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A SPECIAL TRICATEGORY

by J. V. MICHALOWICZ

1. Introduction. The concepts of subobject, quotient object, one-to-one mapping, onto mapping, embedding and quotient map play an important role in many concrete categories. However, these notions are essentially non-categorical; that is, they do not have a categorical (i.e. point-free) definition which is valid for the general category. This paper is directed toward obtaining a method for dealing with these concepts.

Various procedures have been adopted to handle this problem. At one methodological extreme, attention is directed to one specific concrete category and an effort is made to characterize, if possible, the non-categorical concepts with which one is concerned in point-free terms. This type of investigation has been carried out for the category of uniform spaces by J. R. Isbell [4]. At the other end of the spectrum lies the procedure in which one axiomatically defines point-free concepts designed to reflect the intrinsic properties of the non-categorical concepts. The problem which remains in applying the resulting theory to a given concrete category lies in determining how closely the axiomatically defined concepts approximate the intuitive concepts in that category. Of course, the manner in which the axiomatic concepts are defined is generally dependent upon the categories in which one is particularly interested. This method is exemplified by the notion of bicategory as treated by Z. Semadeni [8] and Isbell [3]. An approach which lies somewhere between is the method used by I. Heller [1]. This procedure begins with the intuitive non-categorical concepts and, by developing a system of requirements on the concrete category, transforms the non-categorical characterizations into categorical ones. The resulting characterizations, furthermore, provide motivation for formulations of the intuitive concepts in the general abstract

category. The most well-behaved abstract category mentioned by Heller [2] is the strongly pure tricategory with embeddings and quotients; this notion is readily seen to be equivalent to the concept of *JTK*-category, the letters *J*, *T*, and *K* denoting the classes of morphisms that constitute the tri-basis, formulated in this paper.

Thus a *JTK*-category is essentially a tricategory with a tri-basis satisfying Heller's axioms which are designed to make *J*, *T* and *K* meaningful generalizations of the classes of embeddings, one-to-one onto mappings, and quotient maps, respectively.

The objective of this paper is to investigate the implications of these axioms; that is, to delineate the extent to which the generalizations retain the properties of their progenitors, and to develop the theory of *JTK*-categories. In the next section we define the *JTK*-category and discuss some immediate consequences. Section 3 studies some categorical concepts in *JTK*-categories and Section 4 introduces the concept of semi-exact *JTK*-category. Section 5 discusses products and coproducts and Section 6 considers the notions of projectivity and injectivity in *JTK*-categories. Various examples are given in Section 7.

We shall draw upon the basic concepts of category theory as given in Mitchell [7] and Semadeni [8]. The classes of monomorphisms, epimorphisms, bismorphisms, retractions, coretractions, isomorphisms and identity morphisms in a category \mathcal{C} will be denoted by M , E , B , R , C , S , and I , respectively. We generally indicate that a morphism belongs to one of these classes by using a lower case letter; that is, m means $m \in M$, etc. The notation $AD = \{ad \mid a \in A, d \in D \text{ and } ad \text{ is defined}\}$ for the product of classes of morphisms in \mathcal{C} will also be used.

The papers of Heller [1, 2] are prerequisite in providing motivation and in supplying our notation for a concrete category \mathcal{C}' ; e.g., \bar{M} denotes the class of one-to-one mappings in \mathcal{C}' , \bar{E} the onto mappings, \bar{B} the one-to-one onto mappings, M° the identity injections of subobjects into their parent objects, E° the class of natural projections of objects onto their quotient objects, P the class of embeddings, and Q the class of quotient maps.

We list some essential results for \mathcal{C}' , to which we will later compare the properties of the classes of morphisms in the *JTK*-category.

$$\bar{M} \subset M; \bar{E} \subset E; \bar{B} \subset B; \bar{B} = \bar{M} \cap \bar{E}; \bar{M}\bar{M} \subset \bar{M}; \bar{E}\bar{E} \subset \bar{E}.$$

$$\bar{m} = fg \text{ implies } g \in \bar{M}; \bar{e} = fg \text{ implies } f \in \bar{E}.$$

$$\bar{m} = g\bar{e} \text{ implies } g \in \bar{M}; \bar{e} = \bar{m}b \text{ implies } b \in \bar{E}.$$

$$\text{Im } |gb| \subset \text{Im } |g|; \text{Im } |g\bar{e}| = \text{Im } |g|; \text{Im } |\bar{m}b| = \text{Im } |\bar{m}| \text{ iff } b \in \bar{E}.$$

$$\text{Coim } |\bar{m}b| \subset (\text{viz. , is coarser than or equal to}) \text{Coim } |b|;$$

$$\text{Coim } |\bar{m}b| \subset \text{Coim } |b|; \text{Coim } |g\bar{e}| = \text{Coim } |\bar{e}| \text{ iff } g \in \bar{M}.$$

Some of the assumptions on \mathcal{C}' studied in [1] are repeated here for future reference, as is a result summarizing the consequent pointwise descriptions.

