by parts, we obtain

$$\begin{aligned}\int_\Xi \gamma_\varepsilon \mathcal{M}(h_\varepsilon^M) \cdot \nabla\nabla\varphi \, \mathrm{d}x = -(\widehat{\gamma}_1 - \widehat{\gamma}_0) \left(\int_{\partial\omega_\varepsilon} M^{nn}(\widehat{x}) \partial_n \varphi \, \mathrm{d}s - \int_{\partial\omega_\varepsilon} (\partial_t M^{tn}(\widehat{x}) + \mathrm{div} M(\widehat{x}) \cdot n)\varphi \, \mathrm{d}s \right) \\ + \int_{\partial\Xi} \gamma_0 \mathcal{M}(h_\varepsilon^M) \partial_n\varphi \, \mathrm{d}s - \int_{\partial\Xi} (\partial_t(\gamma_0 \mathcal{M}(h_\varepsilon^M)) + \mathrm{div}(\gamma_0 \mathcal{M}(h_\varepsilon^M)) \cdot n)\varphi \, \mathrm{d}s \\ + \int_{\partial\Omega\cap\Xi} [\![\gamma_0 \mathcal{M}(h_\varepsilon^M)]\!] \partial_n\varphi \, \mathrm{d}s - \int_{\partial\Omega\cap\Xi} [\![\partial_t(\gamma_0 \mathcal{M}(h_\varepsilon^M)) + \mathrm{div}(\gamma_0 \mathcal{M}(h_\varepsilon^M)) \cdot n]\!]\varphi \, \mathrm{d}s.\end{aligned}$$

Then, the difference $e_\varepsilon = w_\varepsilon - h_\varepsilon^M$ satisfies

$$\begin{aligned}\int_\Xi \gamma_\varepsilon \mathcal{M}(e_\varepsilon) \cdot \nabla\nabla\varphi \, \mathrm{d}x = -(\widehat{\gamma}_1 - \widehat{\gamma}_0) \int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x})) \partial_n \varphi \, \mathrm{d}s \\ + (\widehat{\gamma}_1 - \widehat{\gamma}_0) \int_{\partial\omega_\varepsilon} (\partial_t M^{tn} - \partial_t M^{tn}(\widehat{x}) + (\mathrm{div} M - \mathrm{div} M(\widehat{x})) \cdot n)\varphi \, \mathrm{d}s \\ + \int_{\partial\Xi} \gamma_0 \mathcal{M}(h_\varepsilon^M) \partial_n\varphi \, \mathrm{d}s - \int_{\partial\Xi} (\partial_t(\gamma_0 \mathcal{M}(h_\varepsilon^M)) + \mathrm{div}(\gamma_0 \mathcal{M}(h_\varepsilon^M)) \cdot n)\varphi \, \mathrm{d}s \\ + \int_{\partial\Omega\cap\Xi} [\![\gamma_0 \mathcal{M}(h_\varepsilon^M)]\!] \partial_n\varphi \, \mathrm{d}s - \int_{\partial\Omega\cap\Xi} [\![\partial_t(\gamma_0 \mathcal{M}(h_\varepsilon^M)) + \mathrm{div}(\gamma_0 \mathcal{M}(h_\varepsilon^M)) \cdot n]\!]\varphi \, \mathrm{d}s.\end{aligned}$$

We shall now estimate every term in the right hand side of the above equation. From the explicit formulas (3.18)–(3.20), the last four terms are bounded by $c\varepsilon^2 \|\varphi\|_{H^2(\Xi)}$. For the first term we proceed by a change of variable. We obtain

$$\begin{aligned}\left| \int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x})) \partial_n \varphi \, \mathrm{d}s \right| &= \varepsilon \left| \int_{\partial\omega} (M^{nn}(\varepsilon x) - M^{nn}(\widehat{x})) \partial_n \varphi(\varepsilon x) \, \mathrm{d}s \right| \\ &\leq c\varepsilon^2 \|\partial_n \varphi(\varepsilon x)\|_{H^{1/2}(\partial\omega)},\end{aligned}$$

where we have used the Lipschitz continuity of M^{nn} in the vicinity of $\widehat{x}$. Then, from the trace theorem we obtain

$$\left| \int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x})) \partial_n \varphi \, \mathrm{d}s \right| \leq c\varepsilon \|\varphi(\varepsilon x)\|_{H^2(\omega)/\mathbb{R}},$$

with the quotient norm defined by

$$\|u\|_{H^2(\omega)/\mathbb{R}} := \inf_{\beta\in\mathbb{R}} \|u + \beta\|_{H^2(\omega)} \quad \forall u \in H^2(\omega).$$

