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# Commuting limits

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# COMMUTING LIMITS

*by Peter HILTON*

In a previous contribution [ 5 ], we discussed the completion of a filtration in an abelian category $\mathfrak{A}$. Precisely, let

$$\ldots \subseteq X^{p-1} \subseteq X^p \subseteq \ldots \subseteq X$$

be a filtration of $X$ in $\mathfrak{A}$, $-\infty < p < \infty$. Then we may form the completion (in a functorial manner),

$$\ldots \subseteq X^{p-1}_{-\infty} \subseteq X^p_{-\infty} \subseteq \ldots \subseteq X^\infty_{-\infty}$$

without affecting the associated graded object. Moreover if $X^p_q = X^p / X^q$, $p \geq q$, there is a commutative diagram

$$\begin{array}{ccc} X^p_q & \stackrel{\psi}{\rightarrowtail} & X^{p+1}_q \\ \downarrow \varphi & & \downarrow \varphi \\ X^p_{q+1} & \stackrel{\psi}{\rightarrowtail} & X^{p+1}_{q+1} \end{array}$$

and (theorem 2.7 of [ 5 ] )

$$X^\infty_{-\infty} = \varinjlim_p \varprojlim_q X^p_q = \varprojlim_q \varinjlim_p X^p_q .$$

Our interest in this note centres on the phenomenon that we may commute the limit $\varprojlim_q$ with the colimit $\varinjlim_p$. Let us give a simple (but natural) example where these limits do not commute; first, however, we remark that there is always a limit-switching transformation

$$\omega : \varinjlim \varprojlim \to \varprojlim \varinjlim$$

which we will describe precisely in the next section.

Consider now a family of abelian groups $D_{mn}$, doubly-indexed by the positive integers, and let

$$X_i^j = \bigoplus_{\substack{m \leq i \\ n \leq j}} D_{mn} .$$

We then have a commutative diagram, of injections and projections

$$\begin{array}{ccc} X_i^j & \overset{\psi}{\rightarrowtail} & X_i^{j+1} \\ \downarrow \varphi & & \downarrow \varphi \\ X_{i-1}^j & \overset{\psi}{\rightarrowtail} & X_{i-1}^{j+1} \end{array}$$

and $\omega$ is, in this case, the natural morphism

$$\omega : \bigoplus_n \prod_m D_{mn} \to \prod_m \bigoplus_n D_{mn}$$

which is monic but not, in general, epic.

Let us introduce a third exemple. Let

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & E & \end{array}$$

be an exact couple in $\mathfrak{A}$. Associated with this exact couple there is a spectral sequence $(E_n, d_n ; n \geq 0)$ with limit term $E_\infty$. Here

$$E_n = \gamma^{-1}(\alpha^n D)/\beta \alpha^{-r}(0)$$

and $d_n$ is induced by $\beta \alpha^{-n} \gamma$. Now (see [ 3 ]) we may define

$$E_{mn} = \gamma^{-1}(\alpha^n D)/\beta \alpha^{-m}(0),$$

so that $E_n = E_{nn}$, and there is then a commutative diagram,of projections and injections,

$$\begin{array}{ccc} E_{mn} & \overset{\psi}{\twoheadrightarrow} & E_{m+1,n} \\ \varphi \downarrow & & \downarrow \varphi \\ E_{m,n-1} & \overset{\psi}{\twoheadrightarrow} & E_{m+1,n-1} \end{array}$$

and (theorem 4.13 of [ 3 ])

$$E_\infty = \varinjlim_m \varprojlim_n E_{mn} = \varprojlim_n \varinjlim_m E_{mn} .$$

Then in this case the limits do commute, but there is an important difference between this case and that of our original example arising from a filtered object, since in the latter the $\psi$-morphisms are monic and the $\varphi$-morphisms are epic, while in this case the $\psi$-morphisms are epic and the $\varphi$-morphisms are monic.

