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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE *Vol. XVII-4 (1976)*

DOUBLE GROUPOIDS AND CROSSED MODULES

*by Ronald BROWN and Christopher B. SPENCER**

INTRODUCTION

The notion of double category has occured often in the literature (see for example [1, 6, 7, 9, 13, 14]). In this paper we study a more particular algebraic object which we call a *special double groupoid with special connection*. The theory of these objects might be called «2-dimensional groupoid theory». The reason is that groupoid theory derives much of its technique and motivation from the fundamental groupoid of a space, a device obtained by homotopies from paths on a space in such a way as to permit cancellations. The 2-dimensional animal corresponding to a groupoid should have features derived from operations on squares in a space. Thus it should have the algebraic analogue of the horizontal and vertical compositions of squares; it should also permit cancellations. But an extra feature of the 2-dimensional theory is the existence of an analogue of the Kan fibres of semi-cubical theory. This analogue is provided by what we call a *special connection*.

In this paper we cover the main algebraic theory of these objects.

We define a category $\mathfrak{DG}$ of special double groupoids with special connection, and a full subcategory $\mathfrak{DG}^!$ of objects D such that D_0 is a point. We prove in Theorem A that $\mathfrak{DG}^!$ is equivalent to the well-known category of crossed modules [5, 10, 12], while Theorem B gives a description of connected objects of $\mathfrak{DG}$.

Theorem A has been available since 1972 as a part of [4]. It was explained in [4] that the motivation for these double groupoids was to find

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an algebraic object which could express statements and proofs of putative forms of a 2-dimensional van Kampen Theorem. Such applications have now been achieved in [3], which defines the homotopy double groupoid $\rho(X,Y,Z)$ of a triple and applies it, with Theorem A of this paper, to obtain new results on second relative homotopy groups.

The structure of this paper is as follows. In parag. 1 we set up the basic machinery of double categories in the form we require. We do this in complete generality for two reasons:

1° we expect applications to topology of general double groupoids,

2° we expect applications to category theory of the general notion of a connection for a double category.

However the notion of connection is the only real novelty of this section.

In section 2 we show how crossed modules arise from double groupoids. In section 3 we define the category $\mathfrak{DG}$ and prove Theorem A. Section 4 discusses retractions and proves Theorem B. We also discuss «rotations» for objects of $\mathfrak{DG}$.

We are grateful to Philip Higgins for a number of very helpful comments.

1. Double categories.

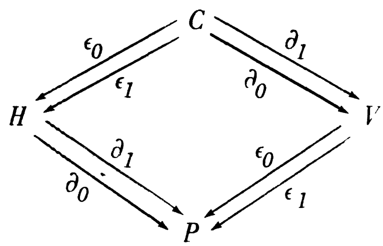
Usually a double category is taken to be a set with two commuting category structures, or, equivalently, a category object in the category of small categories. We shall use another, but again equivalent, definition.

By a *category* is meant a sextuple $(H, P, \partial_0, \partial_1, m, u)$ in which H, P are sets, $\partial_0, \partial_1 : H \to P$ are respectively the initial and final maps, m is the partial composition on H, and $u: P \to H$ is the unit function; these data are to satisfy the usual axioms.

By a *double category* D is meant four related categories

$$(C, V, \partial_0, \partial_1, +, 0), \quad (C, H, \epsilon_0, \epsilon_1, \circ, 1),$$
$$(V, P, \epsilon_0, \epsilon_1, ., e), \quad (H, P, \partial_0, \partial_1, ., f)$$

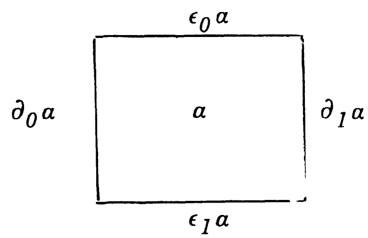
as partially shown in the diagram



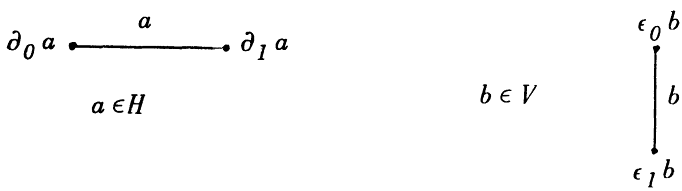
and satisfying the rules (1.1)-(1.5) given below. The elements of C will be called *squares*, of H, V *horizontal* and *vertical edges* respectively, and of P *points*. We will assume the relations

