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# MORITA EQUIVALENCES OF ENRICHED CATEGORIES ★

*by Harald LINDNER*

## 0. Introduction and summary

The «Morita Theorems» treat the question of equivalences of two categories of modules. They give criteria for functors between two categories of modules to be adjoint, coadjoint, or an equivalence. A detailed treatment of these theorems is to be found in the book of H. Bass on *K*-theory (see also [20], 17.9.12). These results have been generalized by D.C. Newell by considering small preadditive categories instead of rings (= preadditive categories with precisely one object).

In the nonadditive version the categories of modules are replaced by functor categories $[\underline{A}, \underline{Ens}]$ and $[\underline{B}, \underline{Ens}]$ for small categories $\underline{A}$ and $\underline{B}$. This question has recently been studied by B. Elkins and J.A. Zilber ([6], to appear).

This paper is a generalization of these results to the case of equivalences of functor categories enriched over a bicomplete closed category $\underline{V}$ as defined in [3]. The «Morita Theorems» have been treated independently in [9] by J. Fisher-Palmquist and P.H. Palmquist (to appear) on the same level of generality. Contrary to [9], this paper also gives the following result on the equivalence of enriched functor categories: Let $\underline{V}$ be a bicomplete closed category. For every small $\underline{V}$-category $\underline{A}$ we define a $\underline{V}$-category $P\underline{A}$ such that the following holds: For any two small $\underline{V}$-categories $\underline{A}$ and $\underline{B}$ the $\underline{V}$-functor categories $[\underline{A}, \underline{V}]$ and $[\underline{B}, \underline{V}]$ are $\underline{V}$-equivalent iff $P\underline{A}$ and $P\underline{B}$ are $\underline{V}$-equivalent. For $\underline{V} = \underline{Ens}$ and for $\underline{V} = \underline{Ab}$ this result is well known and can be found in some of the papers cited above. For $\underline{V} = \underline{Ens}$ see also [1], IV 7.5, [14] and [10], 2.14 a. The fact that $P\underline{A}$ is not known to have a small skeleton causes some trou-

★ Conférence donnée au Colloque d'Amiens (1973)

ble in the proof of this theorem. For this reason and for further applications we introduce the notion of a *small* $\underline{V}$-functor, which generalizes a notion of Ulmer ( [22], 2.29).

In the first part of this paper we define the notion of a small $\underline{V}$-functor and prove a theorem on the $\underline{V}$-representability of $\underline{V}$-functors (1.9). In the second part we consider categories of small $\underline{V}$-functors and prove the existence of Kan (co-)extensions for $\underline{V}$-functors with a small domain (but not necessarily small codomain) (2.4). An important special case is the Yoneda embedding $H_* : \underline{B} \to [\underline{B}^o, \underline{V}]$ for a small $\underline{V}$-category $\underline{B}$. We prove generalizations of well known theorems ( [20], 17.3.1/2) on the Yoneda embedding (2.7 and 2.8). In the third part we apply the results of part one and two to the question of equivalences of $\underline{V}$-functor categories. The theorems 3.7, 3.10, and 3.11 are the announced generalizations of the «Morita Theorems». In part four we consider some examples. We refer particularly to 4.3 (topological modules) and to 4.4: in this case an application of the theory of part three yields a connection with the process of completing a metric space.

Henceforth let $\underline{V}$ denote a closed category as defined in [3]. The reader is assumed to be familiar with the theory of $\underline{V}$-categories as far as presented in [3]. Our notation agrees essentially with that of [3] and [20]. We follow a frequent abuse of notation by omitting the canonical isomorphism $(\underline{A} \otimes \underline{B}) \otimes \underline{C} \to \underline{A} \otimes (\underline{B} \otimes \underline{C})$ in the category of $\underline{V}$-categories (and sometimes in $\underline{V}$ as well). The proofs are either omitted or sketched. This paper is a brief version of the author's doctoral thesis, Düsseldorf 1973.

After this paper had been written, I learned about the existence of a related paper by F. W. Lawvere [24]. Furthermore, I have been informed by G. M. Kelly that the notion of a «small $\underline{V}$-functor» has also been considred by R. Street.

## 1. Small $\underline{V}$-functors

In this first part we introduce the notion of a «small $\underline{V}$-functor» (1.4). The first application is the theorem on the representability of $\underline{V}$-functors (1.9), a further important application is the theorem on Kan coextensions (2.4). In order to fix our notation we recall two definitions and one proposition.

1.1 DEFINITION. Let $U:\underline{B}\to\underline{C}$, $K:\underline{C}\to\underline{D}$, and $G:\underline{B}\to\underline{D}$ be $\underline{V}$-functors and let $k:G\to KU$ be a $\underline{V}$-natural transformation. $(K,k)$ is called a *Kan coextension of* $G$ *along* $U$ iff for every $\underline{V}$-functor $S:\underline{C}\to\underline{D}$ the mapping

$$[K,S]\to[G,SU]\quad(a\mapsto(a_*U)k)$$

is bijective.

We abbreviate «Kan extension» (and «Kan coextension») by «KE» (and «KCE»). Furthermore, we often call $K$ (instead of $(K,k)$) a KCE of $G$ along $U$.

1.2 DEFINITION. Let $T:\underline{B}^o\otimes\underline{C}\otimes\underline{B}\to\underline{D}$ and $K:\underline{C}\to\underline{D}$ be $\underline{V}$-functors and let

$$f=\{f_{B,C}:T(B,C,B)\to KC\mid B\in|\underline{B}|,C\in|\underline{C}|\}$$

be $\underline{V}$-natural. $(K,f)$ is called a *coend of* $T$ iff for every $\underline{V}$-functor $S:\underline{C}\to\underline{D}$ there is a bijection between (the set of) $\underline{V}$-natural transformations $g:K\to S$ and (the set of) $\underline{V}$-natural transformations $d:T\to S$, which is given by $d=gf$.

We often call $K$ (instead of $(K,f)$) a coend of $T$.

1.3 PROPOSITION. *Let* $\underline{D}$ *be a tensored* $\underline{V}$*-category and let* $G:\underline{B}\to\underline{D}$, $U:\underline{B}\to\underline{C}$ *and* $K:\underline{C}\to\underline{D}$ *be* $\underline{V}$*-functors. There is a canonical bijection between the sets of* $\underline{V}$*-natural transformations* $k:G\to KU$ *and*

$$f:Ten_{\underline{D}}\cdot(Hom_{\underline{C}}\otimes 1_{\underline{D}})\cdot(U^o\otimes 1_{\underline{C}}\otimes G)\to K.$$

*If* $k$ *and* $f$ *correspond to each other with respect to this bijection, then* $(K,k)$ *is a* KCE *of* $G$ *along* $U$ *iff:*

$$(K,f)\text{ is a coend of }Ten_{\underline{D}}\cdot(Hom_{\underline{C}}\otimes 1_{\underline{D}})\cdot(U^o\otimes 1_{\underline{C}}\otimes G).$$

We apply 1.3 mainly if $\underline{B}$ is small and $\underline{D}$ is $V$-cocomplete. In that case $(K, f)$ is a coend iff it is a pointwise coend (cf. [4], Prop. I.3.2).

The following definition is fundamental.

1.4 DEFINITION. A $\underline{V}$-functor $F: \underline{C} \to \underline{D}$ is called *small* iff there are a small $\underline{V}$-category $\underline{B}$, $\underline{V}$-functors $U: \underline{B} \to \underline{C}$ and $G: \underline{B} \to \underline{D}$, and a $\underline{V}$-natural transformation $f: G \to FU$ such that $(F, f)$ is a Kan coextension of $G$ along $U$.

