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ABSTRACT HOMOTOPY THEORY IN PROCATEGORIES

by Timothy PORTER

Artin and Mazur in [1] initiated the use of the category of prosimplicial sets as a tool in homotopy theory. Their method involved firstly the formation of the homotopy category of simplicial sets and then passing to the corresponding procategory. This method proves to be unsuitable for certain applications and by analogy with a similar situation with categories of simplicial spectra, it seems to be advisable to reverse the order of construction. This was done in [9] and it allows one to use the homotopy limit construction of Bousfield and Kan [3], or Boardman and Vogt ([2] and [11]). In [9] no attempt was made to investigate other homotopy structures within the procategory.

In this paper an attempt is made to extend a sizeable amount of « abstract homotopy theory » from a category $\mathcal{C}$ to the corresponding category of proobjects. The meaning we attach to the term « abstract homotopy theory » is that of Quillen [10] or Brown [4] ; we only claim « attempt » because the actual result obtained shows that, on extending an abstract homotopy theory in the manner shown, some of the structure is weakened. However, it will be shown that sufficient structure remains to give a sizeable amount of elementary homotopy theory. In future papers particular cases will be examined in more detail and it will be shown just how much structure is retained.

Other extensions are possible; for instance see Hastings [8], Edwards and Hastings [6] and Grossman [7]. These structures retain more of the original homotopy theory on $\mathcal{C}$ but do not allow the same sort of applications.

Finally I would like to thank Professor Wall of Liverpool and Gavin Wraith of Sussex for indirectly suggesting this approach.

1. Categories of fibrant objects.

In [4], Brown showed that a weaker form of homotopy theory than Quillen's «model category» structure [10] gives a sufficient amount of structure to allow a considerable part of homotopy theory to be carried out. Explicitly he made the following definitions.

Let $\mathcal{C}$ be a category with finite products and a final object e. Assume that $\mathcal{C}$ has two distinguished classes of maps called *weak equivalences* and *fibrations*. A map is called an *aspherical* or *trivial fibration* if it is both a fibration and a weak equivalence.

A *path space object* for an object B is an object B^I together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0, d_1)} B \times B$$

where s is a weak equivalence, (d_0, d_1) is a fibration and

$$(d_0, d_1)s = \Delta_B = \text{the diagonal map.}$$

$\mathcal{C}$ will be called a *category of fibrant objects* (for a homotopy theory) if this structure satisfies the axioms :

(F1) If f, g are maps so that gf is defined and two of f, g and fg are weak equivalences, so is the third.

(F2) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(F3) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B$$

with v a fibration, the fibred product $A \times_C B$ exists and the projection $pr: A \times_C B \to A$ is a fibration. If v is aspherical, so is pr.

(F4) For any object B, there exists at least one path space B^I (not necessarily functorial in B).

(F5) For any object B, the map $B \to e$ is a fibration.

We refer the interested reader to Brown's paper [4] for the detailed development of this set of axioms. For the situation in which these axioms will be used here, it is necessary to alter them slightly. We replace (F4) by an axiom which requires that at least one functorial path space object exists; we will however continue to call this one (F4).

It is a well known result (see for instance Artin and Mazur [1] or Duskin [5]) that any map $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ in $\mathrm{pro}(\mathcal{C})$ can be replaced up to isomorphism by a «level map», i.e. a proobject in the category of maps of $\mathcal{C}$. Because of this it helps to look at functor categories $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ before considering the more difficult case of $\mathrm{pro}(\mathcal{C})$. We shall of course assume that $\mathcal{I}$ is small, but no assumption needs be made as to other structure of $\mathcal{I}$.

THEOREM 1.1. *Let* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *be the category of functors from* $\mathcal{I}$ *to a category of fibrant objects with functorial* (F4). *Then* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *is a category of fibrant objects.*

PROOF. We will specify the required structure but will leave it to the reader to verify that this structure works.

