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GERHARD OSIUS

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THE INTERNAL AND EXTERNAL ASPECT OF LOGIC AND SET THEORY IN ELEMENTARY TOPOI *

by Gerhard OSIUS

This paper is concerned with the logical and set-theoretical (rather than the geometrical) aspect of elementary topoi which were introduced by LAWVERE and TIERNEY in [7]. From the logical point of view an elementary topos may be considered as a generalization of the category of sets, and it is quite natural to ask which properties of the category of sets are shared by all elementary topoi. To answer this question (at least partially) let us imagine that objects of arbitrary topoi $\underline{E}$ have unspecified «elements» and allow ourselves to formulate statements about these «elements» in analogy to actual elements of sets. Formally this amounts to the introduction of a «set-theoretical» language $L(\underline{E})$ going back to MITCHELL [8]. The language $L(\underline{E})$ admits a natural «internal» interpretation in the topos $\underline{E}$ which gives rise to a notion of truth, called internal validity, for formulas of $L(\underline{E})$ (internal aspect, see [8]). Furthermore, if $\underline{E}$ is well-opened (i.e. the subobjects of 1 separate maps) one can give an external interpretation of $L(\underline{E})$ by interpreting the abstract «elements» of an object A as partial maps from 1 to A. This in turn gives rise to another notion of truth, called external validity, for formulas of $L(\underline{E})$ (external aspect). Since the internal aspect is developed in detail in [11], we concentrate in this paper on the external aspect and prove as our main result, that internal and external validity coincide if and only if the topos $\underline{E}$ is well-opened. An important application of the external aspect, namely generalizations of results in COLE [1], MITCHELL [8], OSIUS [9] concerning the construction of models for set theory whithin elementary topoi will be treated in a separate paper.

* Conférence donnée au Colloque d'Amiens 1973.

1. Preliminaries.

Throughout the whole paper we will work within a fixed elementary topos $\underline{E}$ or, from a formal point of view, within the elementary theory of elementary topoi. The basic results for elementary topoi which can be found in [2,3,9] are presupposed. To get some notations straight, let us mention a few facts playing an important role for our considerations. For an object A of $\underline{E}$ its *powerobject* is denoted $PA := \Omega^A$ and $A \overset{\eta_A}{\rightarrowtail} \tilde{A}$ denotes the *partial-map-classifier* for A. By a *subobject* of A we understand a map $A \to \Omega$ or sometimes - by a slight abuse of language - a monic map into A (resp. an equivalence class of such monos). The *characteristic map* $A \to \Omega$ of a monomorphism $B \overset{m}{\rightarrowtail} A$ will be denoted by $\chi(m)$.

The subobjects of A form a HEYTING-algebra with the operations $\cap$, $\cup$, $\Longrightarrow$, the partial ordering $\subset$ and greatest resp. smallest element 1_A resp. 0_A.

Any map $A \xrightarrow{f} B$ induces the operation of *inverse image under* f, denoted $f^{-1}(\cdot)$, from subobjects of B to those of A, and three operations from subobjects of A to those of B:

1° *direct existential image under* f, denoted $\exists f(\cdot)$,

2° *direct universal image under* f, denoted $\forall f(\cdot)$,

3° *direct unique-existential image under* f, denoted $\exists! f(\cdot)$.

Since the latter is not well known, let us define $\exists! f(M)$ for a subobject $A \xrightarrow{M} \Omega$: take a monic map $C \overset{m}{\rightarrowtail} A$ with $\chi(m) = M$, then $\exists! f(M)$ is the unique-existentiation part of

$$C \overset{m}{\rightarrowtail} A \xrightarrow{f} B,$$

i.e. the inverse image of $\chi(C \overset{\{\cdot\}}{\rightarrowtail} PC)$ under the map

$$B \overset{\{\cdot\}}{\rightarrowtail} PB \xrightarrow{\Omega^{mf}} PC$$

(cf. FREYD [2], Prop. 2.21). In fact, these operations $f^{-1}(\cdot)$, $\exists f(\cdot)$, $\forall f(\cdot)$, $\exists! f(\cdot)$ induce maps

$$PB \xrightarrow{\Omega^f} PA, \quad PA \xrightarrow{\exists f} PB, \quad PA \xrightarrow{\forall f} PB, \quad PA \xrightarrow{\exists! f} PB$$

representing the operations (on global sections).

Finally let us agree to drop indices and subscripts whenever no

confusion is possible.

2. The set-theoretical language $L(\underline{E})$ of $\underline{E}$.

One of the main tools to translate set-theoretical definitions and arguments involving elements and sets into the theory of elementary topoi is a set-theoretical language defined over (the theory of) topoi. Let us therefore start off with a description of the language $L(\underline{E})$ defined over our base topos $\underline{E}$ which is essentially due to MITCHELL [8].

The idea behind the language $L(\underline{E})$ is that we imagine the objects of the topos $\underline{E}$ to have unspecified «elements» (as if $\underline{E}$ were the topos of sets) in such a way that:

a) maps $A \xrightarrow{f} B$ induce actual operations from «elements» of A to those of B,

b) «elements» of a product $A \times B$ are ordered pairs of the «elements» of A and B,

c) 1 has an (unique) «element»,

d) subobjects $A \xrightarrow{M} \Omega$ induce unitary predicates $(\cdot) \in M$ for «elements» of A.

The formal definition of $L(\underline{E})$ runs as follows. $L(\underline{E})$ is a many-sorted first-order language having the objects of the topos $\underline{E}$ as «types» for the terms of $L(\underline{E})$, i.e. there is a type-operator τ which assigns to any term x of $L(\underline{E})$ an object τx of $\underline{E}$, called the *type of* x.

- The terms of $L(\underline{E})$ are given in the usual way by the following rules 2.1.1-4:

2.1.1. 0_e is a constant term of type 1.

2.1.2. For any object A of $\underline{E}$ there is a countable number of variables of type A,

2.1.3. For any map $A \xrightarrow{f} B$ in $\underline{E}$ there is an «evaluation-operator» $f(\cdot)$ from terms of type A to those of type B:

$$\frac{x \text{ of type } A}{fx \text{ of type } B} \, .$$

2.1.4. For any (ordered) pair (A, B) of objects of $\underline{E}$ there is an «ordered-pair-operator» $<-,->$:

$$\frac{x \text{ of type } A,\ y \text{ of type } B}{<-,-> \text{ of type } A \times B}\ .$$

For intuitive reasons let us now on call the terms of $L(\underline{E})$ simply *elements* (in $\underline{E}$) and for an element x and an object A let us write «$x \in A$» instead of «x is of type A» (i.e. $\tau x = A$).