1.5 REMARKS.

(a) If $\underline{D}$ is $\underline{V}$-cocomplete, one may require without loss of generality that $U$ in 1.4 is $\underline{V}$-fully faithful or a $\underline{V}$-full embedding (cf. 1.8).

(b) If $\underline{C}$ has a small skeleton, then every $\underline{V}$-functor with domain $\underline{C}$ is small. In that case one may choose $U$ as the inclusion of a skeleton of $\underline{C}$.

(c) Every $\underline{V}$-representable $\underline{V}$-functor is small. If $C \in |\underline{C}|$ is a representing object, one may choose $U$ as the inclusion into $\underline{C}$ of the $\underline{V}$-full $\underline{V}$-subcategory with the single object $C$.

(d) Let $\underline{A}$ be a tensored $\underline{V}$-category. Then every generalized $\underline{V}$-representable $\underline{V}$-functor $Ten_{\underline{A}}(-, A) \cdot H^C: \underline{C} \to \underline{A}$ is small. This is a consequence of (c), since $Ten_{\underline{A}}(-, A)$ preserves Kan coextensions (cf. [4], Prop. I.4.2).

The statement (c) is an immediate consequence of the $\underline{V}$-Yoneda lemma because of 1.3. However, 1.5 (c) also follows from proposition 1.6, which is mentioned here for a later application (3.6).

1.6 PROPOSITION. *Let $U: \underline{B} \to \underline{C}$ be the inclusion of a $\underline{V}$-full $\underline{V}$-subcategory with one object $B$. Let $C \in |\underline{C}|$ be a retract of $B$ (in the category $\underline{C}_0$, the underlying category of $\underline{C}$). Then $H^C: \underline{C} \to \underline{V}$ is a* KCE *of $H^C.U$ along $U$.*

In order to prove 1.6, one has to apply 1.3 and one has to see that, for every $X \in |\underline{C}|$,

$$(1) \qquad \mu^{C}_{C,B,X}: \underline{C}[B, X] \otimes \underline{C}[C, B] \to \underline{C}[C, X]$$

is a coend of the $\underline{V}$-functor

$$(2) \qquad \underline{B}^{o} \otimes \underline{B} \xrightarrow{U^{o} \otimes U} \underline{C}^{o} \otimes \underline{C} \xrightarrow{H_X \otimes H^C} \underline{V} \otimes \underline{V} \xrightarrow{Ten_{\underline{V}}} \underline{V}.$$

This is easily seen to be true by writing down the obvious (but large) diagram. The following proposition 1.7 is useful for the proof of 1.8.

1.7 PROPOSITION. *Let* $\underline{A} \xrightarrow{V} \underline{B} \xrightarrow{U} \underline{C} \xrightarrow{K} \underline{D}$, $\underline{A} \xrightarrow{H} \underline{D}$ *and* $\underline{B} \xrightarrow{G} \underline{D}$ *be* $\underline{V}$*-functors.*

*(a) Let* $g : H \to GV$ *be a* $\underline{V}$*-natural transformation such that* $(G, g)$ *is a* KCE *of* $H$ *along* $V$.

*(i) If* $k : G \to KU$ *is a* $\underline{V}$*-natural transformation such that* $(K, k)$ *is a* KCE *of* $G$ *along* $U$, *then* $(K, (k * V) g)$ *is a* KCE *of* $H$ *along* $UV$.

*(ii) If* $a : H \to KUV$ *is a* $\underline{V}$*-natural transformation such that* $(K, a)$ *is a* KCE *of* $H$ *along* $UV$, *then there is a unique* $k : G \to KU$ *such that* $(k * V) g = a$. *Furthermore,* $(K, k)$ *is a* KCE *of* $G$ *along* $U$.

*(b) Let* $\underline{D}$ *be* $\underline{V}$*-cocomplete and let* $\underline{A}$ *be small. Let* $U$ *be* $\underline{V}$*-fully faithful. If* $a : H \to KUV$ *is a* $\underline{V}$*-natural transformation such that* $(K, a)$ *is a* KCE *of* $H$ *along* $UV$, *then* $(KU, a)$ *is a* KCE *of* $H$ *along* $V$.

Part (a) is a purely formal consequence of the definition 1.1, whereas (b) can easily be proved by using 1.3.

1.8 COROLLARY. *Let* $\underline{D}$ *be a* $\underline{V}$*-cocomplete* $\underline{V}$*-category. A* $\underline{V}$*-functor* $F : \underline{C} \to \underline{D}$ *is small iff there is a small* $\underline{V}$*-full* $\underline{V}$*-subcategory* $\underline{B}$ *of* $\underline{C}$ *(let* $U : \underline{B} \to \underline{C}$ *denote the inclusion) such that* $F$ *is a* KCE *of* $FU$ *along* $U$.

As a first application of the notion of «small $\underline{V}$-functor» we state the theorem on the $\underline{V}$-representability of $\underline{V}$-functors (1.9). We call a $\underline{V}$-functor *$\underline{V}$-continuous* iff it preserves cotensor products and all small $\underline{V}$-limits. This notion corresponds to «small $\underline{V}$-continuous» in [4].

1.9 THEOREM *($\underline{V}$-representability of $\underline{V}$-functors). Let* $\underline{C}$ *be a* $\underline{V}$*-complete* $\underline{V}$*-category; let* $T : \underline{C} \to \underline{V}$ *be a* $\underline{V}$*-functor. The following are equivalent :*

*(i)* $T$ *is* $\underline{V}$*-continuous and small.*

*(ii)* $T$ *is* $\underline{V}$*-representable.*

We remark that a representing object for $T$ can be constructed as an end of the $\underline{V}$-functor (3), provided that $T$ is a KCE of $G$ along $U$.

$$(3) \qquad \underline{B}^{o} \otimes \underline{B} \xrightarrow{G^{o} \otimes U} \underline{V}^{o} \otimes \underline{C} \xrightarrow{Cot_{\underline{C}}} \underline{C} \ .$$

For $\underline{V} = \underline{Ens}$, 1.9 reduces to the well known ordinary representability theorem (cf. [20] 10.3.9). This follows from the following remark: Let $\underline{C}$ be a finitely complete (ordinary) category and assume that $T: \underline{C} \to \underline{Ens}$ preserves finite limits. Then $T$ is small iff $T$ is proper. (More precisely: Let $D$ be a subset of $|\underline{C}|$ and let $U: \underline{B} \to \underline{C}$ be the inclusion of the full subcategory $\underline{B}$ such that $|\underline{B}| = D$. $D$ is a dominating set for $T$ iff $T$ is a KCE of $TU$ along $U$.)

## 2. Categories of small $\underline{V}$-functors

In the following we often use the well known «Interchange Theorem for Coends» that is stated below. In its formulation we have used an evident generalization of 1.2.