- *Weak equivalences:* Let

$$\mathbf{X}: \mathcal{I} \to \mathcal{C} \quad \text{and} \quad \mathbf{Y}: \mathcal{I} \to \mathcal{C}$$

be two functors and $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ a natural transformation; $\mathbf{f}$ is a weak equivalence if each $\mathbf{f}(i): \mathbf{X}(i) \to \mathbf{Y}(i)$ is a weak equivalence in $\mathcal{C}$.

- *Fibrations:* With $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ as above, $\mathbf{f}$ is a fibration if each $\mathbf{f}(i): \mathbf{X}(i) \to \mathbf{Y}(i)$ is a fibration in $\mathcal{C}$.

- *Path space object:* Denoting the functorial path space by $(\,)^I$, the path space of $\mathbf{X}: \mathcal{I} \to \mathcal{C}$ is

$$\mathbf{X}^I = (\,)^I \mathbf{X}: \mathcal{I} \to \mathcal{C}.$$

COROLLARY 1.2. *The class of weak equivalences in* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *admits a calculus of right fractions.*

PROOF. This is a direct consequence of the Theorem together with Proposition 2, page 424 of [4].

The corollary implies that the category of fractions formed by formally inverting the weak equivalences is well behaved. A comparison with the work of Vogt [11] suggests that this «homotopy category» is equivalent to some category of coherent $\mathcal{J}$-indexed diagrams in some sense; we will not pursue this idea here.

2. Categories of cofibrant objects.

In [4], Brown only gives explicitly the axioms listed in n° 1; however on page 442 he mentions the duals of these axioms in Proposition 8. Although it is not absolutely necessary, we list below the dual axioms and state the duals of Theorem 1.1 and Corollary 1.2. The reason for stating these axioms for a «category of cofibrant objects» is that any Quillen model category [10] yields both a category of fibrant objects and a category of cofibrant objects. It is this situation which is the most important one from the point of view of the applications.

Let $\mathcal{C}$ be a category with finite coproducts and an initial object e. We assume that $\mathcal{C}$ has two distinguished classes of maps called *weak equivalences* and *cofibrations.* A map in both classes is called a *trivial cofibration.*

A *cylinder object* for an object B is an object $B\times I$ together with maps

$$B\vee B \xrightarrow{\delta_0+\delta_1} B\times I \xrightarrow{\sigma} B$$

with $\sigma(\delta_0+\delta_1)=\nabla_B$, the codiagonal map, $\delta_0+\delta_1$ a cofibration, and σ a weak equivalence.

$\mathcal{C}$ will be called a *category of cofibrant objects* (for a homotopy theory) if this structure satisfies the axioms:

(C1) as (F1),

(C2) as (F2) with «fibration» replaced by «cofibration»,

(C3) Given a diagram

$$A \xleftarrow{u} C \xrightarrow{v} B$$

with v a cofibration, the pushout $A\underset{C}{\vee}B$ exists and $v': A\to A\underset{C}{\vee}B$ is a cofibration which is trivial if v is.

(C4) For each B, there is at least a cylinder object $B\times I$ in $\mathcal{C}$ (which we will need to be functorial in B).

(C5) For any object B, the map $e \to B$ is a cofibration.

EXAMPLES. (a) If $\mathcal{C}$ is a model category in the sense of Quillen [10] and $\mathcal{C}_c$ consists of the cofibrant objects, then $\mathcal{C}_c$ is a category of cofibrant objects in the above sense.

(b) If $\mathcal{C}$ is a category of fibrant objects, then $\mathcal{C}^{op}$ is a category of cofibrant objects in a natural way.

REMARK. Although we will not state or prove any of the dual results, the theorems and proofs of Brown's paper [4] easily dualize and so we shall feel free to use any of these dual results without explicit proof in the remainder of this paper or in sequels to this paper.