(1.1) $$\partial_i \epsilon_j = \epsilon_j \partial_i, \quad i, j = 0, 1,$$

and this allows to represent a square a as having bounding edges pictured as



while the edges are pictured as

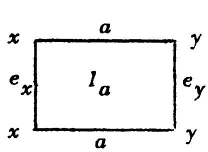


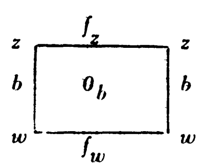
We will assume the relations

(1.2) $$\partial_i (1_a) = e_{\partial_i a}, \quad i = 0, 1, \quad a \in H,$$

$$\epsilon_j (0_b) = f_{\epsilon_j b}, \quad j = 0, 1, \quad b \in V,$$

so that the identities 1_a, 0_b for squares have boundaries





We also require

(1.3) $$0_{e_x} = 1_{f_x}, \quad x \in P,$$

and this square is written $\odot_x$ or simply $\odot$. Similarly, e_x, f_x are written e, f if no confusion will arise.

We assume two further relations:

(1.4) $$\epsilon_i(\alpha+\beta) = \epsilon_i(\alpha).\epsilon_i(\beta), \quad i = 0, 1,$$
$$\partial_j(\alpha \circ \gamma) = \partial_j(\alpha).\partial_j(\gamma), \quad j = 0, 1,$$

for all α, β, $\gamma \in C$ such that both sides are defined, and

(1.5) *the interchange law*

$$(\alpha+\beta)\circ(\gamma+\delta) = (\alpha\circ\gamma)+(\beta\circ\delta),$$

whenever α, β, γ, $\delta \in C$ and both sides are defined.

It is convenient to use matrix notation for compositions of squares. Thus if α, β satisfy $\partial_1 \alpha = \partial_0 \beta$, we write

$$[\alpha, \beta] \quad \text{for} \quad \alpha+\beta,$$

and if $\epsilon_1 \alpha = \epsilon_0 \gamma$, we write

$$\begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \quad \text{for} \quad \alpha \circ \gamma.$$

More generally, we define a *subdivision* of a square α in C to be a rectangular array (α_{ij}), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ of squares in C satisfying

$$\partial_1 \alpha_{i,j-1} = \partial_0 \alpha_{ij} \quad (1 \leqslant i \leqslant m, \quad 2 \leqslant j \leqslant m)$$

and

$$\epsilon_1 \alpha_{i-1,j} = \epsilon_0 \alpha_{ij} \quad (2 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n)$$

such that

$$(\alpha_{11}+\alpha_{12}+\ldots+\alpha_{1n})\circ(\alpha_{21}+\ldots+\alpha_{2n})\circ\ldots\circ(\alpha_{m1}+\ldots+\alpha_{mn}) = \alpha.$$

We call α the *composite* of the array (α_{ij}) and write $\alpha = [\alpha_{ij}]$. The interchange law then implies that, if in the array represented by the subsequent matrix, we partition the rows and columns into blocks B_{kl} and compute the composite β_{kl} of each block, then $\alpha = [\beta_{kl}]$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example we may compute :

$$a = (a_{11} \circ \ldots \circ a_{m1}) + \ldots + (a_{1n} \circ \ldots \circ a_{mn}).$$

We now give two examples of double categories in Topology.

EXAMPLE 1. Let X be a topological space and Y, Z subspaces of X. The double category $\Lambda^2 = \Lambda^2(X; Y, Z)$ will have the set C of «squares» to be the continuous maps $F: [0, r] \times [0, s] \to X$ for some $r, s \geq 0$, satisfying

$$F(\lambda, 0),\ F(\lambda, s) \in Y, \quad \lambda \in [0, r],$$

$$F(0, \mu),\ F(r, \mu) \in Z, \quad \mu \in [0, s].$$

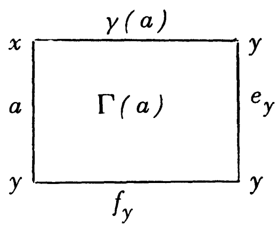
The «horizontal edges» H will consist of the maps $[0, s] \to Z$, $s \geq 0$, and the vertical edges V will consist of the maps $[0, r] \to Y$, $r \geq 0$, while P will be $Y \cap Z$. The obvious horizontal and vertical compositions of squares, together with the usual multiplication of paths and the obvious boundary, zero and unit functions will give us a double category. Note that in this example it is convenient to keep H, V, P as part of the structure, rather than just regard Λ^2 as C with two commuting partially defined compositions.