The quotient space $H^2(\omega)/\mathbb{R}$ can be identified with the space of functions of $H^2(\omega)$ with zero mean, endowed with the $H^2(\omega)$ norm. Hence, by virtue of Lemma 4.1, the quotient norm on $H^2(\omega)/\mathbb{R}$ is equivalent to the semi-norm

$$|u|_{H^2(\omega)} := \left(\|\nabla u\|^2_{L^2(\omega)} + \|\nabla\nabla u\|^2_{L^2(\omega)} \right)^{1/2} \quad \forall u \in H^2(\omega).$$

Therefore, after another change of variable, we get

$$\begin{aligned}\left| \int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x})) \partial_n \varphi \, \mathrm{d}s \right| &\leq c\varepsilon |\varphi(\varepsilon x)|_{H^2(\omega)} \\ &\leq c\varepsilon \|\nabla\varphi\|_{L^2(\omega_\varepsilon)} + c\varepsilon^2 \|\nabla\nabla\varphi\|_{L^2(\omega_\varepsilon)}.\end{aligned}$$

From the Hölder inequality, it comes for any $p > 1$

$$\left|\int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x}))\partial_n\varphi \,\mathrm{d}s\right| \leq c\varepsilon^{1+1/p} \left\|\nabla\varphi\right\|_{L^{2p/(p-1)}(\omega_\varepsilon)} + c\varepsilon^2 \left\|\nabla\nabla\varphi\right\|_{L^2(\omega_\varepsilon)}.$$

Finally, taking into account the Sobolev embedding theorem, we obtain

$$\left|\int_{\partial\omega_\varepsilon} (M^{nn} - M^{nn}(\widehat{x}))\partial_n\varphi \,\mathrm{d}s\right| \leq c\varepsilon^{1+1/p} \left\|\varphi\right\|_{H^2(\Xi)} \quad \forall\varphi\in\mathcal{V}.$$

For the second term we proceed in a similar way, that is

$$\begin{aligned}\left|\int_{\partial\omega_\varepsilon} (\partial_t M^{tn} - \partial_t M^{tn}(\widehat{x}) + (\mathrm{div}M - \mathrm{div}M(\widehat{x}))\cdot n)\varphi \,\mathrm{d}s\right| & \\ &= \varepsilon\left|\int_{\partial\omega} (\partial_t M^{tn}(\varepsilon x) - \partial_t M^{tn}(\widehat{x}) + (\mathrm{div}M(\varepsilon x) - \mathrm{div}M(\widehat{x}))\varphi(\varepsilon x) \,\mathrm{d}s\right| \\ &\leq c\varepsilon^{1+\alpha}\left\|\varphi(\varepsilon x)\right\|_{H^{3/2}(\partial\omega)} \\ &\leq c\varepsilon^{1+\alpha}\left\|\varphi(\varepsilon x)\right\|_{H^2(\omega)} \\ &\leq c\varepsilon^{\alpha}\left\|\varphi\right\|_{L^2(\omega_\varepsilon)} + c\varepsilon^{1+\alpha}\left\|\nabla\varphi\right\|_{L^2(\omega_\varepsilon)} + c\varepsilon^{2+\alpha}\left\|\nabla\nabla\varphi\right\|_{L^2(\omega_\varepsilon)} \\ &\leq c\varepsilon^{1+\alpha}\left\|\varphi\right\|_{L^\infty(\omega_\varepsilon)} + c\varepsilon^{1+\alpha}\left\|\nabla\varphi\right\|_{L^2(\omega_\varepsilon)} + c\varepsilon^{2+\alpha}\left\|\nabla\nabla\varphi\right\|_{L^2(\omega_\varepsilon)} \\ &\leq c\varepsilon^{1+\alpha}\left\|\varphi\right\|_{H^2(\Xi)},\end{aligned}$$