Our object is to find a common generalization (theorem 3.6) of these two cases. We emphasize that until we actually come to theorem 3.6 our definitions and arguments are valid in any category over which we may take the appropriate limits.

Details of this work, which is joint work with B. Eckmann, will appear elsewhere [4]. We refer the reader to the papers of J.E. Roos [6] for a systematic study of the transformation $\omega$ in categories of sheaves (topos).

## 1. The transformation $\omega$.

Let $C$ be a category, $I$ a small connected index category and $F : I \to C$ a functor. Then $F$ is an object of the functor category $C^I$ and there is a full embedding functor $P : C \to C^I$. We suppose $P$ to have a right adjoint

$$P \overset{\tau}{\longmapsto} R, \qquad R : C^I \to C \cdot$$

Thus $\tau : PR \to 1$, $RP = 1$, $\tau P = 1$, $R\tau = 1$. We may write $R$ for $R(F)$ if the functor $F$ may be understood; we may also write $F_i$ for $F(i)$, $i \in |I|$, and $\varphi : F_i \to F_j$ for $F(\varphi) : F_i \to F_j$, if $\varphi : i \to j$ in $I$. We write $\tau_i : R \to F_i$ for the $i^{th}$ component of $\tau_F$ so that

$$\varphi \tau_i = \tau_j \quad \text{if} \quad \varphi : i \to j \text{ in } I.$$

Then $(R, \tau_i)$ is the limit of the functor $F$ and we may write, even, $R = \varprojlim F$.

We adopt similar conventions for the colimit. If $J$ is a small connected index category and $F : J \to C$ a functor, then $F$ is an object of

$C^J$ and we write $Q$ for the embedding functor $Q: C \to C^J$. We suppose $Q$ to have a left adjoint

$$L \overset{\cdot\pi}{\longmapsto} Q, \qquad\qquad L: C^J \to C.$$

Thus $\pi: 1 \to QL$, $LQ = 1$, $\pi Q = 1$, $L\pi = 1$. We may write $L$ for $L(F)$, $F_j$ for $F(j)$, $j \in |J|$, and $\psi: F_i \to F_j$ for $F(\psi): F_i \to F_j$ if $\psi: i \to j$ in $J$. We write $\pi_j: F_j \to L$ for the $j^{th}$ component of $\pi_F$ so that

$$\pi_j \psi = \pi_i \quad \text{if} \quad \psi: i \to j \text{ in } J.$$

Then $(L, \pi_j)$ is the colimit of the functor $F$ and we may write $L = \underset{\to}{lim} F$.

Consider now the product category $I \times J$. We then have the square

$$\begin{array}{ccc} C^{I \times J} & \xrightarrow{L^I} & C^I \\ \Big\downarrow R^J & & \Big\downarrow R \\ C^J & \xrightarrow{L} & C \end{array}$$

which is not, in general, commutative. However,

$$R L^I P^J R^J = R P L R^J = L R^J$$

so we have a natural transformation

$$R L^I \tau^J : L R^J \to R L^I.$$

Dually, we have a natural transformation

$$L R^J \pi^I : L R^J \to R L^I.$$

THEOREM 1.1.

$$R L^I \tau^J = L R^J \pi^I : L R^J \to R L^I.$$

We call this self-dual transformation the *limit-switching transformation*

$$\omega: L R^J \to R L^I.$$

We may even write it as

$$\omega: LR \to RL.$$

If we are given a functor $F: I \times J \to C$ then we may write

$$\omega_F : \varinjlim_J \varprojlim_I F \to \varprojlim_I \varinjlim_J F,$$

even abbreviating $\omega_F$ to $\omega$.

## 2. The index categories.

We now restrict the index categories $I$ and $J$. We will describe explicitly the conditions we impose on the category $J$ and the results which follow from those conditions. It will then be understood that we impose dual conditions on the category $I$ so that the dual results are valid.