2.1 PROPOSITION. *Let* $T: \underline{A}^o \otimes \underline{B}^o \otimes \underline{B} \otimes \underline{A} \to \underline{C}$, $P: \underline{A}^o \otimes \underline{A} \to \underline{C}$ *and* $Q: \underline{B}^o \otimes \underline{B} \to \underline{C}$ *be* $\underline{V}$*-functors. Let* $(P, p)$ *and* $(Q, q)$ *be pointwise coends of* $T$. *For every* $C \in |\underline{C}|$ *there is a bijection between (the sets of)* $\underline{V}$*-natural transformations* $p': P \to C$ *and* $q': Q \to C$, *given by* $p'p = q'q$. *Furthermore,* $p'$ *is a coend iff* $q'$ *is a coend.*

2.2 LEMMA. *Let* $\underline{V}$ *be complete and let* $\underline{D}$ *be a* $\underline{V}$*-cocomplete* $\underline{V}$*-category. If* $S, T: \underline{C} \to \underline{D}$ *are* $\underline{V}$*-functors such that* $S$ *is small, then the* $\underline{V}$*-functor* (4) *has an end:*

$$\underline{C}^o \otimes \underline{C} \xrightarrow{S^o \otimes T} \underline{D}^o \otimes \underline{D} \xrightarrow{Hom_{\underline{D}}} \underline{V}. \tag{4}$$

In order to prove 2.2 one can use 2.1 and the $\underline{V}$-Yoneda lemma.

2.3 COROLLARY. *Let* $\underline{V}$ *be complete. Let* $\underline{C}$ *and* $\underline{D}$ *be* $\underline{V}$*-categories such that* $\underline{D}$ *is* $\underline{V}$*-cocomplete. There is a* $\underline{V}$*-category* $[\underline{C}, \underline{D}]$ *together with an evaluation* $\underline{V}$*-functor* $E: [\underline{C}, \underline{D}] \otimes \underline{C} \to \underline{D}$ *such that:*

*(a) (i) the objects of* $[\underline{C}, \underline{D}]$ *are precisely the small* $\underline{V}$*-functors from* $\underline{C}$ *to* $\underline{D}$.

*(ii)* $E(T, \cdot) = T$ *for every* $T \in |[\underline{C}, \underline{D}]|$.

*(iii) the* $\underline{V}$*-functor* (4) *admits as an end the family*

$$\{E(\cdot, C)_{S,T}: [\underline{C}, \underline{D}][S, T] \to \underline{D}[SC, TC] \mid C \in |\underline{C}|\}.$$

*(b) for every $\underline{V}$-functor $G: \underline{X} \otimes \underline{C} \to \underline{D}$ such that $G(X, \cdot): \underline{C} \to \underline{D}$ is small for all $X \in |\underline{X}|$ there is a unique $\underline{V}$-functor $F: \underline{X} \to [\underline{C}, \underline{D}]$ such that the diagram (5) commutes.*

$$(5) \qquad \begin{tikzcd} \underline{X} \otimes \underline{C} \arrow[r, "F \otimes 1_{\underline{C}}"] \arrow[dr, "G"'] & {[\underline{C}, \underline{D}] \otimes \underline{C}} \arrow[d, "E"] \\ & \underline{D} \end{tikzcd}$$

If $\underline{C}$ is small, $[\underline{C}, \underline{D}]$ is the $\underline{V}$-category whose class of objects consists of all $\underline{V}$-functors from $\underline{C}$ to $\underline{D}$ (cf. [3], 4.1). However, the class of objects of $[\underline{C}, \underline{D}]$ is generally an element of a higher universe (i.e. $[\underline{C}, \underline{D}]$ is an illegitimate category), but one can find a skeleton of $[\underline{C}, \underline{D}]$ whose class of objects is contained in the same universe (cf. [22] 2.29).

Now let $U: \underline{B} \to \underline{C}$ be a $\underline{V}$-functor whose domain $\underline{B}$ is small. If $\underline{V}$ is complete and if $\underline{D}$ is $\underline{V}$-cocomplete then there is a unique $\underline{V}$-functor $\tilde{U} = [U, \underline{D}]: [\underline{C}, \underline{D}] \to [\underline{B}, \underline{D}]$ (composing with $U$) such that the diagram (6) commutes:

$$(6) \qquad \begin{tikzcd} {[\underline{C}, \underline{D}] \otimes \underline{B}} \arrow[r, "\tilde{U} \otimes 1_{\underline{B}}"] \arrow[d, "1_{[\underline{C}, \underline{D}]} \otimes U"'] & {[\underline{B}, \underline{D}] \otimes \underline{B}} \arrow[d, "E"] \\ {[\underline{C}, \underline{D}] \otimes \underline{C}} \arrow[r, "E"] & \underline{D} \end{tikzcd}$$

The $\underline{V}$-functor $\tilde{U}$ is known to be $\underline{V}$-adjoint provided that $\underline{C}$ is small (cf. [3], 6.1). The following theorem is a generalization to the case of arbitrary $\underline{V}$-categories $\underline{C}$.

2.4 THEOREM. *Let $U: \underline{B} \to \underline{C}$ be a $\underline{V}$-functor whose domain $\underline{B}$ is small. If $\underline{V}$ is complete and $\underline{D}$ is $\underline{V}$-cocomplete, the $\underline{V}$-functor $\tilde{U}: [\underline{C}, \underline{D}] \to [\underline{B}, \underline{D}]$ (cf. 2.3 and diagram (6)) has a $\underline{V}$-coadjoint $\underline{V}$-functor $R$ from $[\underline{B}, \underline{D}]$ to $[\underline{C}, \underline{D}]$. If $U$ is $\underline{V}$-fully faithful then so is $R$.*

We remark that for $G \in |[\underline{B}, \underline{D}]|$ and $C \in |\underline{C}|$, $(RG)C$ can be constructed as a coend $(RG)C = \int^{B} \underline{C}[UB, C] \otimes GB$.

2.5 THEOREM. *Let $\underline{V}$ be complete and let $\underline{D}$ be a $\underline{V}$-cocomplete $\underline{V}$-category. For every $\underline{V}$-category $\underline{C}$, the $\underline{V}$-category $[\underline{C}, \underline{D}]$ is $\underline{V}$-cocomplete. Furthermore, the $\underline{V}$-functor $Ten_{[\underline{C},\underline{D}]} : \underline{V} \otimes [\underline{C}, \underline{D}] \to [\underline{C}, \underline{D}]$ can be chosen in such a way that the diagram (7) commutes.*

$$(7) \qquad \begin{array}{ccc} \underline{V} \otimes [\underline{C}, \underline{D}] \otimes \underline{C} & \xrightarrow{Ten_{[\underline{C},\underline{D}]} \otimes 1_{\underline{C}}} & [\underline{C}, \underline{D}] \otimes \underline{C} \\ \downarrow{\scriptstyle 1 \otimes E} & & \downarrow{\scriptstyle E} \\ \underline{V} \otimes \underline{D} & \xrightarrow{Ten_{\underline{D}}} & \underline{D} \end{array}$$

In order to prove that every (ordinary) functor $M : \underline{Q} \to [\underline{C}, \underline{D}]_0$ with a small domain $\underline{Q}$ has a colimit, one proceeds as follows: for every $q \in |\underline{Q}|$ there is, according to 1.8, an inclusion $U_q : \underline{B}_q \to \underline{C}$ of a small $\underline{V}$-full $\underline{V}$-subcategory $\underline{B}_q$ of $\underline{C}$ such that $Mq$ is a KCE of $(Mq).U_q$ along $U_q$. If $U : \underline{B} \to \underline{C}$ denotes the inclusion of the $\underline{V}$-full $\underline{V}$-subcategory $\underline{B}$ of $\underline{C}$, defined by $|\underline{B}| := \bigcup_{q \in |\underline{Q}|} |\underline{B}_q|$, then $Mq$ is a KCE of $(Mq).U$ along $U$ according to 1.7. The assertion follows now from 2.4 and from the cocompleteness of $[\underline{B}, \underline{D}]$. The remaining statements in 2.5 can be verified quite easily.