where we have used again the Sobolev embedding theorem. From these results we get

$$\left|\int_{\Xi} \gamma_\varepsilon \mathcal{M}(e_\varepsilon)\cdot\nabla\nabla\varphi \,\mathrm{d}x\right| \leq c\varepsilon^{1+\min(\alpha,1/p)} \left\|\varphi\right\|_{H^2(\Xi)}. \tag{4.3}$$

Let $\mathbb{R}_1[x]$ be the space of polynomial functions of two variables with degree not greater than one, and C be a neighborhood of ∂D a positive distance away from $\widehat{x}$. By identifying the quotient space $H^2(C)/\mathbb{R}_1[x]$ with the orthogonal complement $\mathbb{R}_1[x]^\perp := \{u \in H^2(C), \langle u, v\rangle_{H^2(C)} = 0\ \forall v \in \mathbb{R}_1[x]\}$, Lemma 4.1 implies that the quotient norm on $H^2(C)/\mathbb{R}_1[x]$ is equivalent to the energy norm $u \mapsto \|\nabla\nabla u\|_{L^2(C)}$. Then there holds

$$\begin{aligned}\|e_\varepsilon\|_{H^{3/2}(\Gamma_{D_h})/\mathbb{R}_1[x]} + \|\partial_n e_\varepsilon\|_{H^{1/2}(\Gamma_{D_g})/\mathbb{R}} &= \|h_\varepsilon^M\|_{H^{3/2}(\Gamma_{D_h})/\mathbb{R}_1[x]} + \|\partial_n h_\varepsilon^M\|_{H^{1/2}(\Gamma_{D_g})/\mathbb{R}} \\ &\leq c\|h_\varepsilon^M\|_{H^2(C)/\mathbb{R}_1[x]} \\ &\leq c\|\mathcal{M}(h_\varepsilon^M)\|_{L^2(C)} \\ &\leq c\varepsilon^2.\end{aligned}$$

For the latter estimate, we have used the explicit formulas (3.18)–(3.20). In (4.3), we make the splitting $e_\varepsilon = e_\varepsilon^1 + e_\varepsilon^2$, where e_ε^1 is a lifting of the first order trace of e_ε on $\partial\Xi$ whose support does not contain the inclusion, and $e_\varepsilon^2 \in \mathcal{V}$. Then, we have by the trace theorem

$$\|e_\varepsilon^1\|_{H^2(\Xi)/\mathbb{R}_1[x]} \leq c\varepsilon^2. \tag{4.4}$$

From (4.3) and (4.4), it comes

$$\left|\int_{\Xi} \gamma_\varepsilon \mathcal{M}(e_\varepsilon^2)\cdot\nabla\nabla\varphi \,\mathrm{d}x\right| \leq c(\varepsilon^{1+\min(\alpha,1/p)} + \varepsilon^2) \left\|\varphi\right\|_{H^2(\Xi)}. \tag{4.5}$$

We shall now distinguish between the cases $\gamma_{\mathrm{out}} > 0$ and $\gamma_{\mathrm{out}} = 0$.

- We first treat the case $\gamma_{\text{out}} > 0$. As $\gamma_\varepsilon(x) \geq \min(\gamma_{\text{in}}, \gamma_{\text{out}}) > 0$ for all $x \in D$, the bilinear form on the left hand side of (4.5) is uniformly coercive on $\mathcal{V}$ with respect to ε. Hence by elliptic regularity, we have

$$\left\|e_\varepsilon^2\right\|_{H^2(D)} \leq c\varepsilon^{1+\min(\alpha,1/p)},$$

  which, together with (4.4), yields (4.2).
- We now turn to the case $\gamma_{\text{out}} = 0$. Using the equivalence of the H^2 norm and the energy norm on the space $\left\{\varphi \in H^2(\Omega) : \varphi|_{\Gamma_{D_h}} = 0 \text{ and } \partial_n\varphi|_{\Gamma_{D_g}} = 0\right\}$, we obtain

$$c\left\|e_\varepsilon^2\right\|_{H^2(\Omega)}^2 \leq \|\nabla\nabla e_\varepsilon^2\|_{L^2(\Omega)}^2 = \|\nabla\nabla e_\varepsilon^2\|_{L^2(\Omega\setminus\overline{\omega}_\varepsilon)}^2 + \|\nabla\nabla e_\varepsilon^2\|_{L^2(\omega_\varepsilon)}^2.$$

