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Filtrations

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CAHIERS DE TOPOLOGIE IX
ET GEOMETRIE DIFFERENTIELLE

FILTRATIONS

by Peter HILTON

We extract from the theory of spectral sequences the following problem. Let $\mathfrak{A}$ be an abelian category, let X be an object of $\mathfrak{A}$, and let

$$\ldots \subseteq X^{p-1} \subseteq X^p \subseteq \ldots \subseteq X$$

be a filtration of X, $-\infty < p < \infty$. Similarly let

$$\ldots \subseteq \tilde{X}^{p-1} \subseteq \tilde{X}^p \subseteq \ldots \subseteq \tilde{X}$$

be a filtration of $\tilde{X} \in \mathfrak{A}$. Let $\varphi: X \to \tilde{X}$ be a morphism of filtered objects, so that φ induces

$$\varphi^p : X^p \to \tilde{X}^p, \; -\infty < p < \infty .$$

Associated with the given filtrations of X and $\tilde{X}$ we have the graded objects GX, $G\tilde{X}$ given by

$$(GX)^p = X^p / X^{p-1}, \; (G\tilde{X})^p = \tilde{X}^p / \tilde{X}^{p-1}$$

and φ obviously induces the morphism

$$G\varphi: GX \to G\tilde{X}$$

of graded objects. We then ask under what circumstances G reflects isomorphisms. Further we are interested in devising a functorial process which will convert the given filtered objects X and $\tilde{X}$ into new filtered objects $\bar{X}$ and $\bar{\tilde{X}}$ in such a way that the associated graded objects are unchanged and that $\bar{\varphi}: \bar{X} \cong \bar{\tilde{X}}$ if $G\varphi: GX \cong G\tilde{X}$. Such a process was described in [2] but under extra hypotheses on $\mathfrak{A}$ which do not appear in this presentation. Moreover, we also describe here a simplification of the process which does not enjoy the strong property of the *completion* $\bar{X}$ above but has the good features *(a)* that it does far less violence to the filtration than the completion process, and *(b)* that, after carrying it out, every non-zero element of the filtered object is represented by a unique non-zero element of the associated graded object. We describe the simplified process as *quasi-completion.*

Details of this work, which is joint work with B. Eckmann, will appear elsewhere [1].

1. Definitions and notations.

Let $\mathbf{N}$ be the ordered set of integers regarded as a category and let $\mathfrak{A}^{\mathbf{N}}$ be the functor category. Thus an object of $\mathfrak{A}^{\mathbf{N}}$ is a sequence $(D^\bullet, \alpha^\bullet)$,

$$\cdots \longrightarrow D^p \xrightarrow{\alpha^p} D^{p+1} \longrightarrow \cdots,$$

and a morphism $\varphi^\bullet : (D^\bullet, \alpha^\bullet) \to (E^\bullet, \beta^\bullet)$ is a sequence of morphisms $\varphi^p : D^p \to E^p$ such that $\beta^p \varphi^p = \varphi^{p+1} \alpha^p$. Note that we may also regard an object of $\mathfrak{A}^{\mathbf{N}}$ as an object $D^\bullet$ of the graded category $\mathfrak{A}^{\mathbf{Z}}$ together with an endomorphism $\alpha^\bullet : D^\bullet \to D^\bullet$ of degree $+1$.

Let $P : \mathfrak{A} \to \mathfrak{A}^{\mathbf{N}}$ be the embedding functor which associates with $X \in \mathfrak{A}$ the sequence

$$\cdots \xrightarrow{1} X \xrightarrow{1} X \longrightarrow \cdots.$$

PROPOSITION 1.1. *P is a full embedding, and preserves monics and epics.*

A *filtration* of X is a monic $\mu^\bullet : (X^\bullet, \xi^\bullet) \rightarrowtail PX$; note that each ξ^p is monic.

A *cofiltration* of X is an epic $\varepsilon_\bullet : PX \twoheadrightarrow (X_\bullet, \xi_\bullet)$; note that each ξ_p is epic.