We say that $J$ is *quasi-filtered* (see [ 1 ]) if it has the following two properties [1]) :

(2.1) given $\cdot \overset{\alpha}{\nearrow} \underset{\beta}{\searrow}$ in $J$, we may find $\overset{\alpha \nearrow \searrow \gamma}{\underset{\beta \searrow \nearrow \delta}{}}$ in $J$ with $\gamma\alpha = \delta\beta$.

(2.2) given $\overset{\rho}{\underset{\sigma}{\rightrightarrows}}$ in $J$, we may find $\overset{\rho}{\underset{\sigma}{\rightrightarrows}} \overset{\eta}{\longrightarrow}$ in $J$ with $\eta\rho = \eta\sigma$.

This is an obvious generalization of a directed set; we are thus led to call a full subcategory $J_o$ of $J$ *cofinal* if, given $j \in |J|$, there exists $\varphi : j \to j_o$ in $J$ with $j_o \in |J_o|$.

PROPOSITION 2.3. *Let $j_1 \in |J|$, where $J$ is quasi-filtered and let $J_o$ be the full subcategory of $J$ such that $j_o \in |J_o|$ if and only if there exists $\varphi : j_1 \to j_o$ in $J$. Then $J_o$ is cofinal in $J$.*

PROPOSITION 2.4. *Let $F : J \to C$ be a functor from the quasi-filtered category $J$ to $C$. Let $J_o$ be cofinal in $J$ and let $F_o = F \mid J_o$. Then*

$$\varinjlim_J F = \varinjlim_{J_o} F_o .$$

More precisely, if $E : C^J \to C^{J_o}$ is the restriction functor and if $L_o \overset{\cdot \pi_o}{\vdash\!\!\!-\!\!\!-\!\!\!-} Q_o$, where $Q_o : C \to C^{J_o}$ is the embedding so that $Q_o = EQ$, then there exists a unique left adjoint extending $\pi_o$

$$L \overset{\cdot \pi}{\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!-} Q$$

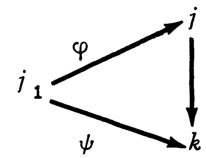
---

1) Notice that we require that a quasi-filtered category be connected.

with $L = L_o E$, $E\pi = \pi_o E$.

We remark that these two propositions remain valid if we only require *(2.1)*; property *(2.2)* does not enter into their proof.

Now, given $j_1 \in |J|$, let $J^1$ be the category under $j_1$. Thus an object of $J^1$ is a morphism $\varphi : j_1 \to j$ in $J$ and a morphism $\theta : \varphi \to \psi$ in $J^1$ is a morphism $\theta : j \to k$ such that $\psi = \theta\varphi$,




There is an evident functor $E_1 : J^1 \to J$, given by $E_1(\varphi) = j$, $E_1(\theta) = \theta$, and $E_1$ maps $J^1$ onto the category $J_o$ of proposition 2.3. Let $E_1$ induce $E_1^* : C^J \to C^{J^1}$.

PROPOSITION 2.5. *Let $F : J \to C$ be a functor from the quasi-filtered category $J$ to $C$ and let $F_1 = FE_1 : J^1 \to C$. Then $J^1$ is also quasi-filtered and*

$$\underset{\substack{\to \\ J}}{lim}\, F = \underset{\substack{\to \\ J^1}}{lim}\, F_1 .$$

More precisely, if $L \overset{\cdot\pi}{\longmapsto} Q$ and $Q^1 = E_1^* Q$, then there exists a unique left adjoint

$$L^1 \overset{\cdot\pi^1}{\longmapsto} Q^1$$

with $L = L^1 E_1^*$, $E_1^* \pi = \pi^1 E_1^*$.