The formulas of $L(\underline{E})$ are given as usual by the following rules 2.2.1-3:

2.2.1. For any subobject $A \xrightarrow{M} \Omega$ there is a unary «membership-predicate» $(\cdot) \in M$ for elements of A: $x \in M$ is an atomic formula provided $x \in A$.

2.2.2. The propositional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (alternation) and $\Longrightarrow$ (implication) are allowed for forming new formulas:

$$\frac{\phi,\ \psi \text{ formulas}}{\neg\phi,\ \phi\wedge\psi,\ \phi\vee\psi,\ \phi \Longrightarrow \psi \text{ formulas}}\ .$$

(Equivalence « $\Longleftrightarrow$ » is defined as usual.)

2.2.3. For any object A and any variable $x \in A$ the quantifiers $\exists x \in A$ (there exists an A-element) and $\forall x \in A$ (for all A-elements) are allowed to form new formulas:

$$\frac{\phi \text{ formula}}{(\exists x \in A)\,\phi,\ (\forall x \in A)\,\phi \text{ formulas}}$$

REMARKS. 1° If the formal point of view is adopted, then the language $L(\underline{E})$ can be constructed over the same alphabet as the theory of elementary topoi, for details see [11].

2° The introduction of the constant element $0_e \in 1$ (which was not mentioned in our abstract [10]) at this stage is useful but not necessary, since 0_e will turn out to be «definable».

3° One should clearly distinguish between the two usages of the symbol $\in$ in $x \in A$ (which is a metastatement) and $x \in M$ (which is a formula of $L(\underline{E})$).

Before introducing a notion of truth for formulas of $L(\underline{E})$ in the next section, let us give a few definitions.

2.3. For $x, y \in A$ we define equality:

$$x = y : \iff \langle x, y \rangle \in \Delta_A$$

where $\Delta_A : A \times A \to \Omega$ is the diagonal of A.

Using equality the unique-existence quantifier can be defined:

2.4. $(\exists! \, x \in A)\, \phi(x) : \iff (\exists x \in A)(\forall y \in A)(\phi(y) \iff x = y)$.

2.5. For $x \in A$ and $y \in PA$ the membership-relation is defined:

$$x \in y : \iff \langle y, x \rangle \in (PA \times A \xrightarrow{ev} \Omega).$$

2.6. For $x \in A$ and $F \in B^A$ we define the value of x under F:

$$Fx := (B^A \times A \xrightarrow{ev} B)\langle F, x \rangle,$$

hence $Fx \in B$.

2.7. For any map $A \xrightarrow{f} B$ in $\underline{E}$ with exponential adjoint $1 \xrightarrow{\bar{f}} B^A$ we define an element $f_e := \bar{f}(0_e) \in B^A$ which «represents» f internally. In particular we have for any $A \xrightarrow{M} \Omega$ an element $M_e \in PA$ and there are two elements true_e, false_e $\in \Omega$.

3. The internal interpretation of $L(\underline{E})$.

The construction of the language $L(\underline{E})$ guarantees that any map $A \xrightarrow{f} B$ induces an operation $f(\cdot)$ on elements and that any subobject $A \xrightarrow{M} \Omega$ induces a predicate $(\cdot) \in M$ for elements. The converse holds as well: any «definable operation» (i.e. a term) in $L(\underline{E})$ defines a map in $\underline{E}$ and any «definable property» (i.e. a formula) in $L(\underline{E})$ defines a subobject in $\underline{E}$.

First, let $t \in A$ be a term of $L(\underline{E})$ such that all (free) variables of t are among the variables $x_1 \in A_1, \dots, x_n \in A_n$. By induction on the length of t we define a map $A_1 \times \dots \times A_n \to A$, denoted

$$\{ < x_1 , \dots , x_n > \mapsto t \},$$

which represents the «operation» t:

3.1. $\{ < x_1 , \dots , x_n > \mapsto 0_e \}$ is the unique map $A_1 \times \dots \times A_n \to 1$.

3.2. $\{ < x_1 , \dots , x_n > \mapsto x_i \}$ is the projection $A_1 \times \dots \times A_n \to A_i$.

3.3. For any map $A \xrightarrow{f} B$:

$$\{ < x_1 , \dots , x_n > \mapsto f t \} := f \mid \{ < x_1 , \dots , x_n > \mapsto t \}.$$

3.4. For terms $t \in A$ and $s \in B$

$$\{ < x_1 , \dots , x_n > \mapsto < t , s > \}$$

is the unique map into $A \times B$ which is induced by the two maps

$$\{ < x_1 , \dots , x_n > \mapsto t \} \text{ and } \{ < x_1 , \dots , x_n > \mapsto s \}.$$

In particular we have:

3.5. If $x_1 , \dots , x_k$ are exactly the variables of t (i.e. $x_{k+1} , \dots , x_n$ do not occur in t) then: $\{ < x_1 , \dots , x_n > \mapsto t \} =$

$$\{ < x_1 , \dots , x_k > \mapsto t \} \mid \{ < x_1 , \dots , x_n > \mapsto < x_1 , \dots , x_k > \}.$$

Now let ϕ be a formula of $L(\underline{E})$ such that all free variables of ϕ are among $x_1 \in A_1 , \dots , x_n \in A_n$. Again by induction on the length of ϕ we define a subobject $A_1 \times \dots \times A_n \to \Omega$, denoted by $\{ < x_1 , \dots , x_n > \mid \phi \}$, which represents the «property» ϕ:

3.6. For any subobject $A \to \Omega$ and any $t \in A$ the subobject

$$\{ < x_1 , \dots , x_n > \mid t \in M \}$$

is the inverse image of M under the map

$$\{ < x_1 , \dots , x_n > \mapsto t \} : A_1 \times \dots \times A_n \to A.$$

In particular $\{ x \mid x \in M \} = M$.

3.7. $\{ < x_1 , \dots , x_n > \mid \neg \phi \} := \neg \{ < x_1 , \dots , x_n > \mid \phi \}$.