The graded object associated with the filtration $\mu^\bullet$ is $\mathcal{G}(\mu^\bullet)$ given by

$$\mathcal{G}(\mu^\bullet)^p = coker\, \xi^{p-1}.$$

The graded object associated with the cofiltration $\varepsilon_\bullet$ is $\mathcal{G}(\varepsilon_\bullet)$ given by

$$\mathcal{G}(\varepsilon_\bullet)^p = ker\, \xi_{p-1}.$$

Note that $\mathcal{G}(\mu^\bullet)$ depends only on $(X^\bullet, \xi^\bullet)$ and $\mathcal{G}(\varepsilon_\bullet)$ depends only on $(X_\bullet, \xi_\bullet)$. We suppose henceforth that $\mu^\bullet$, $\varepsilon_\bullet$ are *mutual annihilators* so that

$$(X^\bullet, \xi^\bullet) \overset{\mu^\bullet}{\rightarrowtail} PX \overset{\varepsilon_\bullet}{\twoheadrightarrow} (X_\bullet, \xi_\bullet) \tag{1.2}$$

is exact in $\mathfrak{A}^{\mathbf{N}}$. We may then think of X_p as the quotient X/X^p.

PROPOSITION 1.3. *If (1.2) is exact then* $\mathcal{G}(\mu^\bullet) = \mathcal{G}(\varepsilon_\bullet)$.

If $\mathfrak{A}$ has countable products (sums) then P has a right (left) adjoint. So we suppose $\mathfrak{A}$ has countable products and sums. Let

$$R, L: \mathfrak{A}^{\mathbf{N}} \to \mathfrak{A}$$

be such that

$$P \vdash R, \quad L \vdash P. \tag{1.4}$$

Then $RP = Id$, and there is a natural transformation $\tau : PR \to Id$ such that $\tau P = 1$, $R\tau = 1$. Also $LP = Id$, and there is a natural transformation $\pi : Id \to PL$ such that $\pi P = 1$, $L\pi = 1$. The *limit* (= limite projective)of $(D^\bullet, \alpha^\bullet)$ is the object $R(D^\bullet, \alpha^\bullet)$ together with the morphism

$$\tau(D^\bullet, \alpha^\bullet) : PR(D^\bullet, \alpha^\bullet) \to (D^\bullet, \alpha^\bullet).$$

The *colimit* (= limite inductive) of $(D^\bullet, \alpha^\bullet)$ is the object $L(D^\bullet, \alpha^\bullet)$ together with the morphism $\pi(D^\bullet, \alpha^\bullet) : (D^\bullet, \alpha^\bullet) \to PL(D^\bullet, \alpha^\bullet)$. We will write

$$X^\infty = L(X^\bullet, \xi^\bullet),\ X^{-\infty} = R(X^\bullet, \xi^\bullet),\ X_\infty = L(X_\bullet, \xi_\bullet),\ X_{-\infty} = R(X_\bullet, \xi_\bullet).$$

PROPOSITION 1.4. *(i) For any* $\varphi^\bullet : (D^\bullet, \alpha^\bullet) \to PX$, $PL\varphi^\bullet \circ \pi(D^\bullet, \alpha^\bullet) = \varphi^\bullet$.
(ii) For any $\psi_\bullet : PX \to (D_\bullet, \alpha_\bullet)$, $\tau(D_\bullet, \alpha_\bullet) \circ PR\psi_\bullet = \psi_\bullet$.

COROLLARY 1.5. *(i) If* $\mu^\bullet$ *is a filtration,* $\pi(X^\bullet, \xi^\bullet) : (X^\bullet, \xi^\bullet) \rightarrowtail PX^\infty$.
(ii) If $\varepsilon_\bullet$ *is a cofiltration,* $\tau(X_\bullet, \xi_\bullet) : PX_{-\infty} \twoheadrightarrow (X_\bullet, \xi_\bullet)$.

We say that $\mu^\bullet$ (or its annihilator $\varepsilon_\bullet$) *generates* X if $L\mu^\bullet$ is epic; $\mu^\bullet$ (or $\varepsilon_\bullet$) *cogenerates* X if $R\varepsilon_\bullet$ is monic; $\mu^\bullet$ (or $\varepsilon_\bullet$) is *quasi-complete* if it generates and cogenerates X.