An important special case of 2.4 can be obtained by taking $U = H_* : \underline{B} \to [\underline{B}^o, \underline{V}]$ for a small $\underline{V}$-category $\underline{B}$. This yields statement (a) in 2.7. The statements (b) - (d) of 2.7 can be found essentially in [3], 6.3. Theorem 2.7 is a literal generalization of theorem 17.3.1 in [20]. In order to be able to formulate 2.7 we need one more definition:

2.6 DEFINITION. Let $\underline{V}$ be complete and let $F : \underline{B} \to \underline{D}$ be a $\underline{V}$-functor with a small domain $\underline{B}$. We denote by $F^{\vee} : \underline{D} \to [\underline{B}^o, \underline{V}]$ the unique $\underline{V}$-functor making the diagram (8) commute:

$$(8) \qquad \begin{array}{ccc} \underline{D} \otimes \underline{B}^o & \xrightarrow{F^{\vee} \otimes 1_{\underline{B}^o}} & [\underline{B}^o, \underline{V}] \otimes \underline{B}^o \\ \downarrow{\scriptstyle 1_{\underline{D}} \otimes F^o} & & \downarrow{\scriptstyle E} \\ \underline{D} \otimes \underline{D}^o & \xrightarrow[Hom_{(\underline{D}^o)}]{} & \underline{V} \end{array}$$

If $\underline{V}$ is also complete (or if $\underline{D}$ has a small skeleton), there is the $\underline{V}$-category $[\underline{D}^o, \underline{V}]$ according to 2.3, together with the Yoneda embedding $H_* : \underline{D} \to [\underline{D}^o, \underline{V}]$ (cf. 1.5(c)). In that case, $F^V$ could also be defined as:

$$F^V := \underline{D} \xrightarrow{H_*} [\underline{D}^o, \underline{V}] \xrightarrow{\tilde{F}^o} [\underline{B}^o, \underline{V}] .$$

2.7 THEOREM. *Let $\underline{V}$ be bicomplete. Let $\underline{B}$ be small and $\underline{D}$ be a $\underline{V}$-cocomplete $\underline{V}$-category.*

*(a) The $\underline{V}$-functor $\tilde{H}_* : [[\underline{B}^o, \underline{V}], \underline{D}] \to [\underline{B}, \underline{D}]$ (composing with $H_* : \underline{B} \to [\underline{B}^o, \underline{V}]$, cf. diagram (6)) has a $\underline{V}$-coadjoint*

$$K : [\underline{B}, \underline{D}] \longrightarrow [[\underline{B}^o, \underline{V}], \underline{D}]$$

*which is $\underline{V}$-fully faithful.*

*(b) Let $F \in |[\underline{B}, \underline{D}]|$. Then $KF$ is $\underline{V}$-coadjoint to $F^V : \underline{D} \to [\underline{B}^o, \underline{V}]$ (cf. 2.6). Furthermore, $(KF).H_* \cong F$.*

*(c) A $\underline{V}$-functor $G : [\underline{B}^o, \underline{V}] \to \underline{D}$ is $V$-coadjoint iff $G$ is $\underline{V}$-cocontinuous. If this is the case, $(GH_*)^V$ is $\underline{V}$-adjoint to $G$.*

*(d) $K$ induces a $\underline{V}$-equivalence of $[\underline{B}, \underline{D}]$ with the $\underline{V}$-full $\underline{V}$-subcategory of $[[\underline{B}^o, \underline{V}], \underline{D}]$, the objects of which are the (equivalence classes of) $\underline{V}$-cocontinuous $\underline{V}$-functors.*

The following theorem is a literal generalization of theorem 17.3.2 in [20] (cf. [2] and [3], 7.3).

2.8 THEOREM (Characterization of the Yoneda embedding). *Let $\underline{V}$ be bicomplete. Let $U : \underline{B} \to \underline{C}$ be a $\underline{V}$-functor such that $\underline{B}$ is small and $\underline{C}$ is $\underline{V}$-cocomplete. The following are equivalent:*

*(a) There is a $\underline{V}$-equivalence $T : \underline{C} \to [\underline{B}^o, \underline{V}]$ such that $TU$ is $\underline{V}$-isomorphic to $H_* : \underline{B} \to [\underline{B}^o, \underline{V}]$.*

*(b) For every $\underline{V}$-functor $F : \underline{B} \to \underline{D}$ with a $\underline{V}$-cocomplete codomain $\underline{D}$ there is a $\underline{V}$-cocontinuous $V$-functor $G : \underline{C} \to \underline{D}$ such that $GU \cong F$, and any two such $\underline{V}$-functors are $\underline{V}$-isomorphic.*

*(c) $U^V : \underline{C} \to [\underline{B}^o, \underline{V}]$ is a $\underline{V}$-equivalence (cf. 2.6).*

*(d) $U$ is $\underline{V}$-dense, $U^V$ is $\underline{V}$-cocontinuous and $U^V . U$ is $\underline{V}$-isomorphic to $H_* : \underline{B} \to [\underline{B}^o, \underline{V}]$.*

*(e) $U$ is $\underline{V}$-fully faithful and $\underline{V}$-dense. Furthermore, for every $B \in |\underline{B}|$ the $\underline{V}$-functor $H^{UB} : \underline{C} \to \underline{V}$ is $\underline{V}$-cocontinuous.*

We call $U$ «$\underline{V}$-*prodense*» iff it has one (hence all) of the properties (a) - (e) of 2.8. This is a slight generalization of a definition in [9] (proof of 3.2), where $\underline{C}$ is required to be of the form $[\underline{A}, \underline{V}]$ for a small $\underline{V}$-category $\underline{A}$. The definition then applies to the $\underline{V}$-functor $\underline{B} \otimes \underline{A} \to \underline{V}$ that uniquely corresponds to $U$ by means of 2.3 (b). The proof of 2.7 and 2.8 is analogous to the proof in the special case $\underline{V} = \underline{Ens}$ (cf. [20], 17.3. 1/2). We proceed to generalize parts of the results of 2.7 and 2.8 to the case of not necessarily small $\underline{V}$-categories $\underline{B}$. In the case $\underline{V} = \underline{Ens}$ this result (existence of a free $\underline{V}$-cocomplete $\underline{V}$-category over every $\underline{V}$-category $\underline{C}$) is due to Ulmer ([22], 2.29). Before proving this generalization we make a definition and state a lemma.

2.9 DEFINITION. Let $\underline{V}$ be bicomplete. Let $\underline{B}$ be a small and $\underline{D}$ be a $\underline{V}$-cocomplete $\underline{V}$-category. Let $K$ denote the $\underline{V}$-coadjoint of

$$\tilde{H}_* : [\underline{B}, \underline{D}] \longrightarrow [[\underline{B}^o, \underline{V}], \underline{D}].$$

$\otimes_{\underline{B}}$ is defined by:

$$\begin{tikzcd}[\underline{B}^o, \underline{V}] \otimes [\underline{B}, \underline{D}] \arrow[rr, "\otimes_{\underline{B}}"] \arrow[d, "\cong"] & & \underline{D} \\ [\underline{B}, \underline{D}] \otimes [\underline{B}^o, \underline{V}] \arrow[rr, "K \otimes 1"] & & [[\underline{B}^o, \underline{V}], \underline{D}] \otimes [\underline{B}^o, \underline{V}] \arrow[u, "E"']\end{tikzcd} \tag{9}$$

For every $G \in |[\underline{B}, \underline{D}]|$, the $V$-functor $- \otimes_{\underline{B}} G$ is $\underline{V}$-coadjoint to $G^{\vee}$. If $F : \underline{B}^o \to \underline{V}$ is another $\underline{V}$-functor, then $F \otimes_{\underline{B}} G$ is a coend of the $\underline{V}$-functor

$$Ten_{\underline{V}} \cdot (F \otimes G) : \underline{B}^o \otimes \underline{B} \to \underline{V}.$$

For $\underline{V} = \underline{Ens}$, $\otimes_{\underline{B}}$ is the well known «tensorproduct over small categories» (cf. [20], 17.7). If $\underline{B} = \underline{I}$, $\otimes_{\underline{I}} \cong Ten_{\underline{D}}$.