In proving this proposition we first use propositions 2.3 and 2.4 to replace $J$ by the category $J_o$ of proposition 2.3 and then use property *(2.2)* to establish that

$$\underset{\substack{\to \\ J_o}}{lim}\, F_o = \underset{\substack{\to \\ J^1}}{lim}\, F_1 .$$

We use proposition 2.5 to establish

THEOREM 2.6. *Let $J$ be a quasi-filtered category and let $\varkappa : A \to B : J \to C$*

*be a natural transformation of functors, thus giving rise, for every* $\psi : j \to k$ *in* $J$, *to a commutative diagram*

$$
(2.7) \qquad
\begin{array}{ccc}
A_j & \xrightarrow{\ \psi\ } & A_k \\
\big\downarrow{\scriptstyle \varkappa_j} & & \big\downarrow{\scriptstyle \varkappa_k} \\
B_j & \xrightarrow{\ \psi\ } & B_k
\end{array}
$$

*and, for every* $j_o$ *in* $|J|$, *a commutative diagram (see* § 1 *for notation)*

$$
(2.8) \qquad
\begin{array}{ccc}
A_{j_o} & \xrightarrow{\ \pi_{j_o}\ } & L\,A \\
\big\downarrow{\scriptstyle \varkappa_{j_o}} & & \big\downarrow{\scriptstyle L\,\varkappa} \\
B_{j_o} & \xrightarrow{\ \pi_{j_o}\ } & L\,B
\end{array}
$$

*Then if (2.7) is a push-out for every* $\psi$, *(2.8) is a push-out for every* $j_o$.

Plainly we could refine this by replacing $J$ by the category $J_o$ of proposition 2.3.

We remark that theorem 2.6 is not true for arbitrary index categories. Thus we may take $J$ to be the category

$$\rightrightarrows$$

and $C$ to be the category of abelian groups and we may consider the squares

$$
(2.9) \qquad
\begin{array}{ccc}
0 & \rightrightarrows & A \\
\big\downarrow & & \big\downarrow{\scriptstyle \varkappa} \\
B & \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} & A \oplus B
\end{array}
$$

where $A$ is cyclic infinite, generated by $a$, $B$ is cyclic infinite generated by $b$,

$$\varkappa(a) = (a, 0), \qquad \psi_1(b) = (0, b), \qquad \psi_2(b) = (a, b).$$

Plainly the two squares in *(2.9)* are push-outs. Passing to the colimit

( = difference cokernel) we obtain

$$(2.10) \qquad \begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow 0 \\ B & = & B \end{array}$$

(since the cokernel of $\psi_1 - \psi_2$ is the projection $A \oplus B \to B$) and plainly (2.10) is not a push-out. We describe the passage from (2.7) to (2.8) as the *L-process;* in fact, if $j = j_o$ there is an evident morphism

$$\begin{pmatrix} 1 & \pi_k \\ 1 & \pi_k \end{pmatrix}$$

from (2.7) to (2.8) but we do not exploit this.

## 3. The main diagram.

Let $F : I \times J \to C$ be a functor and let $\varphi : i_2 \to i_1$ in $I$, $\psi : j_1 \to j_2$ in $J$. Then we have the commutative square (with evident extensions of the notation of § 1)

$$(3.1) \qquad \begin{array}{ccc} F_{i_2 j_1} & \xrightarrow{\psi_{i_2}} & F_{i_2 j_2} \\ \downarrow \varphi_{j_1} & & \downarrow \varphi_{j_2} \\ F_{i_1 j_1} & \xrightarrow{\psi_{i_1}} & F_{i_1 j_2} \end{array}$$