We say that $\mu^\bullet$ (or $\varepsilon_\bullet$) is *L-complete* if $L\mu^\bullet$ is an isomorphism; $\mu^\bullet$ (or $\varepsilon_\bullet$) is *R-complete* if $R\varepsilon_\bullet$ is an isomorphism : $\mu^\bullet$ (or $\varepsilon_\bullet$) is *complete* if $L\mu^\bullet$ and $R\varepsilon_\bullet$ are isomorphisms.

2. The completion procedure.

Consider the exact sequence *(1.2)*. We write $\varepsilon_\bullet \square \mu^\bullet$ to show they are mutual annihilators. We construct filtrations and cofiltrations as follows :

$$(2.1)\quad \begin{array}{llll} \pi^\bullet : (X^\bullet, \xi^\bullet) \rightarrowtail PX^\infty, & & \tau_\bullet : PX_{-\infty} & \twoheadrightarrow (X_\bullet, \xi_\bullet), \\ \zeta^\bullet : (X^\bullet_{-\infty}, \xi^\bullet_{-\infty}) \rightarrowtail PX_{-\infty}, & & \eta_\bullet : PX^\infty & \twoheadrightarrow (X^\infty_\bullet, \xi^\infty_\bullet), \\ \pi^\bullet_{-\infty} : (X^\bullet_{-\infty}, \xi^\bullet_{-\infty}) \rightarrowtail P(X_{-\infty})^\infty, & & \tau^\infty_\bullet : P(X^\infty)_{-\infty} & \twoheadrightarrow (X^\infty_\bullet, \xi^\infty_\bullet). \end{array}$$

Explicitly, $\pi^\bullet = \pi(X^\bullet, \xi^\bullet)$, so $\pi^\bullet$ is certainly L-complete; $\tau_\bullet = \tau(X_\bullet, \xi_\bullet)$ so $\tau_\bullet$ is R-complete : then $\eta_\bullet \square \pi^\bullet$, $\tau_\bullet \square \zeta^\bullet$, so $\zeta^\bullet$ is R-complete, $\eta_\bullet$ is L-complete; finally, $\pi^\bullet_{-\infty} = \pi(X^\bullet_{-\infty}, \xi^\bullet_{-\infty})$, $\tau^\infty_\bullet = \tau(X^\infty_\bullet, \xi^\infty_\bullet)$, so $\pi^\bullet_{-\infty}$ is L-complete, $\tau^\infty_\bullet$ is R-complete.

From Proposition 1.3 we immediately deduce

PROPOSITION 2.2. *All the filtrations and cofiltrations of (2.1) determine the same associated graded object.*

We now explain precisely in what sense we may regard the generation of the morphisms of *(2.1)* as a completion procedure. We state the result as a comprehensive theorem.

THEOREM 2.3. *(i) There is a natural isomorphism* $(X_{-\infty})^\infty = (X^\infty)_{-\infty}$ *under which the two objects may be identified to a single object* $X^\infty_{-\infty}$.

(ii) $\tau^\infty_\bullet \square \pi^\bullet_{-\infty}$, *so that* $\pi^\bullet_{-\infty}$ *and* $\tau^\infty_\bullet$ *are complete.*

(iii) The square

$$(2.4)\qquad \begin{array}{ccc} X^\infty & \xrightarrow{\;L\mu^\bullet\;} & X \\ \big\downarrow {\scriptstyle R\eta_\bullet} & & \big\downarrow {\scriptstyle R\varepsilon_\bullet} \\ X^\infty_{-\infty} & \xrightarrow{\;L\zeta^\bullet\;} & X_{-\infty} \end{array}$$

is bicartesian (pull-back and push-out).

Thus the completion procedure is natural with respect to morphisms of filtrations or cofiltrations and self-dual.