  Yet, a change of variable entails

$$\begin{aligned}
\|\nabla\nabla e_\varepsilon^2\|_{L^2(\omega_\varepsilon)}^2 &= \frac{1}{\varepsilon^2}\|\nabla\nabla(e_\varepsilon^2(\varepsilon x))\|_{L^2(\omega)}^2 \\
&\leq \frac{c}{\varepsilon^2}\|e_\varepsilon^2(\varepsilon x)\|_{H^{3/2}(\partial\omega)/\mathbb{R}_1[x]}^2 \\
&\leq \frac{c}{\varepsilon^2}\|e_\varepsilon^2(\varepsilon x)\|_{H^2(C(1,2))/\mathbb{R}_1[x]}^2 \\
&\leq \frac{c}{\varepsilon^2}\|\nabla\nabla(e_\varepsilon^2(\varepsilon x))\|_{L^2(C(1,2))}^2 = c\|\nabla\nabla e_\varepsilon^2\|_{L^2(C(\varepsilon,2\varepsilon))}^2,
\end{aligned}$$

  where $C(r_1, r_2) = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$. It comes

$$\left\|e_\varepsilon^2\right\|_{H^2(\Omega)}^2 \leq c\|\nabla\nabla e_\varepsilon^2\|_{L^2(\Omega\setminus\overline{\omega}_\varepsilon)}^2 \leq c\int_\Omega \gamma_\varepsilon \mathcal{M}(e_\varepsilon^2)\cdot\nabla\nabla e_\varepsilon^2 \,\mathrm{d}x.$$

  Using (4.5) with $\varphi = e_\varepsilon^2$ and (4.4) leads to the desired result.

□

4.1. **First remainder**

The first remainder $\mathcal{E}_1(\varepsilon)$ in (3.13) is given by

$$\mathcal{E}_1(\varepsilon) = \int_{\omega_\varepsilon} (\mathcal{M}(u_0)\cdot\nabla\nabla v_0 - \mathcal{M}(u_0)(\widehat{x})\cdot\nabla\nabla v_0(\widehat{x}))\,\mathrm{d}x. \tag{4.6}$$

By interior elliptic regularity, u_0 and v_0 are respectively of class $\mathcal{C}^\infty$ and $\mathcal{C}^{4,\alpha}$ in a neighborhood of $\widehat{x}$. From these observations, it comes immediately that

$$|\mathcal{E}_1(\varepsilon)| \leq c\varepsilon^3. \tag{4.7}$$

4.2. **Second remainder**

The second remainder $\mathcal{E}_2(\varepsilon)$ in (3.26) is given by

$$\mathcal{E}_2(\varepsilon) = \int_{\omega_\varepsilon} \mathcal{M}(\tilde{v}_\varepsilon - h_\varepsilon^M)\cdot\nabla\nabla u_0 \,\mathrm{d}x. \tag{4.8}$$

The Cauchy-Schwarz inequality entails

$$\begin{aligned}
|\mathcal{E}_2(\varepsilon)| &\leq \left\|\mathcal{M}(\tilde{v}_\varepsilon - h_\varepsilon^M)\right\|_{L^2(\omega_\varepsilon)}\|u_0\|_{H^2(\omega_\varepsilon)} \\
&\leq c\varepsilon\left\|\mathcal{M}(\tilde{v}_\varepsilon - h_\varepsilon^M)\right\|_{L^2(\omega_\varepsilon)}.
\end{aligned} \tag{4.9}$$