(A2) i.) If X and Y are objects in \mathcal{C}' and X is a subobject of Y , then the identity injection $m^\sigma : X \rightarrow Y$ is in \mathcal{C}' .

ii.) If X and Y are objects in \mathcal{C}' and Y is a quotient object of X , then the natural projection $e^\circ : X \rightarrow Y$ is in \mathcal{C}' .

(A3) i.) $\text{Im } |m|$ is an object in \mathcal{C}' for all $\bar{m} \in \bar{M}$.

ii.) $\text{Coim } |\bar{e}|$ is an object in \mathcal{C}' for all $\bar{e} \in \bar{E}$.

(A5) i.) If f and m° have common codomain and $\text{Im } |f| = \text{Im } |m^\circ|$, then $f = m^\circ b$ for some b in \mathcal{C}' .

ii.) If f and e° have common domain and $\text{Coim } |f| = \text{Coim } |e^\circ|$, then $f = ge^\circ$ for some g in \mathcal{C}' .

(A9) i.) If $m \notin \bar{M}$, then $m = g\bar{b}$ where $g \in M$.

ii.) If $e \notin \bar{E}$, then $e = \bar{b}b$ where $b \in E$.

PROPOSITION A. If \mathcal{C}' satisfies (A2, 3, 5, 9,) then

$$\bar{B} = \{ b \mid b = \bar{b}_1 f \bar{b}_2 \text{ implies } f \in B \},$$

$$\bar{M} = \{ m \mid m = g\bar{b} \text{ implies } g \in M \},$$

$$\bar{E} = \{ e \mid e = \bar{b}b \text{ implies } b \in E \},$$

$$P = \{ m \mid m = g\bar{b} \text{ implies } g \in M \text{ and } \bar{b} \in S \},$$

$$Q = \{ e \mid e = \bar{b}b \text{ implies } b \in E \text{ and } \bar{b} \in S \}.$$

2. Axiomatic description of the JTK -category. A category \mathcal{Q} will be called a JTK -category if there is a class T of morphisms in \mathcal{Q} such that T and the classes J, K, L, N of morphisms defined by

$$(T1) \quad \begin{cases} J = \{ m \mid m = gt \text{ implies } g \in M \text{ and } t \in S \} \\ K = \{ e \mid e = tb \text{ implies } b \in E \text{ and } t \in S \} \end{cases}$$

$$(T2) \quad \begin{cases} L = \{ m \mid m = gt \text{ implies } g \in M \} \\ N = \{ e \mid e = tb \text{ implies } b \in E \} \end{cases}$$

satisfy the following conditions:

$$(T3) \quad T = \{ b \mid b = t_1 f t_2 \text{ implies } f \in B \}.$$

$$(T4) \quad JJ CJ, KK CK, TT CT, \text{ under the law of composition in } \mathcal{Q}.$$

$$(T5) \quad J \cap T = T \cap K = J \cap K = S.$$

$$(T6) \quad TJ CL, KT CN.$$

$$(T7) \quad \text{Every morphism } f \text{ in } \mathcal{Q} \text{ has a representation } f = jtk \text{ which is unique up to isomorphism; that is, if } f = j_1 t_1 k_1, \text{ then } j_1 = js, k_1 = s_1 k, \text{ and } t_1 = s^{-1} t s^{-1}.$$

Observe the notational devices used as indicators; e.g., $t \in T, t_1 \in T$, etc.. Clearly, (T3) is not a definition of the class T but a requirement on it, whereas (T1) and (T2) provide definitions for the classes J, K, L and N . Evidently, $JCLCM, KCNCE$, and $T \subset B$. It should be noted in (T1) that the implications that $g \in M$ and $b \in E$ are redundant.