2.10 LEMMA. Let $\underline{V}$ be bicomplete. Let $\underline{C}$ and $\underline{D}$ be $\underline{V}$-categories, $\underline{D}$ $\underline{V}$-cocomplete. For every $\underline{V}$-functor $F : \underline{C} \to \underline{D}$ there is a KCE

$$KF : [\underline{C}^o, \underline{V}] \longrightarrow \underline{D}$$

of $F$ along $H_* : \underline{C} \to [\underline{C}^o, \underline{V}]$. Let $\underline{B}$ be small and let $U : \underline{B} \to \underline{C}^o$ be a $\underline{V}$-functor. Then the diagram (10) is commutative up to $\underline{V}$-isomorphism ($R$ denotes the $\underline{V}$-coadjoint of $\tilde{U}$ (cf. diagram (6))).

$$(10)\qquad \begin{tikzcd} {[\underline{B},\underline{V}]} \arrow[d, "R"] \arrow[dr, "\cdot\otimes_{\underline{B}^o}(F.U^o)"] & \\ {[\underline{C}^o,\underline{V}]} \arrow[r, "KF"] & \underline{D} \end{tikzcd}$$

2.11 THEOREM. *Let $\underline{V}$ be bicomplete. For every $\underline{V}$-category $\underline{C}$, $[\underline{C}^o,\underline{V}]$ is the free $\underline{V}$-cocomplete $\underline{V}$-category over $\underline{C}$ by means of $H_*:\underline{C}\to[\underline{C}^o,\underline{V}]$, i.e.:*

*(a) $[\underline{C}^o,\underline{V}]$ is $\underline{V}$-cocomplete. For every $\underline{V}$-functor $F:\underline{C}\to\underline{D}$ such that $\underline{D}$ is V-cocomplete, there is a $\underline{V}$-cocontinuous $\underline{V}$-functor*

$$G:[\underline{C}^o,\underline{V}]\to\underline{D}\quad \text{such that}\quad GH_*\cong F.$$

*(b) Let $\underline{D}$ be $\underline{V}$-cocomplete. If $S:[\underline{C}^o,\underline{V}]\to\underline{D}$ is $\underline{V}$-cocontinuous, then $S$ is a* KCE *of $SH_*$ along $H_*$.*

## 3. $\underline{V}$-equivalence of $\underline{V}$-functor categories

Criteria are known for a $\underline{V}$-category to be $\underline{V}$-equivalent to $[\underline{B},\underline{V}]$ for a small $\underline{V}$-category $\underline{B}$ (cf. 2.8; [3], 7.3; [2]). For $\underline{V}=\underline{Ens}$ there are statements on the equivalence of $[\underline{A},\underline{Ens}]$ and $[\underline{B},\underline{Ens}]$ for small categories $\underline{A}$ and $\underline{B}$. There is such an equivalence iff the full subcategories of $[\underline{A},\underline{Ens}]$ and $[\underline{B},\underline{Ens}]$ of all retracts of representable functors are equivalent (cf. [1], IV 7.5; [10], 2.14 a; [14]). This result will be generalized as follows: For each small $\underline{V}$-category $\underline{A}$ we define a $\underline{V}$-category $P\underline{A}$ such that $[\underline{A},\underline{V}]$ and $[\underline{B},\underline{V}]$ are $\underline{V}$-equivalent iff $P\underline{A}$ and $P\underline{B}$ are $\underline{V}$-equivalent (3.7). For $\underline{V}=\underline{Ab}$ this means: Let $R$ and $S$ be two rings (or, more generally, two small preadditive categories). ${}_R\underline{mod}$ and ${}_S\underline{mod}$ are equivalent iff the corresponding full subcategories of all finitely generated projectives are equivalent. In this case more statements on the equivalence of categories of modules are known, the «Morita Theorems» (cf. [20], 17.9.12). These theorems are also generalized (3.10, 3.11) (cf. [9]). From now on we will assume $\underline{V}$ to be bicomplete. $\underline{V}$-equivalence will be denoted by «$\sim$».

3.1 DEFINITION. Let $\underline{A}$ be small. We denote by $P\underline{A}$ the $\underline{V}$-full $\underline{V}$-subca-

tegory of $[\underline{A}^o, \underline{V}]$ of all («atoms», cf. [2]) objects $T$ such that $H^T: [\underline{A}^o, \underline{V}] \to \underline{V}$ is $\underline{V}$-cocontinuous. For every $A \in |\underline{A}|$, the $\underline{V}$-functor $H_A$ is in $P\underline{A}$, because $H^{(H_A)} \cong E(-, A)$, and $E(-, A): [\underline{A}^o, \underline{V}] \to \underline{V}$ is $\underline{V}$-cocontinuous. Therefore $H_*: \underline{A} \to [\underline{A}^o, \underline{V}]$ factors through $P\underline{A} \hookrightarrow [\underline{A}^o, \underline{V}]$ by means of a uniquely determined $\underline{V}$-functor $J_{\underline{A}}: \underline{A} \to P\underline{A}$.

An obvious consequence of this definition is:

3.2 LEMMA. *Let $\underline{A}$ and $\underline{B}$ be small $\underline{V}$-categories. Then*

$$[\underline{A}^o, \underline{V}] \sim [\underline{B}^o, \underline{V}] \quad \textit{implies} \quad P\underline{A} \sim P\underline{B}.$$

We are now going to prove the converse of 3.2, and moreover that $[\underline{A}, \underline{D}]$ is $\underline{V}$-equivalent to $[\underline{B}, \underline{D}]$ for every $\underline{V}$-cocomplete $\underline{V}$-category $\underline{D}$ (cf. 3.5). We also derive a simple description of $P\underline{A}$ (3.4).

3.3 PROPOSITION. *Let $\underline{B}$ be small and let $U: \underline{B} \to \underline{C}$ be a $\underline{V}$-functor. The following are equivalent:*

*(a) $H^C: \underline{C} \to \underline{V}$ is a* KCE *of $H^C. U$ along $U$ for all $C \in |\underline{C}|$.*

*(b) If $\underline{D}$ is $\underline{V}$-cocomplete, every $\underline{V}$-functor $G: \underline{C} \to \underline{D}$ is a* KCE *of $GU$ along $U$.*

*(c) Let $\underline{D}$ be $\underline{V}$-cocomplete. If $l: T \to L$ is a coend of $T: \underline{C}^o \otimes \underline{C} \to \underline{D}$, then $l.(U^o \otimes U): T.(U^o \otimes U) \to L$ is a coend.*

In the proof of 3.3, the $\underline{V}$-Yoneda lemma, 1.3, and 2.1 are used.