3.8. $\{ < x_1 , \dots , x_n > \mid \phi \wedge \psi \} := \{ < x_1 , \dots , x_n > \mid \phi \} \cap \{ < x_1 , \dots , x_n > \mid \psi \}$,

$\{ < x_1 , \dots , x_n > \mid \phi \vee \psi \} := \{ < x_1 , \dots , x_n > \mid \phi \} \cup \{ < x_1 , \dots , x_n > \mid \psi \}$,

$\{ < x_1 , \dots , x_n > \mid \phi \Rightarrow \psi \} := \{ < x_1 , \dots , x_n > \mid \phi \} \Rightarrow \{ < x_1 , \dots , x_n > \mid \psi \}$,

3.9. $\{<x_1,\ldots,x_n>|(\exists x \in A)\phi\} := (\exists pr)\{<x,x_1,\ldots,x_n>|\phi\}$,

$\{<x_1,\ldots,x_n>|(\forall x \in A)\phi\} := (\forall pr)\{<x,x_1,\ldots,x_n>|\phi\}$,

where pr is the projection $A\times A_1\times\ldots\times A_n \to A_1\times\ldots\times A_n$.

Concerning the quantifier $\exists!$ one can prove (see [11]):

3.10. $\{<x_1,\ldots,x_n>|(\exists! x \in A)\phi\} = (\exists! pr)\{<x,x_1,\ldots,x_n>|\phi\}$.

By induction one can establish the analogue of 3.5:

3.11. If $x_1,\ldots,x_k$ are exactly the free variables of ϕ $(k\leqslant n)$ then $\{<x_1,\ldots,x_n>|\phi\}$ is the inverse image of $\{<x_1,\ldots,x_k>|\phi\}$ under the projection

$$\{<x_1,\ldots,x_n>\mapsto<x_1,\ldots,x_k>\}.$$

Let us now give MITCHELL's internal interpretation of $L(\underline{E})$, which assigns to any formula ϕ of $L(\underline{E})$ a map $||\phi||$ in $\underline{E}$ (see [8]):

3.12. Let $x_1\in A_1,\ldots,x_n\in A_n$ be all distinct free variables of the formula ϕ in their natural order (of their first occurance in ϕ), then $||\phi||$ is defined as the map (subobject)

$$\{<x_1,\ldots,x_n>|\phi\}: A_1\times\ldots\times A_n\to\Omega.$$

In the same way one can define for any term t of $L(\underline{E})$ a map $||t||$ in $\underline{E}$.

Using the internal interpretation we introduce a notion of truth in $L(\underline{E})$: a formula ϕ is called *internally valid* (or simply: true), noted

$$||\phi|| = \text{true} \quad \text{or} \quad \vdash\phi,$$

iff $||\phi||$ factors through $1\xrightarrow{true}\Omega$. Among the various interesting properties of internal validity let us only state the most important ones without the proofs (most of them being straight-forward anyway, for details see [11]).

PROPOSITION. *The axioms and deductive rules of intuitionistic logic are internally valid:*

$1°$ $(\phi\Rightarrow\psi)\Rightarrow((\psi\Rightarrow\theta)\Rightarrow(\phi\Rightarrow\theta))$.

$2°$ $\phi\Rightarrow(\phi\vee\psi)$.

3° $\psi \Longrightarrow (\phi \vee \psi)$.

4° $(\phi \Longrightarrow \theta) \Longrightarrow ((\psi \Longrightarrow \theta) \Longrightarrow (\phi \vee \psi \Longrightarrow \theta))$.

5° $(\phi \wedge \psi) \Longrightarrow \phi$.

6° $(\phi \wedge \psi) \Longrightarrow \psi$.

7° $(\theta \Longrightarrow \phi) \Longrightarrow ((\theta \Longrightarrow \psi) \Longrightarrow (\theta \Longrightarrow \phi \wedge \psi))$

8° $(\phi \Longrightarrow (\psi \Longrightarrow \theta)) \Longrightarrow (\phi \wedge \psi \Longrightarrow \theta)$.

9° $(\phi \wedge \psi \Longrightarrow \theta) \Longrightarrow (\phi \Longrightarrow (\psi \Longrightarrow \theta))$.

10° $\phi \wedge \neg \phi \Longrightarrow \psi$.

11° $(\phi \Longrightarrow (\phi \wedge \neg \phi)) \Longrightarrow \neg \phi$.

12° $\phi \Longrightarrow (\exists x \in A)\phi$.

13° $(\forall x \in A)\phi \Longrightarrow \phi$.

14° *If* x *is not free in* ψ :

$$\frac{\phi \Longrightarrow \psi}{(\exists x \in A)\phi \Longrightarrow \psi} \qquad \frac{\psi \Longrightarrow \phi}{\psi \Longrightarrow (\forall x \in A)\phi}$$

15° *If* x *is a variable and* t *a term of same type:*

$$\frac{\phi(x)}{\phi(t)} \qquad (\textit{substitution}).$$

16° *If all free variables of* ϕ *are among those of* ψ :

$$\frac{\phi,\ \phi \Longrightarrow \psi}{\psi} \qquad (\textit{restricted modus ponens}). \qquad \blacksquare$$

It should be pointed out that the restriction of the modus ponens in 16 is essential, indeed we will see later that

$$x = x \quad \text{and} \quad x = x \Longrightarrow (\exists x \in A)\, x = x$$

are internally valid but $(\exists x \in A)\, x = x$ is not (for arbitrary A).

3.14. PROPOSITION. *The following axioms of equality are internally valid:*

1° $x=x$.

2° $x=y \Longrightarrow y=x$.

3° $x=y \wedge y=z \Longrightarrow x=z$.

4° $x=y \Longrightarrow fx=fy$ *(f being a map)*.

5° $x=u \wedge y=v \iff \langle x,y\rangle=\langle u,v\rangle$.

6° $x=y \Longrightarrow (x\in M \iff y\in M)$ *(M being a subobject)*.

7° *For identity maps id: $id(x)=x$.*

8° *For composable maps f, g: $g(fx)=(gf)x$.*

9° $pr_1\langle x,y\rangle=x$, $pr_2\langle x,y\rangle=y$, *where pr_1, pr_2 are the corresponding projections.*

10° *For elements $z\in A\times B$: $z=\langle pr_1 z, pr_2 z\rangle$.* ■

We remarked earlier that the constant element $0_e \in 1$ is «definable» namely because $(\exists!\ x\in 1)x=x$ and $0_e=0_e$ are internally valid. Returning to relationship between maps resp. subobjects in $\underline{E}$ and «definable» operations, resp. properties in $L(\underline{E})$ we note :

3.15. LEMMA. *1° For $A\xrightarrow{M}\Omega$, $A\xrightarrow{f}B$ and $x\in A$ the formulas*

$$x\in M \iff x\in M_e \quad \text{and} \quad fx=f_e(x)$$

are internally valid (making the index «e» superflous).