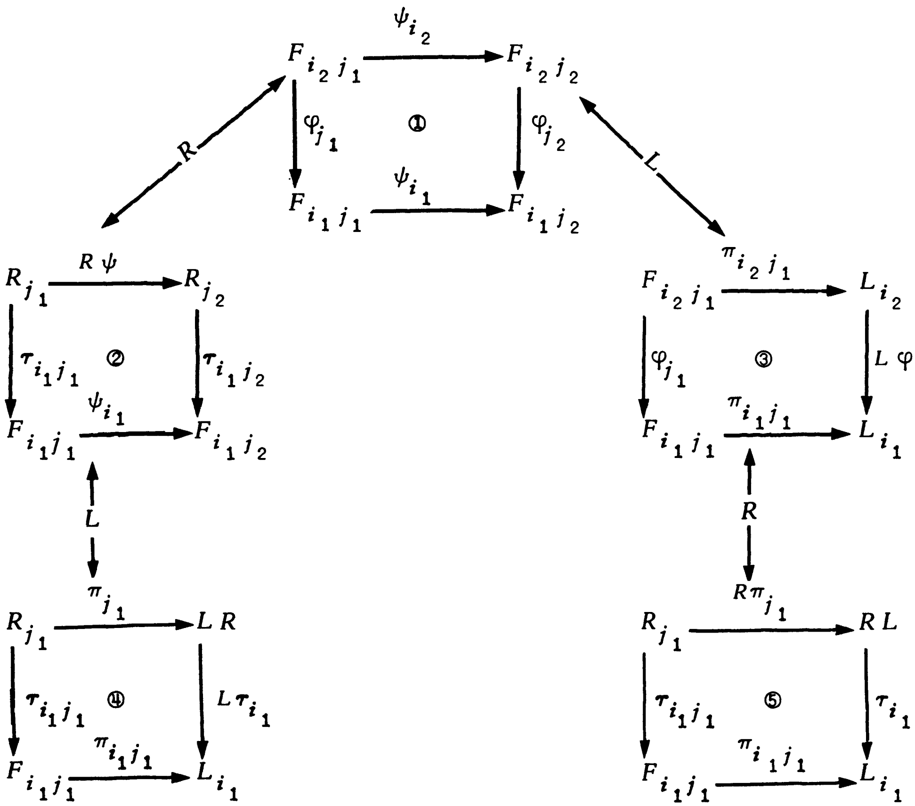
We may then submit (3.1) to either the *L*-process or its dual, the *R*-process. If we carry out the *L*-process we obtain, again with evident notation

$$(3.2) \qquad \begin{array}{ccc} F_{i_2 j_1} & \xrightarrow{\pi_{i_2 j_1}} & L_{i_2} \\ \downarrow \varphi_{j_1} & & \downarrow L\varphi \\ F_{i_1 j_1} & \xrightarrow{\pi_{i_1 j_1}} & {}_{i_1} \end{array}$$

However we may then apply the *R*-process to *(3.2)*, obtaining

$$(3.3)\qquad \begin{array}{ccc} R_{j_1} & \xrightarrow{\;R\,\pi_{j_1}\;} & RL \\ \big\downarrow{\scriptstyle \tau_{i_1 j_1}} & & \big\downarrow{\scriptstyle \tau_{i_1}} \\ F_{i_1 j_1} & \xrightarrow{\;\pi_{i_1 j_1}\;} & L_{i_1} \end{array}$$

Conversely we may apply first the *R*-process to *(3.1)* and then the *L*-process to the result. We thus obtain the *main diagram*



Moreover there is a morphism $\Omega = \begin{pmatrix} 1 & \omega \\ 1 & 1 \end{pmatrix} :$ ④ → ⑤ .

PROPOSITION 3.5. *Let $C$ be an abelian category, let ④ be a push-out and let ⑤ be a pull-back. Then $\omega : LR \to RL$ is an isomorphism.*

THEOREM 3.6. *If ① is bicartesian (pull-back and push-out) in the abelian category $C$ and if the $L$-processes and $R$-processes in (3.4) preserve the bicartesian property, then $\omega : LR \to RL$ is an isomorphism.*

We use theorem 2.6 (and its dual) to show how this theorem provides the common generalization of the two examples of commuting limits quoted in the introduction. We suppose henceforth in this section that $C$ is abelian, that ① is bicartesian for fixed $i_1, j_1$ and variable $\varphi, \psi$, that $J$ is quasi-filtered and that $I$ is quasi-cofiltered. Then invoking theorem 2.6 and its dual we deduce from theorem 3.6 :