Then, by Lemma 4.2, there exists $\delta > 1$, such that

$$|\mathcal{E}_2(\varepsilon)| \leq c\varepsilon^{2+\delta}. \tag{4.10}$$

4.3. **Third remainder**

The third remainder $\mathcal{E}_3(\varepsilon)$ in (3.27) is given by

$$\mathcal{E}_3(\varepsilon) = \int_{\omega_\varepsilon} \mathcal{M}(h_\varepsilon^M) \cdot (\nabla\nabla u_0 - \nabla\nabla u_0(\widehat{x}))\, \mathrm{d}x. \tag{4.11}$$

From the Cauchy-Schwarz inequality and taking into account the regularity of u_0 near $\widehat{x}$ as well as the explicit formulas (3.21)–(3.23), we obtain

$$\begin{aligned} |\mathcal{E}_3(\varepsilon)| &\leq \left\|\mathcal{M}(h_\varepsilon^M)\right\|_{L^2(\omega_\varepsilon)} \|\nabla\nabla u_0 - \nabla\nabla u_0(\widehat{x})\|_{L^2(\omega_\varepsilon)} \\ &\leq c\varepsilon^2 \left\|\mathcal{M}(h_\varepsilon^M)\right\|_{L^2(\omega_\varepsilon)} \\ &\leq c\varepsilon^3. \end{aligned} \tag{4.12}$$

5. Examples of shape functionals

We present two examples of objective functionals which are of interest for practical applications.

Proposition 5.1. *We consider an objective functional of the form*

$$J_\varepsilon(u) := J(u|_{\widetilde{D}}),$$

where $\widetilde{D}$ is an open subset of D (Ω if $\gamma_{\mathrm{out}} = 0$) which does not contain a neighborhood of $\widehat{x}$. In addition, we assume that J admits the following expansion,

$$J(u_0|_{\widetilde{D}} + \varphi) - J(u_0|_{\widetilde{D}}) = \left\langle L(u_0|_{\widetilde{D}}), \varphi\right\rangle + O(\|\varphi\|^2_{H^2(\widetilde{D})}) \quad \forall \varphi \in \widetilde{\mathcal{V}},$$

where $\widetilde{\mathcal{V}} = \{u|_{\widetilde{D}}, u \in \mathcal{V}\}$ and $L(u_0) \in \widetilde{\mathcal{V}}'$. We set

$$\langle L_\varepsilon(u_0), \varphi\rangle = \langle L_0(u_0), \varphi\rangle = \left\langle L(u_0|_{\widetilde{D}}), \varphi|_{\widetilde{D}}\right\rangle \quad \forall \varphi \in \mathcal{V}.$$

Then, the assumptions of Theorem 3.1 are satisfied with

$$\mathcal{B}(u_0) = 0,\ \ \delta J_1 = \delta J_2 = 0.$$

Proof. It is sufficient to verify the conditions (3.5) and (3.6). The second one is straightforward. For the first condition we write

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) &= J(u_\varepsilon|_{\widetilde{D}}) - J(u_0|_{\widetilde{D}}) \\ &= \left\langle L(u_0|_{\widetilde{D}}), u_\varepsilon|_{\widetilde{D}} - u_0|_{\widetilde{D}}\right\rangle + O(\|u_\varepsilon - u_0\|^2_{H^2(\widetilde{D})}) \\ &= \langle L_\varepsilon(u_0), u_\varepsilon - u_0\rangle + O(\|u_\varepsilon - u_0\|^2_{H^2(\widetilde{D})}). \end{aligned}$$

Then we make the splitting

$$\|\mathcal{M}(u_\varepsilon) - \mathcal{M}(u_0)\|_{L^2(\widetilde{D})} = \left\|\mathcal{M}(u_\varepsilon) - \mathcal{M}(u_0) - \mathcal{M}(h_\varepsilon^M)\right\|_{L^2(\widetilde{D})} + \left\|\mathcal{M}(h_\varepsilon^M)\right\|_{L^2(\widetilde{D})},$$

where $M = \mathcal{M}(u_0)$. On the one hand, by setting $w_\varepsilon = u_\varepsilon - u_0$ we deduce from Lemma 4.2 that

$$\left\|\mathcal{M}(u_\varepsilon - u_0 - h_\varepsilon^M)\right\|_{L^2(\widetilde{D})} = O(\varepsilon^{1+\delta}), \quad \delta > 0.$$

On the other hand, taking into account the analytical formulas (3.18)–(3.23) we straightforwardly derive

$$\left\|\mathcal{M}(h_\varepsilon^M)\right\|_{L^2(\widetilde{D})} = O(\varepsilon^2), \qquad \square$$

which completes the proof.