COROLLARY 2.5. *The filtration* $\mu^\bullet$ *is complete if and only if (2.4) is a diagram of isomorphisms. Moreover, the completion procedure leaves a complete filtration unchanged.*

Given (1.2) there are objects $X^p_q = \operatorname{coker} X^q \rightarrowtail X^p = \ker X_q \twoheadrightarrow X_p$, $q \leq p$. There are also morphisms

$$\varepsilon^p_q : X^p_q \twoheadrightarrow X^p_{q+1}, \quad q+1 \leq p,$$
$$\mu^{p\bullet}_q : X^p_q \rightarrowtail X^{p+1}_q, \quad q \leq p,$$

such that the square

$$(2.6) \qquad \begin{array}{ccc} X^p_q & \xrightarrow{\ \varepsilon^p_q\ } & X^p_{q+1} \\ \Big\downarrow \mu^p_q & & \Big\downarrow \mu^p_{q+1} \\ X^{p+1}_q & \xrightarrow{\ \varepsilon^{p+1}_q\ } & X^{p+1}_{q+1} \end{array}$$

is bicartesian, $q+1 \leq p$.

THEOREM 2.7. *The natural map*

$$\varinjlim_p \varprojlim_q (X^p_q;\ \varepsilon^p_q;\ \mu^p_q) \to \varprojlim_q \varprojlim_p (X^p_q;\ \varepsilon^p_q;\ \mu^p_q)$$

is an isomorphism and the common double limit is $X^\infty_{-\infty}$.

Moreover $\varprojlim_q (X^p_q;\ \varepsilon^p_q) = X^p_{-\infty}$ and μ^p_q induces $\xi^p_{-\infty} : X^p_{-\infty} \rightarrowtail X^{p+1}_{-\infty}$;

and dually,

$$\varinjlim_p (X^p_q;\ \mu^p_q) = X^\infty_q \quad \text{and} \quad \varepsilon^p_q \quad \text{induces} \quad \xi^\infty_q : X^\infty_q \twoheadrightarrow X^\infty_{q+1}.$$

COROLLARY 2.8. *Let* $\varphi : \mu^\bullet \to \tilde{\mu}^\bullet$ *be a morphism of filtrations, inducing*

$$(2.9) \qquad \mathcal{G}\varphi : \mathcal{G}(\mu^\bullet) \to \mathcal{G}(\tilde{\mu}^\bullet).$$

Suppose $\mathcal{G}\varphi$ *is an isomorphism and* $\mu^\bullet$, $\tilde{\mu}^\bullet$ *are complete. Then* $\varphi : \mu^\bullet \cong \tilde{\mu}^\bullet$.

The notation means that φ gives a commutative diagram

$$
(2.10)\quad
\begin{array}{ccccccccc}
X^{p-1} & \xrightarrow{\xi^{p-1}} & X^{p} & \xrightarrow{\mu^{p}} & X & \xrightarrow{\varepsilon_p} & X_p & \xrightarrow{\xi_p} & X_{p+1} \\
\downarrow{\scriptstyle \varphi^{p-1}} & & \downarrow{\scriptstyle \varphi^{p}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \varphi_{p+1}} \\
\tilde{X}^{p-1} & \xrightarrow{\tilde{\xi}^{p-1}} & \tilde{X}^{p} & \xrightarrow{\tilde{\mu}^{p}} & \tilde{X} & \xrightarrow{\tilde{\varepsilon}_p} & \tilde{X}_p & \xrightarrow{\tilde{\xi}_p} & \tilde{X}_{p+1}
\end{array}
$$

for each p. Then (2.9) coincides with the morphism $G\varphi : GX \to G\tilde{X}$ of the introduction and the theorem asserts that if $\mathcal{G}\varphi$ is an isomorphism and $\mu^\bullet$, $\tilde{\mu}^\bullet$ are complete then all vertical maps in (2.10) are isomorphisms. To see this we invoke the remarks following theorem 2.7 to deduce that if $\mu^\bullet$, $\tilde{\mu}^\bullet$ are R-complete and $\mathcal{G}\varphi$ is an isomorphism, then

$$\varphi^p : X^p \cong \tilde{X}^p, \text{ all } p.$$

Similarly if $\mu^\bullet$, $\tilde{\mu}^\bullet$ are L-complete and $\mathcal{G}\varphi$ is an isomorphism, then

$$\varphi_p : X_p \cong \tilde{X}_p, \text{ all } p.$$

The corollary now follows from these observations and (2.10). We note that, assuming only that $\mathcal{G}\varphi$ is an isomorphism, it follows that φ induces

$$\varphi^{\infty}_{-\infty} : X^{\infty}_{-\infty} \cong \tilde{X}^{\infty}_{-\infty} ;$$

we note also that we have achieved the objective of the introduction since the completion procedure is functorial and does not change the associated graded object.