The JTK -category is then a tricategory with tri-basis (J, T, K) which satisfies some additional assumptions intended to make T an abstraction of the class of one-to-one onto mappings \overline{B} , J an abstraction of P , the class of embeddings, K of Q , the class of quotient maps, L of \overline{M} , the class of one-to-one mappings, and N of \overline{E} , the class of onto mappings. Proposition A at once presents motivation for these additional axioms and provides the basis for constructing many examples of JTK -categories.

We want to be sure that these JTK -categorical classes of morphisms retain the desired properties of their progenitors in the concrete category. First we note that the classes J and K are dual, as are L and N ,

and T is self-dual. Obviously, $S \subset T$, $S \subset J$, $S \subset K$ and all of the classes are closed under isomorphisms. As should be expected, we have $T = L \cap N$ and $J \cap N = S = K \cap L$. It can also be verified that $L = JT$, $N = TK$ and L and N are closed under composition. That the JTK -categorical classes of morphisms satisfy the first half of the list of properties of their progenitors in the concrete category given in Section 1 can now be readily established; the remaining properties also carry over to the JTK -category, as will be noted in the next section.

Thus the JTK -categorical classes of morphisms are appropriate abstractions of the corresponding non-categorical classes. Therefore, in the special type of tricategory herein called the JTK -category, the concepts of embedding, quotient map, one-to-one mapping, onto mapping and one-to-one onto mapping have point-free abstractions and hence can be discussed categorically. There are a large number of categories for which a JTK -categorical structure can be defined; some examples are given in the last section of this paper. Since the non-categorical concepts with which we are concerned are of the greatest interest in categories which are topological rather than algebraic, the emphasis of this paper will be on categories of a topological nature.

We further note that every JTK -category can be made into a bicategory with either (J, N) or (L, K) as the di-basis.

3. Categorical concepts in the JTK -category. Many of the concepts introduced in the category are based on the underlying specification that an object A is a subobject of an object A' iff there is a monomorphism from A to A' and that an object A'' is a quotient object of an object \hat{A} iff there is an epimorphism from \hat{A} to A'' . The JTK -category is a tricategory, so the underlying notions of subobject and quotient object are based on J and K instead of on M and E ; since J and K are abstractions of the classes of embeddings and quotient maps, these notions will in general closely approximate the intuitive notions and will also guide us in our investigation of special properties of categorical concepts in the JTK -category. In this paper we shall mention only a few of the fundamental results in this investigation; a complete discussion can be found in [5].

Dual results will be left unstated.

Some categorical concepts which are monomorphisms become morphisms in J in the JTK -category, thus corresponding to the underlying JTK -categorical notion of subobject. Examples are the equalizer and the intersection of any family of morphisms in J . However, this does not always occur; when it doesn't, we define a new JTK -categorical concept and relate it to the associated categorical concept. For instance, the image of a morphism has not to be in J (as can be seen in any JTK -category in which $S \nabla T$), so we give the following definition in the JTK -category: a morphism $u: A'' \rightarrow A'$ is called a J -image of $f: A \rightarrow A'$ if $u \in J$, if $f = u \circ f'$ for some $f': A \rightarrow A''$ and if, whenever $f = j \circ b$, then $u = j \circ g$ for some g . Clearly, the J -image is unique up to isomorphism. The dual concept is labelled the K -coimage. That these are the natural concepts in the JTK -category follows from the fact that for a morphism f with the decomposition $f = j \circ t \circ k$, j is the J -image and k is the K -coimage. The J -image of a morphism is the J -image of its image (when the image exists) so the image and J -image of a morphism are the same iff that image is in J . Among the results which now follow readily are those evolved from the remainder of the properties of the concrete category in Section 1. Also the union of morphisms in J has not to belong to J (an example can be found in the JTK -category of topological spaces), so we again define a new concept, called the J -union, in the natural way. The J -union is unique up to isomorphism and is given by the J -image of the union (when the union exists).

Various statements involving these concepts can now be established; e.g., the J -image of the J -union is the J -union of the J -images, the inverse image of the intersection is the intersection of the inverse images.

Now let us consider a JTK -category with a zero object Z_0 . Since RCK and CCJ , the unique morphism from Z_0 into A is in J and the unique morphism from A into Z_0 is in K , for each object A . Kernels are in J and cokernels in K , so it is natural to define J -normal (K -conormal) JTK -categories as those in which every morphism in J is a kernel (every morphism in K is a cokernel). It is immediate that a J -normal JTK -category is normal iff $M = J$, and it follows that, if a JTK -category is normal or

conormal, then it is balanced and the *JTK*-categorical classes reduce to the corresponding categorical classes of morphisms. As examples of results which are valid in a *J*-normal *JTK*-category we include the statements that $J \cap E = S$ and that, if the *JTK*-category also has cokernels, then the *J*-image of a morphism is the kernel of its cokernel.