2° For a term t, resp. formula ϕ, of $L(\underline{E})$ with free variables among $x_1\in A, \dots, x_n\in A$ the formulas

$$\{\langle x_1,\dots,x_n\rangle \mapsto t\}(\langle x_1,\dots,x_n\rangle)=t$$

$$\langle x_1,\dots,x_n\rangle \in \{\langle x_1,\dots,x_n\rangle \mid \phi\} \iff \phi$$

are internally valid. ■

More interesting is a 1-1-correspondance between maps in $\underline{E}$ and functional relations in $L(\underline{E})$ observed by MITCHELL [8] :

3.16. PROPOSITION. *1° For any map $A\xrightarrow{f}B$ the formula*

$$(\forall x\in A)(\exists!\ y\in B)fx=y$$

is internally valid.

2° Let $\phi(x,y)$ be a formula with two free variables $x\in A$, $y\in B$.

If

$$(\forall x \in A)(\exists !\ y \in B)\,\phi(x,y)$$

is internally valid, then there exists a unique map $A \xrightarrow{f} B$ *such that*

$$(\forall x \in A)\ \phi(x, fx)$$

is internally valid. ■

As a consequence we note that internal unique-existence implies actual existence in $\underline{E}$:

3.17. COROLLARY. *Let* $\phi(x)$ *be a formula with one free variable* $x \in A$. *If* $(\exists !\ x \in A)\phi(x)$ *is internally valid, then there exists a unique global section* $1 \xrightarrow{a} A$ *such that* $\phi(a_e)$ *is internally valid.* ■

Furthermore we have some useful criterions

3.18. LEMMA. *1°* $A \xrightarrow{f} B = A \xrightarrow{g} B$ iff $(\forall x \in A) fx = gx$ *is internally valid.*

2° $A \xrightarrow{f} B$ *is monic, resp. epic, iff*

$$(\forall x, y \in A)(fx = fy \Longrightarrow x = y),\ \textit{resp.}\ (\forall z \in B)(\exists x \in A) fx = z$$

is internally valid. ■

The last results briefly indicate how the language $L(\underline{E})$, and hence set-theoretical arguments, can be used to establish results in the topos $\underline{E}$ (e.g. existence and equality of maps). Finally let us mention that the important axioms of (many-sorted) set theory are internally valid (for the proofs the reader is referred to [11]):

3.19. THEOREM. *The following axioms of many-sorted set theory are internally valid:*

1° Extensionality:

$$(\forall X, Y \in PA)((\forall z \in A)(z \in X \Longleftrightarrow z \in Y) \Longleftrightarrow X = Y).$$

2° Existence of empty sets:

$$(\exists !\ Y \in PA)(\forall z \in A) z \notin Y.$$

3° Existence of singletons:

$$(\forall x \in A)(\exists !\ Y \in PA)(\forall z \in A)(z \in Y \Longleftrightarrow z = x).$$

4° *Existence of binary and arbitrary unions :*

$$(\forall X, Y \in PA)(\exists!\ Z \in PA)(\forall z \in A)(z \in Z \Longleftrightarrow z \in X \vee z \in Y),$$

$$(\forall X \in PPA)(\exists!\ Z \in PA)(\forall z \in A)(z \in Z \Longleftrightarrow (\exists Y \in PA) z \in Y \in X).$$

5° *Existence of powersets :*

$$(\forall X \in PA)(\exists!\ Z \in PPA)(\forall Y \in PA)(Y \in Z \Longleftrightarrow (\forall y \in A)(y \in Y \Longrightarrow y \in X)).$$

6° *Separation axioms for formulas* ϕ *with one free variable* $z \in A$ *:*

$$(\exists!\ Z \in PA)(\forall z \in A)(z \in Z \Longleftrightarrow \phi). \quad ■$$

4. Well-opened topoi.

In this section we introduce well-opened topoi and state some of their properties. Our main application for these topoi will be the definition of an external interpretation of the language $L(\underline{E})$.

Let us recall from KOCK-WRAITH [4] that an object U of $\underline{E}$ is called *open* iff the unique map $U \to 1$ is monic, or equivalently iff for any object A there is at most one map $A \to U$. Open objects are closed under forming products and exponentials, and in particular $U \simeq U \times U$ for open U. A map $A \to B$ is called *open* iff its domain A is open (making the map monic). Generalizing FREYD's notion of well-pointed topoi in [2], let us call the topos $\underline{E}$ *well-opened* iff the open objects separate maps, i.e. the following axiom holds :

4.1. *(Open objects separate)* For any pair of distinct maps $A \xrightarrow{f} B \neq A \xrightarrow{g} B$ there exists an open map $U \to A$ separating f and g:

$$U \to A \xrightarrow{f} B \neq U \to A \xrightarrow{g} B.$$

An equivalent version of 4.1 is :

4.2. Any (monic) map $C \xrightarrow{h} A$ is epic if all open maps $U \to A$ factor through h.

To prove $4.1 \Longrightarrow 4.2$ take two maps f, g such that $fh = gh$ and conclude $f = g$ from 4.1. Conversely, for $4.2 \Longrightarrow 4.1$ apply 4.2 to the equalizer of f and g. ■

The following criterions will be needed later:

4.3. LEMMA *for well-opened* $\underline{E}$.

1° $A \xrightarrow{f} B$ *is monic iff for all open maps* $u, v : fu = fv \Longrightarrow u = v$.

2° *Let* $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ *be maps such that for all open maps* $U \xrightarrow{u} A$, fu *factors uniquely through* g. *Then* f *factors uniquely through* g.

PROOF. 1° To show that f is monic, take two maps g, h with $fg = fh$ and prove that their equalizer is iso (using 4.2).