THEOREM 2.1. *Let* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *be the category of functors from* $\mathcal{I}$ *to a category of cofibrant objects* $\mathcal{C}$, *with functorial* (C4). *Then* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *is a category of cofibrant objects.*

PROOF. As before the bulk of the work is left to the reader. If we define weak equivalences as in 1.1 and cofibrations in like manner, the result follows easily.

COROLLARY 2.2. *The class of weak equivalences in* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *admits a calculus of left fractions.*

COROLLARY 2.3. *If* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *is the category of functors from* $\mathcal{I}$ *to a closed model category* $\mathcal{C}$, *then* $\mathcal{H}om(\mathcal{I}, \mathcal{C}_f)$ *is a category of fibrant objects and* $\mathcal{H}om(\mathcal{I}, \mathcal{C}_c)$ *is a category of cofibrant objects. If* $\mathcal{C} = \mathcal{C}_c = \mathcal{C}_f$, *then* $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ *has both structures.*

REMARKS. (a) In 2.3, $\mathcal{C}_f$ is, following Quillen's notation [10], the full subcategory of $\mathcal{C}$ determined by the fibrant objects of $\mathcal{C}$.

(b) Although $\mathcal{H}om(\mathcal{I}, \mathcal{C})$ is both a category of fibrant and of cofibrant objects, this does not mean it is a Quillen model category. In Quillen's axioms, (M1) requires a certain interaction between fibrations and cofibrations; there seems no reason to suppose that fibrations and cofibrations interact in this way in $\mathcal{H}om(\mathcal{I}, \mathcal{C})$. It is for this reason that categories of fibrant and cofibrant objects suggest themselves as suitable for functor categories.

(c) Given more structure in $\mathcal{C}$, Hastings has equipped $\mathcal{H}om(\mathcal{I},\mathcal{C})$ with a full Quillen model category structure (see [8]). His definitions however are not suitable for the applications envisaged for the theory being developed here.

3. Procategories.

As was mentioned before, any map in a procategory $pro(\mathcal{C})$ can be reindexed to give a «level map», and so if $\mathcal{C}$ has a homotopy structure involving weak equivalences, etc..., it is natural to define $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ to be a weak equivalence in $pro(\mathcal{C})$ if, by reindexing, one gets a level map $\mathbf{f}_{\mathcal{J}}$: $\mathbf{X}_{\mathcal{J}} \to \mathbf{Y}_{\mathcal{J}}$ indexed by some cofiltering category $\mathcal{I}$, which is a weak equivalence in the homotopy structure of $\mathcal{H}om(\mathcal{I},\mathcal{C})$, and similarly for fibrations and cofibrations if any. From another point of view, this approach to defining a homotopy theory on $pro(\mathcal{C})$ might be considered as an attempt to «glue» the various homotopy structures in the functor categories $\mathcal{H}om(\mathcal{I},\mathcal{C})$, for cofiltering $\mathcal{I}$, together via final (or coinitial) functors. Unfortunately this naive approach fails; for instance the class of weak equivalences defined in this way is not closed under composition.

The obvious way is to «generate» an abstract homotopy from the «basic» material given by the $\mathcal{H}om(\mathcal{I},\mathcal{C})$. We will thus make the following definitions :

- *Basic weak equivalence:* $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a basic weak equivalence if by reindexing one obtains a level weak equivalence, i.e. a weak equivalence in the image of some $\mathcal{H}om(\mathcal{I},\mathcal{C})$ with $\mathcal{I}$ cofiltering. $\mathbf{f}$ will also be called a basic weak equivalence if it is an isomorphism.
- *Basic fibration* (or *cofibration*): $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a basic fibration if it can be replaced (by reindexing) by a level fibration (similarly for cofibration).
- *Basic trivial fibration:* as above but with a level trivial fibration.
- *Path space:* If $\mathbf{X}: \mathcal{I} \to \mathcal{C}$ then as before $\mathbf{X}^I$ is the composite

$$(\ )^I\mathbf{X}: \mathcal{I} \to \mathcal{C}.$$

- *Cylinder object:* If $\mathbf{X}: \mathcal{I} \to \mathcal{C}$, then :

$$\mathbf{X} \times I = ((\ )\times I)\mathbf{X} : \mathcal{I} \to \mathcal{C}.$$

- *Weak equivalence:* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a weak equivalence if it is a composite of basic weak equivalences.