3. The quasi-completion procedure.

We again consider the exact sequence (1.2) and describe the quasi-completion procedure. Set

$\alpha^\bullet = \tau(X^\bullet, \xi^\bullet) : PX^{-\infty} \to (X^\bullet, \xi^\bullet)$. Then $\mu^\bullet \alpha^\bullet = PR\mu^\bullet$ and $PR\mu^\bullet$ is monic since R has a left adjoint. Thus we have

$$
\begin{array}{ccc}
PX^{-\infty} & = & PX^{-\infty} \\
\downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle PR\mu^\bullet} \\
(X^\bullet, \xi^\bullet) & \rightarrowtail & PX
\end{array}
$$

and passing to quotients we obtain the filtration

$$(3.1) \qquad \mu_R^\bullet : (X_R^\bullet, \xi_R^\bullet) \rightarrowtail PX_R ;$$

its annihilator is

$$(3.2) \qquad \varepsilon_{R\bullet} : PX_R \twoheadrightarrow (X_\bullet, \xi_\bullet).$$

We say that $\mu_R^\bullet$ is obtained from $\mu^\bullet$ by *killing* $X^{-\infty}$. Dually we may obtain a new cofiltration from $\varepsilon_\bullet$ or $\varepsilon_{R\bullet}$ by *killing* X_∞. We call these cofiltrations $\varepsilon_\bullet^L$ and $\varepsilon_{R\bullet}^L$ respectively,

$$(3.3) \qquad \varepsilon_\bullet^L : PX^L \twoheadrightarrow (X_\bullet^L, \xi_\bullet^L),$$

$$(3.4) \qquad \varepsilon_{R\bullet}^L : P(X_R)^L \twoheadrightarrow (X_\bullet^L, \xi_\bullet^L),$$

with annihilators

$$(3.5) \qquad \mu^{L\bullet} : (X^\bullet, \xi^\bullet) \rightarrowtail PX^L$$

$$(3.6) \qquad \mu_R^{L\bullet} : (X_R^\bullet, \xi_R^\bullet) \rightarrowtail P(X_R)^L .$$

THEOREM 3.7. *(i)* $\mu_R^\bullet$ *cogenerates* X_R *and* $\varepsilon_\bullet^L$ *generates* X^L.

(ii) The processes of killing $X^{-\infty}$ *and* X_∞ *commute. Precisely there is a natural isomorphism* $(X_R)^L \cong (X^L)_R$ *under which the two objects may be identified to* X_R^L *and then (3.6) is obtained from (3.5) by killing* $X^{-\infty}$. *Thus* $\mu_R^{L\bullet}$ *(or* $\varepsilon_{R\bullet}^L$*) is quasi-complete.*

Thus we may describe the quasi-completion procedure as killing X_∞ and $X^{-\infty}$. As a result we replace the original $(X^\bullet, \xi^\bullet)$ by its quotient by $X^{-\infty}$ and $(X_\bullet, \xi_\bullet)$ by its subobject by X_∞ and X is replaced by its subquotient X_R^L. A filtration is quasi-complete if and only if $X_\infty = X^{-\infty} = 0$ and the procedure leaves such a filtration unchanged. Moreover the relation of the completion and quasi-completion procedures is described precisely in the following theorem.

THEOREM 3.8. *Let each morphism in (2.4) be factorized as epic followed by monic. This divides (2.4) into four bicartesian squares of which the top right hand square is*

$$(3.9) \qquad \begin{array}{ccc} X^L & \overset{\lambda}{\rightarrowtail} & X \\ \Big\downarrow{\scriptstyle \kappa^L} & & \Big\downarrow{\scriptstyle \kappa} \\ X^L_R & \underset{\lambda_R}{\rightarrowtail} & X_R \end{array}$$

where $\kappa \,\square\, R\mu^\bullet$, $L\,\varepsilon_\bullet \,\square\, \lambda$, $\kappa^L \,\square\, R\mu^{L\bullet}$, $L\,\varepsilon_{R\bullet} \,\square\, \lambda_R$. *In particular (3.9) gives a self-dual description of* X^L_R *as* $Im\,\kappa\,\lambda$, *or equivalently* $Im\,R\,\varepsilon_\bullet \circ L\mu^\bullet$.