2° From 4.2 we conclude that pulling g along f yields an iso, and the uniqueness of the factorization follows from 4.1. ■

Let us give some examples for well-opened topoi, first «internal» ones:

4.4. PROPOSITION *for well-opened* $\underline{E}$:

1° *For any object* A *the topos* $\underline{E}/A$ *defined over* A *is well-opened.*

2° *For any topology* $\Omega \xrightarrow{j} \Omega$ *the topos* $Sh_j(\underline{E})$ *of* j*-sheaves is well-opened.*

PROOF. 1° is straight forward, and to prove 2° we only observe that the reflector $\underline{E} \to Sh_j(\underline{E})$ preserves open objects since it preserves finite limits (see KOCK-WRAITH [4]). ■

4.5. EXAMPLES. Let $\underline{S}$ be the category of sets and $\underline{A}$ a small category. If $\underline{A}$ is a (partially) ordered set, then the topos $\underline{S}^{\underline{A}}$ is well-opened. Thus in particular the topos of set-valued presheaves resp. sheaves over a fixed topological space is well-opened (use 4.4.2). However if $\underline{A}$ is a non-trivial group, then the topos $\underline{S}^{\underline{A}}$ is not well-opened although - according to [2] - it is boolean and two-valued.

A further possible axiom for topoi - considered in [5,8] - is:

4.6. (*Support splits*) The epic part of any map $A \to 1$ splits.

«Support splits» is in fact equivalent to the converse of 4.2:

4.7. For any epic map $A \overset{f}{\twoheadrightarrow} B$ all open maps into B factor through f.

PROOF. «4.6 $\Longrightarrow$ 4.7»: Pulling f along an open map gives an epic which splits by 4.6. Conversely, for $A \overset{e}{\twoheadrightarrow} V \rightarrowtail 1$ the map $id : V \to V$ is open and

factors by 4.7 through e. ■

Note that if support splits in E, then internal existence in $L(\underline{E})$ implies actual existence in $\underline{E}$ (i.e. 3.17 holds if unique existence is replaced by simple existence). «Support splits» is a weak form of the axiom of choice for $\underline{E}$ (i.e. all epis split):

4.8. REMARK. The axiom of choice holds in $\underline{E}$ iff for all objects A of $\underline{E}$ support splits in $\underline{E}/A$. ■

4.9. PROPOSITION. *If $\underline{E}$ is boolean and support splits, then $\underline{E}$ is well-opened.*

PROOF. To prove 4.2, let $C \overset{h}{\rightarrowtail} A$ be a mono such that all open maps to A factor through h, and let $B \overset{g}{\rightarrowtail} A$ be the complement of h. Since $\underline{E}$ is boolean, h will be iso (resp. epi) if $B \simeq 0$. Now let $U \overset{u}{\rightarrowtail} B$ be a split of the epic part of $B \twoheadrightarrow U \rightarrowtail 1$. Then $U \overset{u}{\rightarrowtail} B \overset{g}{\rightarrow} A$ is open and factors through h and g which proves $U \simeq 0$ and hence $B \simeq 0$. ■

As to the converse of 4.9 let us exhibit a boolean well-opened topos in which support does not split. Take the topos $\underline{S}$ of sets for a model of set theory in which the axiom of choice fails (e.g. a FRAENKEL-MOSTOWSKI or a COHEN-model). Then by 4.8 there is a set A such that support does not split in $\underline{S}/A$, but $\underline{S}/A$ is of course boolean and well-opened (since $\underline{S}$ is).

To conclude this section we introduce the notion of support for objects A of $\underline{E}$: the characteristic map of the monic part of $A \to 1$ is called the *support of* A, denoted $1 \xrightarrow{Spt(A)} \Omega$. The existence of a map $A \xrightarrow{f} B$ implies $Spt(A) \subset Spt(B)$, and even $Spt(A) \doteq Spt(B)$ if f was epic. In terms of the internal interpretation we have

$$Spt(A) = \|(\exists x \in A) x = x\|.$$

We say that A has *full support* iff $Spt(A) = true$ i.e. $(\exists x \in A) x = x$ is internally valid.

4.10. PROPOSITION *for well-opened $\underline{E}$:*

$$Spt(A) = sup\ \{ Spt(U) \mid \text{there exists an open map } U \to A \}.$$

Here «sup» means an ordinary supremum in the (external) HEYTING-*algebra* $\underline{E}(1, \Omega)$ *of subobjects of* 1.

PROOF. $Spt(A)$ is clearly an upper bound for the family. For any upper bound $Spt(V)$, V open, of the family we wish to show the existence of a map $A \to V$, which implies $Spt(A) \subset Spt(V)$. For this it is sufficient to prove that pulling $V \rightarrowtail 1$ along $A \to 1$ gives an epic map $B \rightarrowtail A$ (using 4.2), which is then iso. Now, for any open map $U \rightarrowtail A$, the map $U \rightarrowtail A \to 1$ factors through $V \rightarrowtail 1$ since $Spt(V)$ was an upper bound. Hence $U \to A$ factors through $B \to A$.

$$\begin{tikzcd}
U \arrow[dr, dashed] \arrow[drr, dashed] \arrow[d, phantom] \arrow[dr, bend right=0, shift right=0, phantom] & & \\
& B \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "p.b."] & V \arrow[d, tail] \\
& A \arrow[r] & 1
\arrow[from=1-1, to=3-2]
\end{tikzcd}$$

■

Note that the suprema in 4.10 become maxima iff support splits in $\underline{E}$.

5. The external interpretation of $L(\underline{E})$.

Although the main results in this section require that $\underline{E}$ is well-opened we work as far as possible without this assumption. Our aim is to define an actual external interpretation of the language $L(\underline{E})$ within the topos $\underline{E}$, and we start as follows. For any object A the elements $x \in A$, i.e. terms of type A in $L(\underline{E})$, are interpreted as partial maps from 1 to A, called *A-elements* (short: *A-El*). Thus A-elements (already considered by MITCHELL [8]) generalize the notion of global sections $1 \to A$ which are the «natural» elements for well-pointed $\underline{E}$, as shown in [9].

5.1. A-elements may be viewed either as maps $1 \xrightarrow{a} \tilde{A}$ or as (equivalence classes of) open maps $U \overset{u}{\rightarrowtail} A$, which are related by the pullback:

$$\begin{tikzcd} U \arrow[r, tail, "u"] \arrow[d, tail] & A \arrow[d, tail, "\eta_A"] \\ 1 \arrow[r, "a"] & \tilde{A} \end{tikzcd}$$

The passage from u to its character a and conversely from a to an inverse image u under η_A will be denoted by $a = \tilde{u}$ and $u = \bar{a}$.