- *Fibration:* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a fibration if it is a composite of basic fibrations and isomorphisms.

Even with these definitions, $\mathrm{pro}(\mathcal{C})$ is not a category of fibrant (or cofibrant) objects, but by careful use of the reindexing results ([1] Appendix) available in $\mathrm{pro}(\mathcal{C})$, it is possible to recover the most important results of Part 1 of Brown's paper [4]. We will not prove all the intermediate steps in the proof of these more general results, but merely indicate how to adapt the proofs given in [4].

First note that there is a weak form of axiom system satisfied by the classes described above. The important differences are that the map $(\mathbf{d}_0, \mathbf{d}_1)$ is a basic fibration and in axiom (F3) we can only claim that, if $\mathbf{v}$ is a basic trivial fibration, then so is $\mathbf{pr}$ (similarly for (C3)).

LEMMA 3.1. (Factorization lemma). *If* $\mathbf{u}$ *is any map in* $\mathrm{pro}(\mathcal{C})$, *then* $\mathbf{u}$ *can be factored* $\mathbf{u} = \mathbf{p}\,\mathbf{i}$, *where* $\mathbf{p}$ *is a basic fibration and* $\mathbf{i}$ *is right inverse to a basic trivial fibration.*

PROOF. Represent $\mathbf{u}$ by a level map and apply the factorization lemma in the relevant $\mathcal{H}om(\mathcal{I}, \mathcal{C})$.

The definition of «homotopic» is interesting in its own right and indicates the sort of structure that is involved here.

If $\mathbf{f}, \mathbf{g} : \mathbf{A} \rightrightarrows \mathbf{B}$ are two maps in $\mathrm{pro}(\mathcal{C})$ then they are *homotopic* if there is a «homotopy»

$$\mathbf{h} : \mathbf{A} \to \mathbf{B}^I \quad \text{such that} \quad \mathbf{d}_0 \mathbf{h} = \mathbf{f} \ \text{ and } \ \mathbf{d}_1 \mathbf{h} = \mathbf{g}.$$

We will write $\mathbf{f} \simeq \mathbf{g}$ or $\mathbf{h} : \mathbf{f} \simeq \mathbf{g}$.

There is immediately the problem of proving «$\simeq$» is transitive. We can reinterprete «$\simeq$» by noting that $\mathbf{f} \simeq \mathbf{g}$ if and only if by reindexing one gets a diagram

$$\mathbf{A}_{\mathcal{I}} \underset{\mathbf{g}_{\mathcal{I}}}{\overset{\mathbf{f}_{\mathcal{I}}}{\rightrightarrows}} \mathbf{B}_{\mathcal{I}}$$

so that $f_{\mathcal{I}} \simeq g_{\mathcal{I}}$ in $\mathcal{H}om(\mathcal{I}, \mathcal{C})$.

LEMMA 3.2 (Transitivity of « $\simeq$ »). *If* $f \simeq g$ *and* $g \simeq h$, *then* $f \simeq h$.

PROOF. Suppose $f_{\mathcal{I}} \simeq g_{\mathcal{I}}$ and $g_{\mathcal{I}'} \simeq h_{\mathcal{I}'}$; then there is a small cofiltering category with objects pairs (i, j) with i in $\mathcal{I}$, $j \in \mathcal{I}'$ such that there is a diagram

$$\begin{tikzcd}[column sep=large]
A_{\mathcal{I}}(i) \arrow[r, shift left, "f_{\mathcal{I}}(i)"] \arrow[r, shift right, "g_{\mathcal{I}}(i)"'] \arrow[d, "p^i_j"'] & B_{\mathcal{I}}(i) \arrow[d, "p^i_j"] \\
A_{\mathcal{I}'}(j) \arrow[r, shift left, "g_{\mathcal{I}'}(j)"] \arrow[r, shift right, "h_{\mathcal{I}'}(j)"'] & B_{\mathcal{I}'}(j)
\end{tikzcd} \qquad *(i,j)$$

representing the diagram

$$\begin{tikzcd}[column sep=large]
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & B \arrow[d] \\
A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B
\end{tikzcd}$$