3.4 PROPOSITION. *Let $\underline{A}$ be small and let $T: \underline{A}^o \to \underline{V}$ be a $\underline{V}$-functor. The following are equivalent:*

*(a) $T \in |P\underline{A}|$, i.e. $H^T: [\underline{A}^o, \underline{V}] \to \underline{V}$ is $\underline{V}$-cocontinuous.*

*(b) $H^T: [\underline{A}^o, \underline{V}] \to \underline{V}$ is a* KCE *of $H^T. H_*$ along $H_*: \underline{A} \to [\underline{A}^o, \underline{V}]$.*

*(c) Let $\underline{C}$ denote $[\underline{A}^o, \underline{V}]$. Then*

$$\{\underline{C}[H_A, T] \otimes \underline{C}[T, H_A] \xrightarrow{\mu\, \underline{C}} \underline{C}[T, T] \mid A \in |\underline{A}|\}$$

*is a coend of*

$$\underline{A}^o \otimes \underline{A} \xrightarrow{(H_*)^o \otimes H_*} [\underline{A}^o, \underline{V}]^o \otimes [\underline{A}^o, \underline{V}] \xrightarrow{H_T \otimes H^T} \underline{V} \otimes \underline{V} \xrightarrow{Ten_{\underline{V}}} \underline{V}.$$

3.5 COROLLARY. *Let $\underline{A}$ be small and let $\underline{D}$ be $\underline{V}$-cocomplete. Then $\tilde{J}_{\underline{A}}: [P\underline{A}, \underline{D}] \to [\underline{A}, \underline{D}]$ is a $\underline{V}$-equivalence.*

This follows immediately from 2.4, 3.3, and 3.4.

3.6 COROLLARY. *$P\underline{A}$ is closed with respect to retracts (in $[\underline{A}^o, \underline{V}]$).*

This is a consequence of 1.6 and 3.5.

3.7 COROLLARY. *Let $\underline{A}$ and $\underline{B}$ be small $\underline{V}$-categories. The following are equivalent:*

*(a) $[\underline{A}, \underline{D}] \sim [\underline{B}, \underline{D}]$ for every $\underline{V}$-cocomplete $\underline{V}$-category $\underline{D}$.*

*(b) $[\underline{A}^o, \underline{D}] \sim [\underline{B}^o, \underline{D}]$ for every $\underline{V}$-cocomplete $\underline{V}$-category $\underline{D}$.*

*(c) $P\underline{A} \sim P\underline{B}$.*

*(d) $P(\underline{A}^o) \sim P(\underline{B}^o)$.*

For $\underline{D} = \underline{V}$, (c) follows from (b) according to 3.2, whereas (c) $\Longrightarrow$ (a) follows from 3.5. The implications (a) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (b) are dual.

$P(\underline{A}^o)$ is known to be $\underline{V}$-equivalent to $(P\underline{A})^o$ if $\underline{V} = \underline{Ens}$ or if $\underline{V} = \underline{Ab}$. However, this equivalence is valid in general, and we shall prove this fact below (3.9). In order to do so we first define a certain pair of $\underline{V}$-adjoint functors. By restricting these adjoint functors to certain $\underline{V}$-subcategories we will obtain the desired $\underline{V}$-equivalence.

3.8 DEFINITION. Let $\underline{A}$ be a small $\underline{V}$-category. The KE (not KCE! ) of $H_* : \underline{A} \to [\underline{A}^o, \underline{V}]$ along $(H^*)^o : \underline{A} \to [\underline{A}, \underline{V}]^o$ will be denoted by $D_{\underline{A}}$. Furthermore, we introduce the abbreviations

$$* := D_{\underline{A}}, \quad + := (D_{(\underline{A}^o)})^o.$$

The functors $*$ and $+$ have been considered by Isbell [11, 12] and Lambek [16] in the special case $\underline{V} = \underline{Ens}$. By definition, the following formula holds for $F \in |[\underline{A}, \underline{V}]|$ and $A \in |\underline{A}|$: $F^*A = \int_B \underline{V}\,[FB, \underline{A}[A, B]]$.

The $\underline{V}$-functor $+$ is $\underline{V}$-coadjoint to $*$.

3.9 PROPOSITION. *Let $\underline{A}$ be a small $\underline{V}$-category. The $\underline{V}$-functors $*$ and $+$ (3.8) induce a $\underline{V}$-equivalence between $P\underline{A}$ and $(P(\underline{A}^o))^o$.*

We remark that it suffices to prove:

(i) $u_T : T \to T^{+*}$ ($u$ denotes the adjunction transformation) is $V$-isomorphic for all $T \in |P\underline{A}|$.

(ii) $T \in |P\underline{A}| \Longrightarrow T^+ \in |P(\underline{A}^o)|$.

Now we state the generalization of the «Morita Theorems» (3.10 and 3.11). Closely related results can be found in [9]. For the interpretation of these theorems in the case $\underline{V} = \underline{Ab}$ the reader is referred to [20], 17.9.12 and to the book of H. Bass on $K$-theory.

3.10 PROPOSITION. *Let $\underline{A}$ and $\underline{B}$ be small $\underline{V}$-categories and suppose that $[A^o, V]$ and $[B^o, V]$ are $\underline{V}$-equivalent. Then the $\underline{V}$-centers of $\underline{A}$ and $\underline{B}$ are $\underline{V}$-isomorphic (as monoids in $\underline{V}$).*

We recall the definition of the $\underline{V}$-center of a $\underline{V}$-category as the end of $Hom_{\underline{C}}: \underline{C}^o \otimes \underline{C} \to \underline{V}$, together with its structure of a commutative monoid in $\underline{V}$, induced by the $\underline{V}$-category structure of $\underline{C}$. The proof of 3.10 uses the fact that every $\underline{V}$-dense and $\underline{V}$-fully faithful $\underline{V}$-functor $U: \underline{B} \to \underline{C}$ induces a $\underline{V}$-isomorphism between the $\underline{V}$-centers of $\underline{B}$ and $\underline{C}$.

Let $\underline{X}$, $\underline{Y}$ be small and let $\underline{Z}$ be arbitrary. We denote the canonical $\underline{V}$-isomorphism $[\underline{X}, [\underline{Y}, \underline{Z}]] \to [\underline{Y}, [\underline{X}, \underline{Z}]]$ by «$\wedge$» $(F \mapsto \hat{F})$.

3.11 THEOREM. *Let $\underline{A}$ and $\underline{B}$ be small $\underline{V}$-categories.*

*(a) If $T: \underline{C} \to [\underline{B}^o, \underline{V}]$ is $\underline{V}$-adjoint then there is a $\underline{V}$-functor $F: \underline{B} \to \underline{C}$ such that $T \cong F^\vee$ (cf. 2.6).*

*(b) Let $\underline{D}$ be $\underline{V}$-cocomplete and let $T: [\underline{A}^o, \underline{V}] \to \underline{D}$ be $\underline{V}$-cocontinuous. Then there is a $\underline{V}$-functor $G: \underline{A} \to \underline{D}$ such that $T \cong (- \otimes_{\underline{A}} G)$.*

*(c) If $T: [\underline{A}^o, \underline{V}] \to [\underline{B}^o, \underline{V}]$ is $\underline{V}$-adjoint, $\underline{V}$-cocontinuous, and $\underline{V}$-fully faithful then $\otimes_{\underline{B}} . (G \otimes \hat{F}) \cong Hom_{(\underline{A}^o)}$.*

*(d) Let $T: [\underline{A}^o, \underline{V}] \to [\underline{B}^o, \underline{V}]$ be a $\underline{V}$-equivalence.*

*(i) $(- \otimes_{\underline{B}} F)$ is an equivalence inverse of $(- \otimes_{\underline{A}} G)$ and of $F^\vee$. Furthermore,*

$$\otimes_{\underline{A}^o} . (\hat{G} \otimes F) \cong Hom_{\underline{B}}, \quad F^\vee H_* \cong G, \quad G^\vee H_* \cong F.$$