4. Remarks.

We remark that the completion procedure depends only on $(X^\bullet, \xi^\bullet)$ (or $(X_\bullet, \xi_\bullet)$) and not on $\mu^\bullet$ (or $\varepsilon_\bullet$). That is, provided $(X^\bullet, \xi^\bullet)$ is the domain of some filtration $\mu^\bullet$, then the completion is independent of the choice of $\mu^\bullet$. On the other hand the quasi-completion procedure does depend in general on $\mu^\bullet$ and not simply on $(X^\bullet, \xi^\bullet)$. For let $\mu^\bullet$ generate X without being L-complete; that is $L\mu^\bullet$ is epic with non-zero kernel K,

$$K \rightarrowtail X^\infty \overset{L\mu^\bullet}{\twoheadrightarrow} X .$$

Then $\mu^\bullet$ and $\pi^\bullet$ both generate; but the object we get in quasi-completing $\mu^\bullet$ is $X/X^{-\infty}$, while the object we get in quasi-completing $\pi^\bullet$ is $X^\infty/X^{-\infty}$ and the kernel of $X^\infty/X^{-\infty} \twoheadrightarrow X/X^{-\infty}$ is again K.

In a category of modules $L\mu^\bullet$ is always monic (but $R\,\varepsilon$ is not always epic !). Thus we have

PROPOSITION 4.1. *If* $\mathfrak{A}$ *is a category of modules then*

(i) $\mu^\bullet$ *is* L*-complete if it generates* X;

(ii) the quasi-completion of $\mu^\bullet$ *depends only on* $(X^\bullet, \xi^\bullet)$.

Reverting to the general case, we may consider criteria of completeness (or quasi-completeness) instead of the procedures. Plainly the question whether $\mu^\bullet : (X^\bullet, \xi^\bullet) \rightarrowtail PX$ is complete depends on $\mu^\bullet$ itself; but, given that $(X^\bullet, \xi^\bullet)$ is the domain of some filtration, one may give necessary and sufficient conditions for it to be the domain of a complete filtration. Let us call $(X^\bullet, \xi^\bullet)$ a *prefiltration* if it is the domain of some filtration (i.e., if $\pi^\bullet : (X^\bullet, \xi^\bullet) \to PX^\infty$ is monic), and a *complete* prefiltration if it is the domain of some complete filtration.

THEOREM 4.2. *If $(X^{\bullet}, \xi^{\bullet})$ is a prefiltration, then it is complete if and only if $R\eta_{\bullet}$ is an isomorphism. If $R\eta_{\bullet}$ is an isomorphism then $\pi^{\bullet}$ is the unique complete filtration with domain $(X^{\bullet}, \xi^{\bullet})$.*

Here $\eta_{\bullet} \square \pi^{\bullet}$, $\eta_{\bullet} : PX^{\infty} \twoheadrightarrow (X^{\infty}_{\bullet}, \xi^{\infty}_{\bullet})$.

Insofar as quasi-completeness is concerned we have the following result.

THEOREM 4.3. *The prefiltration $(X^{\bullet}, \xi^{\bullet})$ is quasi-complete if and only if $R(X^{\bullet}, \xi^{\bullet}) = 0$. If $R(X^{\bullet}, \xi^{\bullet}) = 0$, then $\mu^{\bullet} : (X^{\bullet}, \xi^{\bullet}) \rightarrowtail PX$ is quasi-complete if and only if $L\mu^{\bullet}$ is epic. If also $\mathfrak{A}$ is a category of modules then $\pi^{\bullet}$ is the unique quasi-complete filtration with domain $(X^{\bullet}, \xi^{\bullet})$.*

5. Examples.

Since we wish to exhibit the difference between completion and quasi-completion by examples within a category of modules, we are content (see Proposition 4.1 *(i)*) to consider L-complete filtrations.