COROLLARY 3.7. *If $\psi_{i_1}$ is epic for all $\psi$ and $\varphi_{j_1}$ is monic for all $\varphi$, then $\omega$ is an isomorphism.*

COROLLARY 3.8. *If $\psi_i$ is a prefiltration[2] for every $\psi$ and every $i \to i_1$ in $I$, and if $\varphi_j$ is a precofiltration for every $\varphi$ and every $j_1 \to j$ in $J$, then $\omega$ is an isomorphism.*

Corollary 3.7 incorporates the spectral sequence example; corollary 3.8 incorporates the filtration example. In the remaining example of the introduction (commuting direct sum with direct product) all the conditions of corollary 3.8 are satisfied except that ① is not bicartesian.

## 4. Commuting limits in a category of modules.

Plainly the situation is very much simplified if $C$ is an abelian category in which $L$ is exact. This is certainly so in a category of modules. Thus, to apply the main diagram to a functor $F$ from $I \times J$ to a category of modules we need information as to when $R$ preserves exactness. We state the result for a functor to groups. It is evidently equally valid for a functor to modules.

Let $F : I \to \mathfrak{G}$ be a functor from the quasi-cofiltered category $I$ to

2) See p. 250 of [ 5 ]; *precofiltration* is just the dual notion.

the category $\mathcal{G}$ of groups. We say (generalizing the definition in [2]) that $F$ has the *Mittag-Leffler property* if, given any $i \in |I|$, there exists $\varphi : j \to i$ in $I$ such that

$$\varphi \psi (F_k) = \varphi(F_j)$$

for all $k$ and all $\psi : k \to j$ in $I$; notice that this property certainly holds if for some cofinal subcategory $I_o$ of $I$, $F(\psi)$ is epic for all $\psi$ in $I_o$.

We say that $I$ is *special* if

*(i)* $|I|$ is countable,

*(ii)* given $i, j \in |I|$ we may simultaneously equalize the whole of $I(i, j)$.

Precisely, *(ii)* asserts the existence of $\varphi$ in $I$, with codomain $i$, such that $\psi \varphi$ is independent of the choice of $\psi \in I(i, j)$; of course, if $I(i, j)$ is finite, *(ii)* follows from the dual of *(2. 2)*. We then may prove (compare *(3. 8)* of [2]).

THEOREM 4. 1. *Let $I$ be a special quasi-cofiltered category and let*

$$F \rightarrowtail^{\varkappa} G \twoheadrightarrow^{\lambda} H$$

*be an exact sequence of functors $I \to \mathcal{G}$. Then if $F$ has the Mittag-Leffler property the limit sequence*

$$RF \rightarrowtail^{R\varkappa} RG \twoheadrightarrow^{R\lambda} RH$$

*is exact.*

From this theorem and the main diagram *(3. 4)*, we may deduce

COROLLARY 4. 2. *Let $F : I \times J \to M$ be a functor such that $I$ is a special quasi-cofiltered category, $J$ is a quasi-filtered category, and $M$ is a category of modules. Suppose, for fixed $i_1, j_1$ and variable $\varphi, \psi$ that ① in (3.4) is bicartesian and that $F_{j_1} : I \to M$ has the Mittag-Leffler property. Then $\omega : LR \to RL$ is an isomorphism.*

Of course there are also evident simplifications of corollaries 3.7, 3.8 for functors to $M$. Under the same presuppositions as stated in the prologue to those corollaries [3)] but also assuming that $C$ *is a category of modules* we have, referring to *(3. 4)*:

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3) Thus in theorem 4. 3, I need not be special; we do not know to what extent the hypotheses of speciality are essential for the conclusion of theorem 4.1.

THEOREM 4.3. *If* $\psi_i$ *is epic for all* $\psi$ *or if* $\varphi_j$ *is a precofiltration for every* $\varphi$ *and every* $j_1 \to j$ *in* $J$, *then* $\omega : LR \to RL$ *is an isomorphism.*

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