One further remark is necessary. Since equivalence classes of open maps are not an elementary notion, A-elements should be considered as maps $1 \to \tilde{A}$. However frequently the open maps are more flexible to handle. So let us view A-elements both ways and from the context it will always be clear which view is adopted. If it becomes necessary to distinguish A-elements from elements $x \in A$, we call the former *external* and the latter *internal elements.*

5.2. The constant $0_e \in 1$ is interpreted as $1 \xrightarrow{true} \Omega = \tilde{1}$ resp. $1 \xrightarrow{id} 1$, again denoted by 0_e.

5.3. For a map $A \xrightarrow{f} B$ the operation $f(\cdot)$ is interpreted by

$$f(1 \to \tilde{A}) := 1 \to \tilde{A} \xrightarrow{\tilde{f}} \tilde{B} \text{ resp. } f(U \rightarrowtail A) := U \rightarrowtail A \xrightarrow{f} B,\ U \text{ open.}$$

5.4. Ordered pairs are interpreted as follows:

$$< 1 \xrightarrow{a} \tilde{A},\, 1 \xrightarrow{b} \tilde{B} > := 1 \xrightarrow{(a,b)} \tilde{A} \times \tilde{B} \xrightarrow{h_{AB}} (A \times B)^{\sim},$$

$$\text{resp. } < U \overset{u}{\rightarrowtail} A,\, V \overset{v}{\rightarrowtail} B > := U \times V \overset{u \times v}{\rightarrowtail} A \times B,\ U, V \text{ open.}$$

Here h_{AB} is the character of the partial map $(\eta_A \times \eta_B; id)$.

5.5. The *support* of external elements is defined:

$$|U \rightarrowtail A| := Spt(U) \subset Spt(A),\ U \text{ open,}$$

$$\text{resp. } |1 \xrightarrow{a} A| := |\bar{a}| \subset Spt(A).$$

We note: $|fa| = |a|$, $|<a, b>| = |a| \cap |b|$.

For any object A we always have a unique A-element $0 \rightarrowtail A$ with minimal support $1 \xrightarrow{false} \Omega$. A-elements are called *full*, resp. *proper* iff their support is «true», resp. not «false». Hence the full A-elements are of the form $1 \to A \rightarrowtail \tilde{A}$, resp. $1 \simeq U \rightarrowtail A$ and are thus precisely the

global sections of A. Concerning the external interpretation of formulas of $L(\underline{E})$ let us first interprete the predicates of $L(\underline{E})$ in the following way.

5.6. For a subobject $A \xrightarrow{M} \Omega$ the predicate $(\cdot) \in M$ will be interpreted by assigning to any A-element $1 \xrightarrow{a} \tilde{A}$ a truth value in the HEYTING-algebra $\underline{E}(1, \Omega)$ of subobjects of 1.

$$|a \in M| := 1 \xrightarrow{a} \tilde{A} \xrightarrow{\tilde{M}} \Omega,$$

where $\tilde{M}$ is the existential image of M under $A \rightarrowtail \tilde{A}$. Equivalently, let $B \overset{m}{\rightarrowtail} A$ be a monic with $\chi(m) = M$ and $U \overset{u}{\rightarrowtail} A$ an open map, then

$$|u \in M| := |U \cap B \xrightarrow{u \cap m} A| = Spt(U \cap B) \subset |u|.$$

Note, that $|u \in M| = |u|$ holds iff u factors through m. In particular, $|u \in M| = true$ holds iff u is full and $1 \xrightarrow{u} A \xrightarrow{M} \Omega = true$.

According to definition 2.3 we have the interpretation of equality:

5.7. $$|1 \xrightarrow{a} \tilde{A} = 1 \xrightarrow{b} \tilde{A}| = |<a, b> \in \Delta_A| = |\chi(eq(a,b))| \cap |<a,b>|$$

where $eq(a,b)$ is an equalizer of a and b, resp.

$$|U \overset{u}{\rightarrowtail} A = V \overset{v}{\rightarrowtail} A| = |U \cap V \xrightarrow{u \cap v} A| = Spt(U \times V).$$

Note, that $|u = v| \subset |u| \cap |v|$ and equality holds iff

$$U \times V \rightarrowtail U \overset{u}{\rightarrowtail} A = U \times V \rightarrowtail V \overset{v}{\rightarrowtail} A.$$

In particular $|u = v| = true$ holds iff u, v are full and $u = v$.

5.8. PROPOSITION.

$1°$ $|a = a| = |a|$.

$2°$ $|a = b| = |b = a|$.

$3°$ $|a = b| \cap |b = c| \subset |a = c|$.

$4°$ $|a = b| \subset |fa = fb|$.

$5°$ $|a = c| \cap |b = d| = |<a, b> = <c, d>|$.

$6°$ $|a = b| \cap |a \in M| = |a = b| \cap |b \in M|$, *for subobjects* M.

$7°$ $|id(a) = a| = |a|$, *for identity maps* id.

$8°$ $|g(fa) = (gf)a| = |a|$, *for composable maps* f, g.

$9°$ $|pr_1 <a, b> = a| = |<a, b>| = |pr_2 <a, b> = b|$,

where pr_1, pr_2 *are the corresponding projections.*

10^o $|c = \langle pr_1 c, pr_2 c\rangle| = |c|$.

The straight-forward proof is omitted. ■

Let us now state the following easily established properties of external «membership»:

5.9. PROPOSITION. *For*

$$1 \xrightarrow{a} \tilde{A},\ 1 \xrightarrow{b} \tilde{B},\ A \xrightarrow{f} B,\ A \xrightarrow{M} \Omega,\ A \xrightarrow{L} \Omega,\ B \xrightarrow{N} \Omega:$$

1^o $|a \in 0_A| = false$.

2^o $|a \in 1_A| = |a|$.

3^o $|a \in \neg M| = |a| \cap \neg\, |a \in M|$.

4^o $|a \in M \cap N| = |a| \cap (\,|a \in M| \cap |a \in N|\,)$.

5^o $|a \in M \cup N| = |a| \cap (\,|a \in M| \cup |a \in N|\,)$.

6^o $|a \in M \Longrightarrow N| = |a| \cap (\,|a \in M| \Longrightarrow |a \in N|\,)$.