4. Semi-exact *JTK*-categories. We define a *JTK*-category to be semi-exact if it is *J*-normal and *K*-conormal with kernels and cokernels. First we take a look at two finite *JTK*-categories, which serve to provide counterexamples and to illustrate the structure of the *JTK*-category.

Category \mathcal{A}_1 :	Objects: $X = \{1, 2, 3\}$,
	$Y = \{1, 2\}$,
	$Z = \{1\}$,
	$V = \{\{1, 2\}, \{3\}\}$,

Morphisms: see Table I.

Law of Composition of morphisms: Usual composition of mappings. As we have just formulated it, the objects of \mathcal{A}_1 are considered as sets, and the morphisms as mappings. However, if X is given the topology with $X, \emptyset, \{1, 3\}$ and $\{2\}$ as the open sets, Y and Z the discrete topology, V the indiscrete topology, and $1, 1, 1$, and $\{1, 2\}$ are taken as the distinguished points of X, Y, Z and V , respectively, then the morphisms in Table I are precisely the continuous mappings which preserve distinguished points, so that \mathcal{A}_1 can be viewed as a full subcategory of the category of topological spaces with distinguished points. \mathcal{A}_1 is a *JTK*-category with $S = I, T = B = S \cup \{b_{12}\}, L = M$ and $N = E$, which is *J*-normal with kernels and cokernels, but is not *K*-conormal since $b_{30} \in K$ but is not a cokernel.

The second *JTK*-category \mathcal{A}_2 is the subcategory of \mathcal{A}_1 obtained by eliminating the morphisms b_{24} and b_{30} . It can be shown that \mathcal{A}_2 is a semi-exact *JTK*-category with $S = I, T = B = S \cup \{b_{12}\}, L = M$, and $N = E$. Details are provided in [5].

For a semi-exact *JTK*-category \mathcal{A} there is for each object A a one-to-one correspondance between the equivalence classes of morphisms in J with codomain A and the equivalence classes of morphisms in K

Table 1

Morphisms in \mathcal{A}_1 :		
$b_1 : Z \rightarrow X$	$b_1(1) = 1$	$\in J$
$b_2 : Z \rightarrow Y$	$b_2(1) = 1$	$\in J$
$b_3 : Z \rightarrow Z$	$b_3(1) = 1$	$= i_Z$
$b_4 : Z \rightarrow V$	$b_4(1) = \{1, 2\}$	$\in J$
$b_5 : Y \rightarrow X$	$b_5(1) = b_5(2) = 1$	
$b_6 : Y \rightarrow X$	$b_6(1) = 1, b_6(2) = 2$	$\in J$
$b_7 : Y \rightarrow X$	$b_7(1) = 1, b_7(2) = 3$	$\in M - J$
$b_8 : Y \rightarrow Y$	$b_8(1) = 1, b_8(2) = 2$	$= i_Y$
$b_9 : Y \rightarrow Y$	$b_9(1) = b_9(2) = 1$	
$b_{10} : Y \rightarrow Z$	$b_{10}(1) = b_{10}(2) = 1$	$\in K$
$b_{11} : Y \rightarrow V$	$b_{11}(1) = b_{11}(2) = \{1, 2\}$	
$b_{12} : Y \rightarrow V$	$b_{12}(1) = \{1, 2\}, b_{12}(2) = 3$	$\in (T - S) \cap (M - J) \cap (E - K)$
$b_{13} : V \rightarrow X$	$b_{13}(\{1, 2\}) = b_{13}(\{3\}) = 1$	
$b_{14} : V \rightarrow X$	$b_{14}(\{1, 2\}) = 1, b_{14}(\{3\}) = 3$	$\in J$
$b_{15} : V \rightarrow Y$	$b_{15}(\{1, 2\}) = b_{15}(\{3\}) = 1$	
$b_{16} : V \rightarrow Z$	$b_{16}(\{1, 2\}) = b_{16}(\{3\}) = 1$	$\in K$
$b_{17} : V \rightarrow V$	$b_{17}(\{1, 2\}) = \{1, 2\}, b_{17}(\{3\}) = \{3\}$	$= i_V$
$b_{18} : V \rightarrow V$	$b_{18}(\{1, 2\}) = b_{18}(\{3\}) = \{1, 2\}$	
$b_{19} : X \rightarrow X$	$b_{19}(1) = 1, b_{19}(2) = 2, b_{19}(3) = 3$	$= i_X$
$b_{20} : X \rightarrow X$	$b_{20}(1) = b_{20}(2) = b_{20}(3) = 1$	
$b_{21} : X \rightarrow X$	$b_{21}(1) = b_{21}(2) = 1, b_{21}(3) = 3$	
$b_{22} : X \rightarrow X$	$b_{22}(1) = b_{22}(3) = 1, b_{22}(2) = 2$	
$b_{23} : X \rightarrow X$	$b_{23}(1) = b_{23}(3) = 1, b_{23}(2) = 3$	
$b_{24} : X \rightarrow X$	$b_{24}(1) = 1, b_{24}(2) = b_{24}(3) = 3$	
$b_{25} : X \rightarrow Y$	$b_{25}(1) = b_{25}(2) = b_{25}(3) = 1$	
$b_{26} : X \rightarrow Y$	$b_{26}(1) = b_{26}(3) = 1, b_{26}(2) = 2$	$\in K$
$b_{27} : X \rightarrow Z$	$b_{27}(1) = b_{27}(2) = b_{27}(3) = 1$	$\in K$
$b_{28} : X \rightarrow V$	$b_{28}(1) = b_{28}(2) = b_{28}(3) = \{1, 2\}$	
$b_{29} : X \rightarrow V$	$b_{29}(1) = b_{29}(3) = \{1, 2\}, b_{29}(2) = \{3\}$	$\in E - K$
$b_{30} : X \rightarrow V$	$b_{30}(1) = \{1, 2\}, b_{30}(2) = b_{30}(3) = \{3\}$	$\in K$
$b_{31} : X \rightarrow V$	$b_{31}(1) = b_{31}(2) = \{1, 2\}, b_{31}(3) = \{3\}$	$\in K$