(with the vertical maps the respective identities) in $\mathrm{pro}(\mathcal{C})$. A map from (i, j) to (i', j') is a map of the corresponding diagrams. If we denote this small cofiltering category by $\mathfrak{M}$ we get a prodiagram by assigning to each (i, j) the corresponding diagram $*(i, j)$.

Each diagram thus formed simplifies to give

$$\begin{tikzcd}[column sep=huge]
A_{\mathcal{I}}(i) \arrow[r, shift left=2, "p^i_j f_{\mathcal{I}}(i)"] \arrow[r, "p^i_j g_{\mathcal{I}}(i)" description] \arrow[r, shift right=2, "h_{\mathcal{I}'}(j)\, p^i_j"'] & B_{\mathcal{I}'}(j)
\end{tikzcd}$$

and on noting that $p^i_j g_{\mathcal{I}}(i) = g_{\mathcal{I}'}(j)\, p^i_j$ we can join the two homotopies

$$p^i_j f_{\mathcal{I}}(i) \simeq p^i_j g_{\mathcal{I}}(i) \qquad \text{and} \qquad g_{\mathcal{I}'}(j)\, p^i_j \simeq h_{\mathcal{I}'}(j)\, p^i_j$$

together to give a homotopy

$$\mathbf{A}_{\mathcal{J}}(i) \to \mathbf{B}^{I}_{\mathcal{J}'}(j) = (\mathbf{B}_{\mathcal{J}'}(j))^{I}$$

and this is the required homotopy, $\mathbf{f} \simeq \mathbf{h}$.

LEMMA 3.3 *(cf. Brown [4] Lemma 1). Any diagram*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "\mathbf{i}"'] & \mathbf{E} \arrow[d, "\mathbf{p}"] \\ \mathbf{X} \arrow[r] & \mathbf{B} \end{tikzcd}$$

with $\mathbf{i}$ *a basic weak equivalence and* $\mathbf{p}$ *a fibration can be imbedded (up to isomorphism) in a diagram*

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[dr, "\mathbf{i}"'] & \mathbf{X'} \arrow[r] \arrow[d, "\mathbf{t}"] & \mathbf{E} \arrow[d, "\mathbf{p}"] \\ & \mathbf{X} \arrow[r] & \mathbf{B} \end{tikzcd}$$

with $\mathbf{t}$ *a composite of basic trivial fibrations.*

PROOF. First apply in case $\mathbf{p}$ is a basic fibration and then using induction on the length of the composite. Again reindexing is used to enable each stage to be done in a functor category.

It is only this weaker form of Lemma 3.3 which is needed, one never needs the stronger form where $\mathbf{i}$ is a weak equivalence.

Brown's Lemma 2 and Propositions 1 and 2 now follow easily using induction on the lengths of composite required in the weak equivalences. As a result of this, Theorem 1 page 425 holds and we can form the homotopy category $\mathcal{H}o\,pro(\mathcal{C})$ using

$$[\mathbf{A}, \mathbf{B}] = \varinjlim Hom_{\pi\, pro(\mathcal{C})}(\mathbf{A'}, \mathbf{B})$$

(where the limit is over the category of weak equivalences over $\mathbf{A}$) as the *hom*-set.

Rather than dwell too long on the minor changes needed in Brown's subsequent proofs to effect the extension to $pro(\mathcal{C})$ - basically the changes are managed by careful reindexing along the lines of Lemma 3.2 above and induction on the lengths of fibrations and weak equivalences - we will leave the statement and proofs of intermediate results to the reader and will merely state the theorem which is most useful in the applications.