*(ii) $(- \otimes_{\underline{A}^o} \hat{F}): [\underline{A}, \underline{V}] \to [\underline{B}, \underline{V}]$ is a $\underline{V}$-equivalence with inverses $(- \otimes_{(\underline{B}^o)} \hat{G})$ and $(\hat{F})^\vee$. Furthermore,*

$$(\hat{F})^\vee . H^* \cong \hat{G} \quad \text{and} \quad (\hat{G})^\vee . H^* \cong \hat{F}.$$

*(e) Let $G: \underline{A} \to [\underline{B}^o, \underline{V}]$ be a $\underline{V}$-functor. $G^\vee: [\underline{B}^o, \underline{V}] \to [\underline{A}^o, \underline{V}]$ is a $\underline{V}$-equivalence iff $G$ is $\underline{V}$-prodense (cf. remark after 2.8). If this is true, then $\hat{G}: \underline{B}^o \to [\underline{A}, \underline{V}]$ is $\underline{V}$-prodense and $(\hat{G})^\vee: [\underline{A}, \underline{V}] \to [\underline{B}, \underline{V}]$*

*is a $\underline{V}$-equivalence.*

If $\underline{V} = \underline{Ens}$ or if $\underline{V} = \underline{Ab}$, the condition «$T$ is $\underline{V}$-adjoint» in 3.11 (a) may be weakened to «$T$ is $\underline{V}$-continuous», provided that $\underline{C} \cong [\underline{A}^o, \underline{V}]$ for a small $\underline{V}$-category $\underline{A}$. In general, this need not be possible:

3.12 PROPOSITION. *The following are equivalent:*

*(a) Every $\underline{V}$-continuous $\underline{V}$-functor $S : \underline{V} \to \underline{V}$ is $\underline{V}$-representable.*

*(b) Let $\underline{A}$ be a small $\underline{V}$-category. Every $\underline{V}$-continuous $\underline{V}$-functor $T$: $[\underline{A}, \underline{V}] \to \underline{V}$ is $\underline{V}$-representable.*

*(c) Let $\underline{A}$ be a small $\underline{V}$-category. Every $\underline{V}$-continuous $\underline{V}$-functor $T$ whose domain is $[\underline{A}, \underline{V}]$ is $\underline{V}$-adjoint.*

A sufficient condition for each of the three equivalent statements in 3.12 to hold is the following: $\underline{V}$ is well-powered and has a cogenerator.

## 4. Examples

4.1. $\underline{V} = \underline{Ens}$. $\underline{V}$-categories are (in bijection with) ordinary categories. If $\underline{A}$ is a small category, $P\underline{A}$ consists of the closure of $\underline{A}$ in $[\underline{A}^o, \underline{Ens}]$ with respect to retracts (cf. [1], IV 7.5; [6]; [10], 2.14 a). In [14], categories with precisely one object, i.e. monoids, are considered.

4.2. $\underline{V} = \underline{Ab}$. $\underline{V}$-categories ($\underline{V}$-functors) are preadditive categories (- functors). If $\underline{A}$ is a small preadditive category, $P\underline{A}$ consists of the closure of $\underline{A}$ in $[\underline{A}^o, \underline{Ab}]$ with respect to retracts of finite coproducts. If $\underline{A}$ is an $\underline{Ab}$-category wiht precisely one object, i.e. $\underline{A}$ is a ring, then $[\underline{A}^o, \underline{Ab}]$ is the category of all $\underline{A}$-right modules. The objects of $P\underline{A}$ are precisely the finitely generated projective $\underline{A}$-right modules. For the interpretation of 3.10 and 3.11 we refer to [20], 17.9.12.

4.3. Every cartesian closed, bicomplete category $\underline{V}$ is an example for the results of part 3. According to A.Kock, [15], and another (unpublished) manuscript of A. Kock, Halifax 1970, for every such category $\underline{V}$ there is another bicomplete closed category $\underline{Ab}(\underline{V})$, the category of

abelian group objects in $\underline{V}$. For example, $\underline{Ab}$ is the category of abelian group objects in the bicomplete, cartesian closed category $\underline{Ens}$. We mention as further examples:

a) Let $\underline{V}$ be a topos in the sense of A. Grothendieck [1], or let $\underline{V}$ be a bicomplete topos in the sense of F.W. Lawvere and M. Tierney. Then $\underline{Ab}(\underline{V})$ is a bicomplete closed category (cf. [1], II-6.7 and IV-12.8).

b) Let $\underline{V}$ be the category $\underline{CG}$ of compactly generated topological spaces or the category $\underline{Ke}$ of compactly generated topological Hausdorff spaces, i.e. Kelley spaces. Then $\underline{Ab}(\underline{V})$ is a bicomplete closed category (cf. [7]).

Let $\underline{V}$ be a cartesian closed category, which is also a *Top*-category over $\underline{Ens}$ by means of a topological functor $T:\underline{V}\to\underline{Ens}$ in the sense of Wyler (cf. [7]). This includes the two cases $\underline{V}=\underline{CG}$ and $\underline{V}=\underline{Ke}$. Let $\underline{A}$ be a small $\underline{V}$-category. We claim that $P\underline{A}$ consists of all retracts of $\underline{V}$-representables. This can easily be derived from the fact that $T$ carries the structure of a monoidal functor in a canonical way and from the fact that $T$ is cocontinuous, using 3.4 and 3.6.

We remark finally that the category $\underline{Ab}(\underline{V})$ may be trivial for a cartesian closed, bicomplete category $\underline{V}$ (cf. 4.5).

4.4. Let $\underline{V}$ be defined as follows: the objects of $\underline{V}$ are all non-negative real numbers and $\infty$. For all $a, b\in|\underline{V}|$ there is at most one morphism from $a$ to $b$, and $hom(a,b)\neq\emptyset$ iff $a\geqslant b$. The following definitions define on $\underline{V}$ the structure of a closed category (*):

$$I:=0,\quad a\otimes b:=a+b \qquad \text{and} \qquad \underline{V}[a,b]:=\begin{cases} b-a & \text{for } a<b \\ 0 & \text{for } a\geqslant b. \end{cases}$$

$\underline{V}$ is bicomplete. Small $\underline{V}$-categories are semiquasimetric spaces, i.e. pairs $(X,d)$ such that $X$ is a set and $d:X\times X\to[0,\infty]$ is a mapping such that:

(i) $d(x,x)=0$ for all $x\in X$,

(ii) $d(x,z)\leqslant d(x,y)+d(y,z)$ for all $x,y,z\in X$.

(*) Metric spaces as $V$-based categories have been considered by F.W. Lawvere, who lectured on that topic in Brunswick (U.S.A) and Halifax (Canada) in 1969 (see also [24]).

$\underline{V}$-functors $f$ from $(X,d)$ to $(X',d')$ are all nonexpansive mappings $f: X \to X'$, i.e. mappings $f$ such that (iii) holds :

(iii) $d'(fx,fy) \leqslant d(x,y)$ for all $x,y \in X$.