7^o $|a \in f^{-1}(N)| = |fa \in N|$.

8^o $|fa \in (\forall f)M| \subset |a \in M| \subset |fa \in (\exists f)M|$.

If f *is monic, the equality holds instead of* « $\subset$ ». ■

However, the most important properties of external membership require that $\underline{E}$ is well-opened:

5.10. THEOREM *for well-opened* $\underline{E}$. *For any*

$$A \xrightarrow{f} B,\ A \xrightarrow{M} \Omega,\ 1 \xrightarrow{b} \tilde{B}$$

we have:

1^o $|b \in (\exists f)M| = \sup\limits_{a\ A\text{-}El} (\,|fa = b| \cap |a \in M|\,)$.

2^o $(\,|b| \Longrightarrow |b \in (\forall f)M|\,) = \inf\limits_{a\ A\text{-}El} (\,|fa = b| \Longrightarrow |a \in M|\,)$, *hence*

$$|b \in (\forall f)M| = |b| \cap \inf_{a\ A\text{-}El} (\,|fa = b| \Longrightarrow |a \in M|\,).$$

Here «*sup*» *and* «*inf*» *mean ordinary suprema and infima in the (external)* HEYTING-*algebra* $\underline{E}(1, \Omega)$ *of subobjects of* 1. *Note that* 1 *and* 2 *still hold if the supremum and infimum is taken only for proper A-elements* a.

PROOF. We take $C \overset{m}{\rightarrowtail} A$, $V \overset{v}{\rightarrowtail} B$, V open, such that $M = \chi m$, $b = \tilde{v}$.

1° $|b \in (\exists f)M|$ is an upper bound by 5.9.8, 5.8.6. We pull the epi-mono-factorization of $C \overset{m}{\rightarrowtail} A \overset{f}{\to} B$ along $V \overset{v}{\rightarrowtail} B$ to get the diagram

$$\begin{tikzcd}
& X \arrow[rr, two heads] \arrow[dl] \arrow[dd, tail] & & V \cap D \arrow[r, tail] \arrow[dl] \arrow[dd, tail] & 1 \\
Y \arrow[rr] \arrow[dd, tail] & & V \arrow[dd, tail, "v"] & & \\
& C \arrow[rr, two heads] \arrow[dl, "m"'] & & D \arrow[dl, "(\exists f)m"] & \\
A \arrow[rr, "f"] & & B & &
\end{tikzcd}$$

Then

$$|b \in (\exists f)M| = Spt(V \cap D) = Spt(X) = \sup_{U \rightarrowtail X \text{ open}} Spt(U)$$

by 4.10. Now for any open map $U \rightarrowtail X$ we conclude for the composition

$$U \overset{u}{\rightarrowtail} A := U \rightarrowtail X \rightarrowtail C \overset{m}{\to} A$$

immediately

$$|fu = v| = Spt(U) = |u \in M|.$$

Hence for any upper bound $Spt(W)$ of the family, W open, we have

$$Spt(U) \subset Spt(W).$$

Thus

$$|b \in (\exists f)M| = Spt(X) \subset Spt(W).$$

2° We prove the first equation which implies the second by intersecting with $|b|$. To show that $|b| \Rightarrow |b \in (\forall f)M|$ is a lower bound, we have to establish

$$|fa = b| \cap (|b| \Rightarrow |b \in (\forall f)M|) \subset |a \in M|$$

which follows from 5.9.8, 5.8.6 since $|fa = b| \subset |b|$ implies that the left term is included in $|fa = b| \cap |b \in (\forall f)M|$. Now let $Spt(W)$, W open, be any lower bound. We wish to show

$$Spt(W) \subset (|b| \Rightarrow |b \in (\forall f)M|),$$

$$\text{resp.} \quad Spt(W \times V) = Spt(W) \cap |b| \subset |b \in (\forall f)M|.$$

Thus it is sufficient to show the existence of a map $W \times V \dashrightarrow E$ such that

$$
\begin{tikzcd}
W \times V \arrow[drr, dashed] \arrow[ddr] & & \\
& V \cap E \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\text{p.b.}"] & E \arrow[d, tail, "(\forall f)m"] \\
& V \arrow[r, tail, "v"'] & B
\end{tikzcd}
$$

commutes.

By adjointness this is equivalent to the existence of a map $Z \dashrightarrow C$ such that the following diagram commutes :

$$
\begin{tikzcd}
& C \arrow[dr, tail, "m"] & \\
Z \arrow[ur, dashed] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & Y \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & A \arrow[d, "f"] \\
W \times V \arrow[r, tail] & V \arrow[r, tail, "v"'] & B
\end{tikzcd}
$$

We use the criterion 4.3.2 to prove that such a map exists. Given any open map $U \rightarrowtail Z$ we have to show that the map

$$U \overset{u}{\rightarrowtail} A := U \rightarrowtail Z \rightarrowtail Y \rightarrowtail A$$

factors through m, i.e. $Spt(U) \subset |u \in M|$. Since $Spt(W)$ was a lower bound we have

$$Spt(W) \cap |fu = v| \subset |u \in M|,$$

and

$$|fu = v| = Spt(U) \subset Spt(W)$$

yields $Spt(U) \subset |u \in M|$. ■

Returning to the definition of the external interpretation (cf. 5.1-5.6) let us now give the interpretation for arbitrary formulas of $L(\underline{E})$:

5.11. For any formula $\phi(x_1, \dots, x_n)$ of $L(\underline{E})$ with the free variables $x_1 \in A_1, \dots, x_n \in A_n$ and any A_i-elements a_i, $i = 1, \dots, n$, we define the

external truth-value (extending 5.6)

$$|\phi(a_1,\dots,a_n)| := |<a_1,\dots,a_n> \in \{<x_1,\dots,x_n> \mid \phi(x_1,\dots,x_n)\}|.$$

Note that the definition is independent of the listed order of the variables resp. external elements.

For well-opened $\underline{E}$ one has the following important characterization of the external truth-values :

5.12. THEOREM *for well-opened* $\underline{E}$:

1° $|\neg\phi(a_1,\dots,a_n)| = |<a_1,\dots,a_n>| \cap \neg|\phi(a_1,\dots,a_n)|$

2° $|\phi(a_1,\dots,a_n) \Longrightarrow \psi(b_1,\dots,b_m)| =$

$|<a_1,\dots,a_n,b_1,\dots,b_m>| \cap (|\phi(a_1,\dots,a_n)| \Rightarrow |\psi(b_1,\dots,b_m)|)$

And similar if «$\Longrightarrow$» *is replaced by* «$\wedge$, $\cap$» *resp.* «$\vee$, $\cup$».