with domain A , obtained by associating with each equivalence class of morphisms in J the equivalence class of the cokernel of any representative. This leads to a one-to-one correspondance between all the equivalence classes of morphisms in J and all the equivalence classes of morphisms in K . Therefore, in a skeleton of a semi-exact JTK -category there is a one-to-one correspondance between the morphisms from J and those from K . This is conspicuous in our finite JTK -categories for each of these is a skeleton of itself and in the semi-exact \mathcal{Q}_2 there are 9 morphisms in J and 9 in K whereas in \mathcal{Q}_1 , which is J -normal and has kernels and cokernels but is not semi-exact, there are again 9 morphisms in J but 10 in K .

A sequence of morphisms

$$\dots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \longrightarrow \dots$$

in a semi-exact JTK -category is called semi-exact if the kernel of f_{i+1} is the J -image of f_i for each i . This is weaker than exactness in the semi-exact JTK -category but stronger than the order two property. We can show that

$$\begin{aligned} 0 \longrightarrow A \xrightarrow{f} A' & \text{ is semi-exact iff } f \in L, \\ A \xrightarrow{f} A' \longrightarrow 0 & \text{ is semi-exact iff } f \in N, \\ 0 \longrightarrow A \xrightarrow{f} A' \longrightarrow 0 & \text{ is semi-exact iff } f \in T, \end{aligned}$$

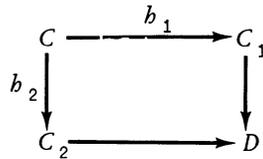
It follows that, in any semi-exact JTK -category, $L = M$, $N = E$, and $T = B$ so the tri-basis is unique. Furthermore, it can be verified that the semi-exact JTK -category has finite J -unions.

Any exact category is a balanced semi-exact JTK -category in which the JTK -categorical classes of morphisms are just the corresponding categorical classes. Hence the usual categorical classes of morphisms provide the desired abstractions in exact categories. Further, the concept of semi-exact JTK -category is an important one. We have seen one example of a semi-exact JTK -category which is not exact, viz. \mathcal{Q}_2 , and there are more interesting such examples, e.g., the category of abelian topological groups, which is also an additive category with products. Many theorems for exact categories, such as diagram lemmas and isomorphism theo-

rems, can be generalized for the semi-exact *JTK*-category (see [6]); the principal difference in the two structures lies in the three-part decomposition of each morphism in the semi-exact *JTK*-category as opposed to the two-part decomposition in the exact category.