Corollary 5.1. *For the total potential energy functional*

$$J_\varepsilon(u) = \frac{1}{2} a_\varepsilon(u,u) - \ell_\varepsilon(u),$$

the topological derivative reads

$$D_T J_\Omega = \frac{\gamma_1 - \gamma_0}{2} \left[(\mathbb{I} - \rho\mathbb{T})\mathcal{M}(u_0) \cdot \nabla\nabla u_0\right]. \tag{5.1}$$

Proof. Let $w \in \mathcal{U}$ be a lifting of the Dirichlet boundary conditions on Γ_{D_g} and Γ_{D_h} whose support does not contain a neighborhood of $\widehat{x}$. Using that

$$a_\varepsilon(u_\varepsilon, \varphi) = \ell_\varepsilon(\varphi) \quad \forall \varphi \in \mathcal{V},$$

we can rewrite the objective functional as

$$J_\varepsilon(u_\varepsilon) = \widetilde{J}_\varepsilon(u_\varepsilon),$$

where

$$\widetilde{J}_\varepsilon(u) = \frac{1}{2}\left(a_\varepsilon(u,w) - \ell_\varepsilon(u) - \ell_\varepsilon(w)\right).$$

Clearly, this functional satisfies the assumptions of Proposition 5.1, with for all $\varepsilon \geq 0$,

$$\langle L_\varepsilon(u_0), \varphi\rangle = \frac{1}{2}\left(a_\varepsilon(\varphi, w) - \ell_\varepsilon(\varphi)\right) \quad \forall \varphi \in \mathcal{V}.$$

The adjoint problem reads: find $v_0 \in \mathcal{V}$ such that

$$\begin{aligned} a_0(v_0, \varphi) &= -\frac{1}{2}\left(a_0(\varphi, w) - \ell_0(\varphi)\right) \\ &= -\frac{1}{2}\left(a_0(w, \varphi) - a_0(u_0, \varphi)\right) \quad \forall \varphi \in \mathcal{V}. \end{aligned}$$

By uniqueness we get

$$v_0 = \frac{1}{2}(u_0 - w).$$

Then

$$\nabla\nabla v_0(\widehat{x}) = \frac{1}{2}\nabla\nabla u_0(\widehat{x}),$$

which concludes the proof. $\square$

Proposition 5.2. *We consider an objective functional of the form*

$$J_\varepsilon(u) := \frac{1}{2}\int_{\widetilde{D}} \gamma_\varepsilon \mathbb{K}\mathcal{M}(u_\varepsilon)\cdot\mathcal{M}(u_\varepsilon),$$

where $\mathbb{K}$ *is a symmetric fourth order tensor and* $\widetilde{D}$ *is an open subset of* D *containing* $\widehat{x}$*. We set for all* $\varphi \in \mathcal{V}$*:*

$$\langle L_\varepsilon(u_0),\varphi\rangle = \int_{\widetilde{D}} \gamma_\varepsilon \mathbb{K}\mathcal{M}(u_0)\cdot\mathcal{M}(\varphi) = \int_{\widetilde{D}} \gamma_\varepsilon k\mathbb{C}\mathbb{K}\mathcal{M}(u_0)\cdot\nabla\nabla\varphi,$$

that is,

$$\mathcal{B}(u_0) = k\mathbb{C}\mathbb{K}\mathcal{M}(u_0).$$

Then, the assumptions of Theorem 3.1 are satisfied with the contributions δJ_1 *and* δJ_2 *at the point* $\widehat{x}$ *given by*

$$\begin{aligned}\delta J_1 &= \frac{1}{2}\widehat{\gamma}_1\int_\omega \mathbb{K}\mathcal{M}\cdot\mathcal{M} + \frac{1}{2}\widehat{\gamma}_0\int_{\mathbb{R}^2\setminus\overline{\omega}} \mathbb{K}\mathcal{M}\cdot\mathcal{M},\\ \delta J_2 &= \frac{1}{2}(\widehat{\gamma}_1-\widehat{\gamma}_0)\mathbb{K}\mathcal{M}(u_0)(\widehat{x})\cdot\mathcal{M}(u_0)(\widehat{x}).\end{aligned}$$