EXAMPLE 1. Let $\theta : D \to D$ be an endomorphism of the module D. We consider the filtration $\ldots \subseteq \theta^n D \subseteq \theta^{n-1} D \subseteq \ldots \subseteq \theta D \subseteq D$ of D. Since the filtration is right-finite it is evidently L-complete. We render it quasi-complete by factoring out $\lim\limits_{\leftarrow n} \theta^n D = \bigcap\limits_n \theta^n D$; we pass to the cofiltration and we have an exact sequence

$$\bigcap_n \theta^n D \rightarrowtail D \xrightarrow{\omega} \lim_{\leftarrow n} D/\theta^n D \,.$$

Then the filtration obtained by factoring out $\bigcap\limits_n \theta^n D$ is complete if and only if ω is epic. Plainly ω is not always epic. For example, let D be the graded abelian group $D = \bigoplus\limits_{k \geq 0} D_k$, with

$$D_k = (a_k, a_{k+1}, \ldots), \quad k \geq 0,$$

the free abelian group on generators $a_k, a_{k+1}, \ldots$. Further let θ be of degree -1, $\theta \mid D_{k+1} : D_{k+1} \to D_k$ being the obvious embedding. Then

$\bigcap_n \theta^n D = 0$, so $\omega : D \rightarrowtail \lim\limits_{\leftarrow n} D/\theta^n D$. Moreover,

$$(D/\theta^n D)_k = (a_k, a_{k+1}, \dots, a_{k+n-1}), \quad k \geq 0,$$

so $$(\lim_{\leftarrow n} D/\theta^n D)_k = \Pi(a_k, a_{k+1}, \dots),$$

the direct product of cyclic groups generated by a_k, $a_{k+1}, \dots$. On the other hand D_k is the direct *sum*, so ω is not epic, and the original filtration, although quasi-complete, is not complete. To complete it we must replace D by $D_{-\infty} = \lim\limits_{\leftarrow} D/\theta^n D$ and annihilate the cofiltration

$$D_{-\infty} \twoheadrightarrow \dots \twoheadrightarrow D/\theta^n D \twoheadrightarrow D/\theta^{n-1} D \twoheadrightarrow \dots \twoheadrightarrow D/\theta D \twoheadrightarrow 0.$$

This amounts to replacing D by $D_{-\infty}$ and extending θ to $\theta_{-\infty} : D_{-\infty} \to D_{-\infty}$ in the obvious way.

EXAMPLE 2. Let h be a cohomology theory defined on the category of CW-complexes. Let $\{K_n\}$ be the skeleton decomposition of the complex K and let $X^n = \ker h(K) \to h(K_n)$, $X = h(K)$. Then we have the filtration

$$(5.1) \qquad \dots \subseteq X^n \subseteq X^{n-1} \subseteq \dots \subseteq X^0 \subseteq X$$

which is again evidently L-complete. We make it quasi-complete by factoring out $\bigcap X^n$; that is, the subgroup of X consisting of cohomology classes which vanish on every skeleton. Call this subgroup $h'(K)$. Moreover let $h_n(K)$ be the image of $h(K)$ in $h(K_n)$. Then we pass to the annihilating cofiltration of (5.1) and obtain the exact sequence

$$h'(K) \rightarrowtail h(K) \xrightarrow{\omega} \lim_{\leftarrow n} h_n(K)$$

and the question at issue is whether ω is epic. This is certainly the case if h is representable (by an Ω-spectrum). Thus in the case of the application of a representable cohomology theory to a skeleton decomposition the quasi-completion of the resulting filtration coincides with the completion [1], and consists of factoring out $h'(K)$. However it is easy to construct examples of cohomology theories wherein ω is not in general epic.

[1] We need not confine attention to a skeleton decomposition; we could take any filtration of K by subcomplexes K_n such that $K_n = 0$, $n < 0$, and $\bigcup_n K_n = K$.

References.

[1] B. ECKMANN and P.J. HILTON. Filtrations, associated graded objects and completions, Math. Zeit. (1967).

[2] S. EILENBERG and J.C. MOORE. Limits and spectral sequences, Topology I (1962), 1-24.