5. Product and coproduct. In this section we will obtain a generalization in the *JTK*-category of the Embedding Lemma of point-set topology. Suppose $\{f_\lambda : A' \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ is a family of morphisms in a *JTK*-category and assume that the product $A = \prod A_\lambda$ with projections $\{p_\lambda : A \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ exists. Then there is a unique morphism $f : A' \rightarrow A$ (the generalization of the evaluation map) such that $p_\lambda f = f_\lambda$ for all $\lambda \in \Lambda$. We wish to determine under what conditions $f \in J$, the abstraction of the class of embeddings.

We call a family of morphisms $\{f_\lambda : A' \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ in a category an *M*-family (monic family) if, for every pair of morphisms g, g' with codomain A' , $f_\lambda g = f_\lambda g'$ for all λ implies $g = g'$. For example, the family of projections for a product is an *M*-family as is the family $\{b_1, b_2\}$ in the pullback diagram



It follows readily that in any category the morphism f under consideration is in *M* iff $\{f_\lambda \mid \lambda \in \Lambda\}$ is an *M*-family.

A family of morphisms $\{f_\lambda : A' \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ in a *JTK*-category is called a *J*-family if it is an *M*-family such that, whenever each f_λ can be written as $f_\lambda = g_\lambda t$ for all λ where $t \in T$, then $\{g_\lambda \mid \lambda \in \Lambda\}$ is an *M*-family and $t \in S$. This definition is just the generalization of that part of (T1) which defines *J*. We can then prove that the morphism f is in *J* iff $\{f_\lambda \mid \lambda \in \Lambda\}$ is a *J*-family. This is the desired generalization of the Embedding Lemma; note that our condition is in fact necessary and sufficient.

The derivation of the basic properties of *M*-families, *J*-families, and *L*-families (obvious definition) will be left to the reader. However, two other important properties concerning products should be pointed out. It is known, for a family of morphisms $\{f_\lambda : A'_\lambda \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ in a category where the products $\prod A_\lambda$ and $\prod A'_\lambda$ exist, that $\prod f_\lambda \in M$ if each

$f_\lambda \in M$. In the *JTK*-category it can be shown that $\bigcap_\lambda f_\lambda \in J$ if each $f_\lambda \in J$; in other words, if A_λ is a subobject of A'_λ for each λ , then $\bigcap_\lambda A_\lambda$ is a subobject of $\bigcap_\lambda A'_\lambda$. Another set-theoretic result which generalizes for the *JTK*-category is as follows. Suppose $\{j_\lambda : A_\lambda \rightarrow A'_\lambda \mid \lambda \in \Lambda\}$ is a family of morphisms in J in a *JTK*-category where the products $A = \prod_\lambda A_\lambda$ with projections $\{p_\lambda : A \rightarrow A_\lambda \mid \lambda \in \Lambda\}$ and $A' = \prod_\lambda A'_\lambda$ with projections $\{q_\lambda : A' \rightarrow A'_\lambda \mid \lambda \in \Lambda\}$ and the inverse images $q_\lambda^{-1}(A_\lambda)$ all exist. Then $\bigcap_\lambda j_\lambda : A \rightarrow A'$ is the intersection for the family $\{j_\lambda : q_\lambda(A_\lambda) \rightarrow A' \mid \lambda \in \Lambda\}$ and so we can write $A = \bigcap_\lambda A_\lambda = \bigcap q_\lambda^{-1}(A_\lambda)$.

6. *K*-projective and *J*-injective objects. An object P in a *JTK*-category will be called *K*-projective if for every diagram

$$\begin{array}{ccc} & P & \\ & \downarrow g & \\ A & \xrightarrow{k} & A' \end{array}$$

there is a morphism $b : P \rightarrow A$ which makes the diagram commute. An object with the dual property will be called *J*-injective. Obviously every projective object is *K*-projective and every injective object is *J*-injective. The converse is not true; for example, the object V in the semi-exact *JTK*-category \mathcal{U}_2 is *K*-projective but not projective.

The analogues of many properties of projective objects hold for *K*-projective objects in the *JTK*-category. For example, a retract of a *K*-projective objects is *K*-projective; the coproduct of *K*-projective objects is *K*-projective, and conversely, if the *JTK*-category has a zero object. Also we can show that, if the *JTK*-category is semi-exact and has *K*-projectives, then it has *K*-projective resolutions for each of its objects.