We denote by ${}^{sq}\underline{Met}$ the category of small $\underline{V}$-categories and $\underline{V}$-functors. The full subcategory ${}^{q}\underline{Met}$ of all quasimetric spaces, i.e. *commutative* semiquasimetric spaces $(X,d)$ (such that $d(x,y)=d(y,x)$ for all $x,y \in X$), is bireflective in ${}^{sq}\underline{Met}$. The coadjoint $L$ for the inclusion functor ${}^{q}\underline{Met} \to {}^{sq}\underline{Met}$ is given by

$$L(X,d):=(X,\bar{d}), \text{ such that } \bar{d}(x,y):=max(d(x,y),d(y,x)).$$

Let $(X,d)$ be a semiquasimetric space. The distance $s$ on $P(X,d)$ is given by (11):

$$(11) \qquad s(f,g)=\int_x \underline{V}\,[fx,gx]=\sup_{x\in X}\underline{V}\,[fx,gx].$$

In general, $P(X,d)$ is not commutative even if $(X,d)$ is commutative. However, $P(X,d)$ is always separated, i.e. $s(f,g)=s(g,f)=0$ implies $f=g$. By 3.4, $f\in P(X,d)$ is equivalent to (12):

$$(12) \qquad \inf_{x\in X}(fx+\sup_{y\in X}\underline{V}\,[fy,d(y,x)])=0.$$

For every semiquasimetric space $(X,d)$, $P(X,d)$ together with the embedding $J_{(X,d)}:(X,d)\to P(X,d)$ (cf. 3.1) is the completion of $(X,d)$ in the following sense (we abbreviate $J_{(X,d)}$ by «$i$»):

THEOREM. *Let $(X,d)$ be a semiquasimetric space.*

*a) $i:(X,d)\to P(X,d)$ is an isometry, i.e. $s(ix,iy)=d(x,y)$ for all $x,y\in X$. Furthermore, $ix=iy$ iff $d(x,y)=d(y,x)=0$ for all $x,y\in X$.*

*b) $i:(X,d)\to P(X,d)$ is dense, i.e. for every $f\in P(X,d)$ there is a sequence $x_n\in X$ such that $f=\lim_{n\to\infty} ix_n$*

*c) $P(X,d)$ is complete, i.e. every Cauchy sequence in $P(X,d)$ has a limit in $P(X,d)$.*

If $(X,d)$ is a metric space, $L(P(X,d))$ is the completion in the usual sense of the metric space $(X,d)$.

4.5. Let $M$ be a set. Let $\underline{V}$ be the category whose objects are the subsets of $M$ and whose morphisms are the inclusion mappings of the-

se subsets. The definitions

$$I := M,\quad A \otimes B := A \cap B \quad \text{and} \quad \underline{V}[A, B] := (M \setminus A) \cup B$$

define on $\underline{V}$ the structure of a cartesian closed category. $\underline{V}$ is bicomplete.

a) $M = \emptyset$, i.e. $\underline{V} = \underline{1}$. A $\underline{V}$-category is an ordinary category $\underline{A}$ such that every object of $\underline{A}$ is a zero object. If $\underline{A}$ is any small $\underline{V}$-category, then $P\underline{A} = [\underline{A}, \underline{V}] = \underline{1}$. Two $\underline{V}$-categories $\underline{A}$, $\underline{B}$ are $\underline{V}$-equivalent iff either both are empty (i.e. $|\underline{A}| = |\underline{B}| = \emptyset$) or both are nonempty.

b) $M = \{\emptyset\}$, hence $\underline{V} = \underline{2}$. $\underline{V}$-categories are partially preordered classes. $\underline{V}$-functors are orderpreserving mappings. $P\underline{A}$ is $\underline{V}$-equivalent to $\underline{A}$ for every small $\underline{V}$-category $\underline{A}$.

c) Let $M$ be arbitrary. A $\underline{V}$-category $\underline{A}$ consists of a class $|\underline{A}|$, and for all $X, Y \in |\underline{A}|$ a subset $\underline{A}[X, Y]$ of $M$ such that:

(i) $\underline{A}[X, X] = M$ for all $X \in |\underline{A}|$.

(ii) $\underline{A}[Y, Z] \cap \underline{A}[X, Y] \subseteq \underline{A}[X, Z]$ for all $X, Y, Z \in |\underline{A}|$.

A mapping $T : |\underline{A}| \to |\underline{B}|$ is a $\underline{V}$-functor iff (iii) holds:

(iii) $\underline{A}[X, Y] \subset \underline{B}[TX, TY]$ for all $X, Y \in |\underline{A}|$.

According to 3.4, the condition $T \in |P\underline{A}|$ for a $\underline{V}$-functor $T : \underline{A}^o \to \underline{V}$ is equivalent to

$$(13) \qquad \bigcup_{X \in |A|} \Big(TX \cap \bigcap_{Y \in |A|} \big((M \setminus TY) \cup \underline{A}[Y, X]\big)\Big) = M.$$

In particular, we will consider a discrete $\underline{V}$-category $\underline{A}$, i.e.

$$\underline{A}[X, Y] = M \text{ for } X = Y,\ \emptyset \text{ otherwise.}$$

This implies certainly that $\underline{A}_0$, the underlying category of $\underline{A}$, is discrete in the ordinary sense. Every mapping from $|\underline{A}|$ to $|\underline{V}|$ is a $\underline{V}$-functor from $\underline{A}^o$ (or $\underline{A}$) to $\underline{V}$. The objects of $P\underline{A}$ are all «partitions» of $M$ with respect to the index set $|\underline{A}|$, i.e. all mappings $T : |\underline{A}| \to |\underline{V}|$ such that

$$\bigcup_{X \in |A|} TX = M \quad \text{and} \quad TX \cap TY = \emptyset \quad \text{for} \quad X \neq Y$$

($TX = \emptyset$ for arbitrarily many $X \in |\underline{A}|$ is permitted). This fact is an immediate consequence of (13). The category $(P\underline{A})_0$, the underlying cate-

gory of the $\underline{V}$-category $P\,\underline{A}$, is discrete, too.

This example demonstrates that the class of objects of $P\underline{A}$ cannot be obtained as a closure of $\underline{A}$ in $[\underline{A}^o, \underline{V}]$ with respect to limits or colimits of any kind, contrary to $\underline{V} = \underline{Ens}$ and $\underline{V} = \underline{Ab}$ (cf. 4.1 and 4.2).

The category of abelian group objects in $\underline{V}$ is equivalent to the terminal category $\underline{1}$ (cf. 4.3).

4.6. We mention some other examples without giving detailed comments.

a) $\underline{V} = \underline{Ens_*}$. $\underline{V}$-categories are ordinary categories with a specified set of zero morphisms (cf. [20], 5.5.6/7). For $|\underline{A}| \neq \emptyset$, $P\,\underline{A}$ consists of all retracts of representables, but if $|\underline{A}| = \emptyset$, $P\,\underline{A} = [\underline{A}^o, \underline{V}] = \underline{1}$.

b) $\underline{V} = \underline{NVS_1}$, the category of real (or complex) normed vector spaces and linear contractions (cf. [21], 20.1.10). For $F, G \in |\underline{NVS_1}|$, $F \otimes G$ is the algebraic tensor product with the norm

$$\|h\| := \inf \{ \sum_{i=1}^{n} \|f_i\| \, \|g_i\| \mid h = \sum_{i=1}^{n} f_i \otimes g_i, \quad f_i \in F, \quad g_i \in G \}.$$

c) $\underline{V} = \underline{Ban_1}$, the category of real (or complex) Banach spaces and linear contractions (cf. [21], 20.1.10). $\underline{V}$-categories with one object are Banach algebra's $\underline{A}$ (cf. [21], 3.2.1), and $[\underline{A}, \underline{V}]$ is the category of $\underline{A}$-modules.

d) $\underline{V} = \underline{Bo}$, the category of bornological vector spaces (cf. [20], 16. 7.2).

e) $\underline{V} = \underline{n\text{-}cat}$, the category of small $n$-categories (cf. [5]). For a small $\underline{V}$-category $\underline{A}$, $P\,\underline{A}$ consists of all retracts of representable functors.

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