Above, $\mathcal{M}(x) = \mathcal{M}(h_\varepsilon^M)(\varepsilon x)$ *is given by the explicit formulas (3.18)–(3.23), with* $M = \mathcal{M}(u_0)(\widehat{x})$*.*

Proof. A simple calculation results in

$$VJ_1(\varepsilon) := J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - \langle L_\varepsilon(u_0), u_\varepsilon - u_0\rangle = \frac{1}{2}\int_{\widetilde{D}} \gamma_\varepsilon\mathbb{K}\mathcal{M}(\tilde{u}_\varepsilon)\cdot\mathcal{M}(\tilde{u}_\varepsilon),$$

with $\tilde{u}_\varepsilon = u_\varepsilon - u_0$. Then we write

$$VJ_1(\varepsilon) = \frac{1}{2}\int_{\widetilde{D}} \gamma_\varepsilon\mathbb{K}\mathcal{M}(h_\varepsilon^M)\cdot\mathcal{M}(h_\varepsilon^M) + \mathcal{E}_4(\varepsilon),$$

with $\mathcal{E}_4(\varepsilon) = o(\varepsilon^2)$. Indeed, it stems from the Cauchy-Schwarz inequality that

$$\begin{aligned}|\mathcal{E}_4(\varepsilon)| &\le c\|\mathcal{M}(\tilde{u}_\varepsilon - h_\varepsilon^M)\|_{L^2(\widetilde{D})}\|\mathcal{M}(\tilde{u}_\varepsilon)+\mathcal{M}(h_\varepsilon^M)\|_{L^2(\widetilde{D})}\\ &\le c\|\mathcal{M}(\tilde{u}_\varepsilon - h_\varepsilon^M)\|_{L^2(\widetilde{D})}\Big(\|\mathcal{M}(\tilde{u}_\varepsilon - h_\varepsilon^M)\|_{L^2(\widetilde{D})} + 2\|\mathcal{M}(h_\varepsilon^M)\|_{L^2(\widetilde{D})}\Big).\end{aligned}$$

According to Lemma 4.2, $\|\mathcal{M}(\tilde{u}_\varepsilon - h_\varepsilon^M)\|_{L^2(\widetilde{D})} = o(\varepsilon)$, and, in view of the explicit expression of $\mathcal{M}(h_\varepsilon^M)$, one easily checks that $\|\mathcal{M}(h_\varepsilon^M)\|_{L^2(\widetilde{D})} = O(\varepsilon)$. Now take a ball $B(\widehat{x},R)$ in which γ_0 and γ_1 are constant. Due to the decrease of $\mathcal{M}(h_\varepsilon^M)$ we have

$$VJ_1(\varepsilon) = \frac{1}{2}\int_{\omega_\varepsilon}\widehat{\gamma}_1\mathbb{K}\mathcal{M}(h_\varepsilon^M)\cdot\mathcal{M}(h_\varepsilon^M) + \int_{B(\widehat{x},R)\setminus\overline{\omega}_\varepsilon}\widehat{\gamma}_0\mathbb{K}\mathcal{M}(h_\varepsilon^M)\cdot\mathcal{M}(h_\varepsilon^M) + o(\varepsilon^2).$$

Using again the decrease of $\mathcal{M}(h_\varepsilon^M)$, it appears that replacing in the second integral the domain $B(\widehat{x},R)\setminus\overline{\omega}_\varepsilon$ by $\mathbb{R}^2\setminus\overline{\omega}_\varepsilon$ produces an error of order $O(\varepsilon^4)$. A change of variable completes the calculation of δJ_1. The calculation of δJ_2 is straightforward. □

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