3° $|(\exists x \in A)\phi(x,a_1,\dots,a_n)| =$

$|<a_1,\dots,a_n>| \cap \sup_{a\ A\text{-}El}(|a| \cap |\phi(a,a_1,\dots,a_n)|).$

4° $|(\forall x \in A)\phi(x,a_1,\dots,a_n)| =$

$|<a_1,\dots,a_n>| \cap \inf_{a\ A\text{-}El}(|a| \Longrightarrow |\phi(a,a_1,\dots,a_n)|).$

The proof by induction on the length of the formula is a fairly straight-forward consequence of 5.9.3-6 , 5.10 using 5.7-8. ■

This theorem and definition 5.6 characterize the external truth-values. Hence, for well-opened $\underline{E}$, these truth-values could have been defined (by induction on the length) through 5.12, 1-4 and 5.6, thus slightly generalizing the standard procedure to lift truth-values from atomic to arbitrary formulas, as described in RASIOWA-SIKORSKI [12] (for type-free languages and *complete* algebras of truth-values). Therefore the external interpretation appears as an actual interpretation in the sense of [12].

For the external truth-values defined in 5.11 we always have

$$|\phi(a_1,\dots,a_n)| \subset |<a_1,\dots,a_n>| = |a_1 = a_1 \wedge \dots \wedge a_n = a_n|.$$

Now, the formula $\phi(x_1,\dots,x_n)$ is said to be *externally valid* iff for all

external elements $a_1, \dots, a_n$ the equation

$$| \phi(a_1, \dots, a_n) | = | < a_1, \dots, a_n > |$$

holds, or equivalently iff

$$| a_1 = a_1 \wedge \dots \wedge a_n = a_n | \Longrightarrow | \phi(a_1, \dots, a_n) |) = true.$$

From the definition 5.11 we deduce :

5.13. Internally valid formulas of $L(\underline{E})$ are externally valid. ■

Concerning the converse, we have as our main result:

5.14. THEOREM. *The notion of internal and external validity for formulas of* $L(\underline{E})$ *coincide if and only if* $\underline{E}$ *is well-opened.*

PROOF. Suppose $\underline{E}$ is well-opened. Let ϕ be an externally valid formula with free variables $x_1 \in A_1, \dots, x_n \in A_n$ (in their natural order) and let $B \overset{m}{\rightarrowtail} A_1 \times \dots \times A_n$ be a monic map with character

$$|| \phi || = \{ < x_1, \dots, x_n > | \phi \}.$$

Using 4.2 we show that m is epic and hence ϕ is internally valid. For any open map $U \overset{u}{\rightarrowtail} A_1 \times \dots \times A_n$ we put $u_i := pr_i\, u$ and have by 5.11 (since ϕ is externally valid):

$$| < u_1, \dots, u_n > \in || \phi || \; | = | \phi(u_1, \dots, u_n) | = | < u_1, \dots, u_n > |.$$

Hence $< u_1, \dots, u_n > \simeq u$ factors through m.

Conversely, suppose externally valid formulas are internally valid. To establish 4.2 let $C \overset{m}{\rightarrowtail} A$ be a monic map with $M := \chi m$, such that all open $U \overset{u}{\rightarrowtail} A$ factor through m, i.e. $| u \in M | = | u |$. Hence the formula $x \in M$ is externally and thus internally valid which proves $M = || x \in M || = true$, making m an iso. ■

The last theorem has interesting applications for well-opened $\underline{E}$, namely that internal validity can be replaced by external validity which often is easier to establish. To illustrate the general method let us give a simple example. A straight-forward argument gives (using 3.11):

5.15. For a relation $C \overset{r}{\rightarrowtail} A \times B$ with character $R := \chi r$ the formula

$$(\forall x \in A)(\exists !\, y \in B) < x, y > \in R$$

is externally valid iff for any open map $U \overset{u}{\rightarrowtail} A$ there exists a unique $U \overset{v}{\rightarrowtail} B$ such that $U \xrightarrow{(u,v)} A \times B$ factors through r. ■

Now, by 5.14-15 we conclude the external version of 3.16.2:

5.16. PROPOSITION *for well-opened $\underline{E}$. Let $C \overset{r}{\rightarrowtail} A \times B$ be a relation such that for all open $U \overset{u}{\rightarrowtail} A$ there exists a unique $U \overset{v}{\rightarrowtail} B$ so that*

$$U \xrightarrow{(u,v)} A \times B$$

factors through r. Then there exists a unique map $A \xrightarrow{f} B$ whose graph is the character of r. ■

Of course, 5.16 could as well be established straight-forward (without 5.14-15).

Finally let us briefly indicate another interesting application of the external interpretation.

First of all we note that for well-pointed $\underline{E}$ (i.e. $\underline{E}$ is well-opened and two-valued) the proper A-elements are full and hence global sections $1 \to A$. Furthermore $(1 \to A) \in (A \to \Omega)$ is externally valid iff

$$1 \to A \to \Omega = true .$$

Hence the external interpretation restricted to proper external elements gives for well-pointed $\underline{E}$ the usual interpretation of «elements» and «membership» considered in COLE [1], MITCHELL [8], OSIUS [9]. However since the powerful properties of the external interpretation can already be proved for well-opened $\underline{E}$ (i.e. without $\underline{E}$ being two-valued) it is natural trying to generalize the constructions for models of set theory (and the resulting characterizations of the category of sets) given in [1,8,9]. In fact, the method of identifying elements by using transitive-set-objects and inclusion maps between them (introduced in [9]) does work for external elements as well. Along this line one can construct an actual HEYTING-valued model for (a weak) set theory within well-opened topoi, and furthermore the categories of sets arising from HEYTING-valued models of set theory may be characterized as a certain type of

well-opened topoi (generalizing the corresponding result of MITCHELL [8] for boolean-valued models and boolean topoi in which support splits). A detailed exposition of these ideas will be given in a separate paper (see also [11]).

Fachsektion Mathematik
Universität Bremen
Achterstrasse
28 BREMEN R. F. A.

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