We first need a definition from [4] page 432 for $\mathcal{C}$ pointed. A *fibration sequence* is a diagram $\mathbf{F} \to \mathbf{E} \to \mathbf{B}$ in $\mathcal{H}o\,pro(\mathcal{C})$ together with an action in $\mathcal{H}o\,pro(\mathcal{C})$, $\mathbf{F} \times \Omega\,\mathbf{B} \to \mathbf{F}$, which is isomorphic to a diagram and action obtained from a fibration in $pro(\mathcal{C})$.

THEOREM 3.4. *If $\mathcal{C}$ is a pointed category of fibrant objects and Ω denotes the loop space functor, then for any fibration sequence*

$$\mathbf{F} \xrightarrow{i} \mathbf{E} \xrightarrow{p} \mathbf{B}, \quad \mathbf{F} \times \Omega\,\mathbf{B} \xrightarrow{a} \mathbf{F}$$

in $pro(\mathcal{C})$, the sequence

$$\to [\mathbf{A}, \Omega^q \mathbf{B}] \to [\mathbf{A}, \Omega^{q-1} \mathbf{F}] \to [\mathbf{A}, \Omega^{q-1} \mathbf{E}] \to [\mathbf{A}, \Omega^{q-1} \mathbf{B}] \to$$

is exact (in the sense of Quillen [10] for $q = 1$) for any $\mathbf{A}$ in $pro(\mathcal{C})$.

Dualising the whole of this section we get the dual result.

THEOREM 3.5. *If $\mathcal{C}$ is a pointed category of cofibrant objects and Σ denotes the suspension functor, then for any cofibration sequence*

$$\mathbf{A} \xrightarrow{i} \mathbf{X} \xrightarrow{c} \mathbf{C}, \quad \mathbf{C} \xrightarrow{a} \mathbf{C} \vee \Sigma \mathbf{A}$$

in $pro(\mathcal{C})$ the sequence

$$\to [\Sigma^n \mathbf{X}, \mathbf{B}] \to [\Sigma^n \mathbf{A}, \mathbf{B}] \to [\Sigma^{n-1} \mathbf{C}, \mathbf{B}] \to [\Sigma^{n-1} \mathbf{X}, \mathbf{B}] \to$$

is exact (as before) for any $\mathbf{B}$ in $pro(\mathcal{C})$.

The interesting more special case is where $\mathcal{C}$ is a pointed Quillen model category and as before $\mathcal{C}_f$ is the category of fibrant objects and $\mathcal{C}_c$ the category of cofibrant objects in $\mathcal{C}$. Theorems 3.4 and 3.5 apply in $pro(\mathcal{C}_f)$ and $pro(\mathcal{C}_c)$ respectively and the $\mathbf{A}$ of 3.4 (and $\mathbf{B}$ of 3.5) can be chosen from amongst those objects weakly equivalent to some object in $pro(\mathcal{C}_f)$ (respectively in $pro(\mathcal{C}_c)$).

As mentioned in Section 2, no such axiom as Quillen's (M1) can be satisfied in $pro(\mathcal{C})$ even if $\mathcal{C} = \mathcal{C}_c = \mathcal{C}_f$, but by using a stronger form of fibration or cofibration such a development is possible (see [6] or [8]). If, in addition, $(\,)^I$ and $(\,)\times I$ are adjoint on $\mathcal{C}$ there is some linking between the two forms of structure (again assuming for simplicity $\mathcal{C} = \mathcal{C}_c = \mathcal{C}_f$). In the cases

(i) $\mathcal{C} = \mathcal{K}an_0$, the category of pointed Kan complexes,

and

(ii) $\mathcal{C} = \mathcal{C}^+(\mathcal{M}od\text{-}A)$, the category of chain complexes of A-modules which are bounded below,

such a linking occurs. We will investigate those two cases in future papers.

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