Defining the concepts of separator and generator much like in [8] except that only categorical, rather than bicategorical, notions are used, and using the concepts of basic free object and basic direct object in [8], we arrive at two versions of the Universality Theorem, one for any category and one for the *JTK*-category, analogous to that proved by Semadeni for the bicategory.

THEOREM. *Suppose the category \mathcal{C} has free objects and the basic free object F is projective. Then*

- 1.) Every free object is projective.
- 2.) For each object A in \mathcal{C} there is a cardinal number α and an epimorphism $e: {}^\alpha F \rightarrow A$.
- 3.) An object P is projective iff there is a retraction $r: A \rightarrow P$ where A is a free object.
- 4.) An object P is projective iff for each object A' every epimorphism $e: A' \rightarrow P$ is a retraction.

THEOREM. Suppose the JTK -category \mathcal{Q} has free objects and that the basic free object F is strict and K -projective. Then

- 1.) Every free object is K -projective.
- 2.) For each object A in \mathcal{Q} there is a cardinal number α and a morphism $k: {}^\alpha F \rightarrow A$ in K .
- 3.) An object P is K -projective iff there is a retraction $r: A \rightarrow P$ where A is a free object.
- 4.) An object P is K -projective iff for each object A' every $k: A' \rightarrow P$ is a retraction.

As a consequence, we see that in a JTK -category with free objects in which the basic free object is strict and projective the concepts of projectivity and K -projectivity coincide. The duals to these results should likewise be noted.

6. Examples. We'll now look at a few examples of JTK -categories; construction of various concepts in these JTK -categories are provided in [5].

The category \mathcal{S} of all sets and mappings is a JTK -category with $T=B=S=\bar{B}$, $J=L=M=\bar{M}$, and $K=N=E=\bar{E}$. Each morphism $f: A \rightarrow A'$ has the representation $f = jtk$ where $k: A \rightarrow \text{Coim } |f|$ is the projection onto the quotient set, $j: \text{Im } |f| \rightarrow A'$ is the inclusion map, and $t: \text{Coim } |f| \rightarrow \text{Im } |f|$ is the one-to-one onto mapping defined by $t([a]) = f(a)$ for each equivalence class $[a]$ in $\text{Coim } |f|$. Every set is projective and every nonempty set is injective; any one-element set is a basic free object and any two-element set is a basic direct object; every set is a free object and every set of cardinality 2^α is a direct object. Both versions of the Universality Theorem are applicable, and the notions of K -projectivity and J -injectivity coincide with those of projectivity and injectivity.

The category \mathcal{J} of topological spaces and continuous mappings becomes a JTK -category with $T = B = \overline{B}$, $L = M = \overline{M}$, $N = E = \overline{E}$, J the class of embeddings and K the class of quotient maps. Each morphism f has the "jtk" representation as in the preceding example, if $\text{Coim } |f|$ is given the quotient topology and $\text{Im } |f|$ the relative topology. An object in this category is projective iff it is discrete and injective iff it is nonempty and indiscrete; any single-element space is a basic free object and any two-element indiscrete space is a basic direct object. The first version of the Universality Theorem is applicable, but not the second, since the basic free object is not a strict generator; likewise for the duals. It can be shown that the notions of projectivity and K -projectivity coincide in this JTK -category; on the other hand, Sierpinski space gives an example of a J -injective object which is not injective.

The next example of a JTK -category is that constructed from the category of topological spaces with distinguished points and continuous mappings which preserve the distinguished points. The JTK -categorical structure is like that of the preceding example and the discussion of projectivity and injectivity is the same except that the basic free object is any two-point discrete space. Any single-point space is a zero object in this category. It is easily verified that this is a J -normal JTK -category (but not a normal category) with kernels and cokernels; it is not K -conormal.

The category \mathcal{G} of all abelian groups and group homomorphisms is an exact category and thus a balanced JTK -category in which the JTK -categorical classes of morphisms are just the corresponding categorical classes.

Next we consider the category \mathcal{P} of pathwise connected topological spaces with distinguished points and continuous mappings which preserve the distinguished points. In this category $E = \overline{E}$ and $\overline{M} \subseteq M$; for example, the orthogonal projection of a helix wound on a right circular cylinder in three-dimensional space onto the circular cross-section of the cylinder in the xy -plane is a monomorphism which is not one-to-one. To define a JTK -categorical structure on \mathcal{P} we use Proposition A, where assumption (A9 i.) is verified as follows: Suppose $m: (X, x_0) \rightarrow (Y, y_0)$ is a

monomorphism which is not one-to-one. Then there are points $x_1 \neq x_2$ in X with $m(x_1) = m(x_2)$ where say $x_1 \neq x_2$. We define a new topology on the set underlying the space X by taking those open sets in X which contain both x_1 and x_2 or neither of x_1 and x_2 and denote the resulting topological space by X' . Then (X', x_0) is a pathwise connected topological space with distinguished point x_0 . Thus $m = gf$ where $f: (X, x_0) \rightarrow (X', x_0)$, defined by $f(x) = x$, is in \bar{B} and $g: (X', x_0) \rightarrow (Y, y_0)$ is defined by $g(x) = m(x)$. Now g is a morphism which is not in M for if $h: (X, x_0) \rightarrow (X', x_0)$ is defined by $h(x) = x$ for $x \neq x_1$ and $h(x_1) = x_2$, then $h \neq f$ but $gh = gf$. Therefore, \mathcal{P} becomes a *JTK*-category with $T = \bar{B}$, $L = \bar{M}$, $N = E = \bar{E}$, J the class of embeddings and K the class of quotient maps. Note that it follows from (A9) that, since $E = \bar{E}$, \bar{M} is the class of pure monomorphisms in \mathcal{P} . This *JTK*-category is J -normal and has products, cokernels, coequalizers, cointersections and coimages.

A further example is the *JTK*-category obtained from the category \mathcal{N} of uniform spaces and uniformly continuous functions. In this category we have $M = \bar{M}$ and $\bar{E} \subseteq E$ since a uniform space (as in [4]) is Hausdorff in the uniform topology so that any morphism the image of which is dense in its codomain is epimorphic. Again we use Proposition A to obtain a *JTK*-categorical structure for \mathcal{N} . Using the uniform subspaces as the intuitive subobjects of a uniform space μX and the quotient spaces $\mu X/R$ where R is a uniform relation as the intuitive quotient objects, we easily check assumptions (A2, 3, 5) since the relation determining $\text{Coim } |f|$ is a uniform relation. The verification of (A9 ii.) proceeds as follows: Suppose $e: \mu X \rightarrow \nu Y$ is an epimorphism which is not onto; so $e(X) \subset Y$ and $X \neq \emptyset$. Then $\nu[e(X)]$ and $\nu[Y - e(X)]$ are uniform subspaces of νY . Let $\lambda Z = \nu[e(X)] \oplus \nu[Y - e(X)]$ be their coproduct (i.e., disjoint union); so a covering for λZ is uniform iff its inverse images under the injections $u_1: \nu[e(X)] \rightarrow \lambda Z$ and $u_2: \nu[Y - e(X)] \rightarrow \lambda Z$ are uniform coverings. Then $e = fb$ where $f: \lambda Z \rightarrow \nu Y$, defined by $f(z) = z$, is in \bar{B} and $b: \mu X \rightarrow \lambda Z$ is the morphism defined by $b(x) = e(x)$. Now b is not an epimorphism, for $f_1: \lambda Z \rightarrow \nu Y$, defined by $f_1(z) = z$ for $z \in e(X)$ and $f_1(z) = t$, where t is some fixed element in $e(X)$, for $z \in Y - e(X)$, is a morphism with $f_1 \neq f$ and

$f_1 b = fb$. Therefore, \mathcal{N} becomes a JTK -category with $T = B$, $L = M = \overline{M}$, $N = E$, J the class of embeddings and K the class of quotient maps. It follows from (A9) that E is the class of pure epimorphisms in \mathcal{N} as has already be noted by Isbell. It is easily seen that this JTK -category has equalizers, pullbacks, inverse images, finite intersections, and J -unions for every family of morphisms in J .

In a similar way we can show that the category \mathcal{I}_2 of Hausdorff spaces and continuous mappings is a JTK -category with $T = B$, $L = M = \overline{M}$, $N = E$, J the class of embeddings and K the class of quotient maps; once again E is the class of pure epimorphisms in \mathcal{I}_2 .

It is hoped that this introduction to the JTK -categories will stimulate interest in the subject. Many important questions remain to be answered. For example, is the JTK -categorical structure unique in any sense (if it exists) in a given category? Further investigation of semi-exact JTK -categories and the study of functors on JTK -categories are immediate areas of future research. The construction of many additional examples of JTK -categories will play an important role in this discussion.

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