EXAMPLE 2. Let $T = (H, P, \partial_0, \partial_1, \cdot, f)$ be a topological category (by which we mean that H, P are topological spaces and all the structure functions are continuous). Let ΛX denote the set of Moore paths on a topological space X. Then we obtain a double category ΛT whose squares are the elements of ΛH, whose horizontal edges and vertical edges are H, ΛP respectively, and whose points are P. The category structure on ΛP is that given by the usual multiplication of Moore paths, while the two compositions on ΛH are the multiplication of Moore paths and the addition +

induced by the composition $\circ$ on H.

This last example arises in the theory of connections in differential geometry. In order to explain this, suppose given as in the beginning of this Section a double category D. By a *connection for* D we mean a pair (Γ, γ) in which $\gamma : V \to H$ is a functor of categories and $\Gamma : V \to C$ is a function such that:

(1.6) the bounding edges of $\gamma(a)$ and $\Gamma(a)$, for $a : x \to y$ in V, are given by the diagram



and

(1.7) the *transport law holds*, viz if $a, b \in V$ and $a.b$ is defined, then

$$\Gamma(a.b) = \begin{bmatrix} \Gamma(a) & 1_{\gamma(b)} \\ 0_b & \Gamma(b) \end{bmatrix}.$$

We call Γ the *transport* of the connection and γ the *holonomy* of the connection. The reason for this terminology is the following

EXAMPLE 3. Let T be the topological category of Example 2 and ΛT the double category of paths on T. A connection (Γ, γ) for ΛT then consists of the transport $\Gamma : \Lambda P \to \Lambda H$ and the holonomy $\gamma : \Lambda P \to H$. The «transport law» is essentially equation (10) of the Appendix to [11] (where Γ is called a «path-connection»).

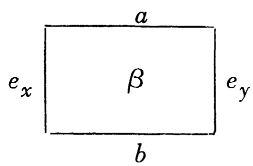
2. Double groupoids.

From now on our interest will be in double groupoids, that is double categories in which each of the four underlying categories is a groupoid. In this section we consider simple aspects of their homotopy theory, in particular their «homotopy groups».

Let D be a double category as in Section 1 such that D is a double groupoid. Then we have two *sets of components of* D, namely

$$\pi_0^h D = \pi_0 H \quad \text{and} \quad \pi_0^v D = \pi_0 V .$$

We also have two *fundamental groupoids* $\pi_1^h D$, $\pi_1^v D$. Each of these has object set P, and $\pi_1^h D$ for example has arrows $x \to y$ the equivalence classes of elements of $H(x,y)$ where a, b in $H(x,y)$ are equivalent if there is a square β whose bounding edges are given by



The multiplication in $\pi_1^h D$ is induced by $+$, and the verification that $\pi_1^h D$ is a groupoid is easy. Similarly, we obtain the «vertical» fundamental groupoid $\pi_1^v D$ and so for each x in P we have two *fundamental groups*

$$\pi_1^h(D,x), \quad \pi_1^v(D,x).$$

The *second homotopy group* $\pi_2(D,x)$ at a point x of D is the set of squares a of D whose bounding edges are e_x or f_x. It is a consequence of the interchange law that, when they are restricted to $\pi_2(D,x)$, the operations $+$, $\circ$ coincide and are abelian.

The groups $\pi_2(D,x)$, $x \in P$, form a local system over each of the groupoids $\pi_1^h D$, $\pi_1^v D$. Suppose for example that

$$\gamma \in \pi_2(D,x), \quad a \in H(x,y).$$

We then define

$$\gamma^a = -1_a + \gamma + 1_a \text{ , an element of } \pi_2(D,y),$$

and this gives on operation of H on the family $\{\pi_2(D,x)\}_{x \in P}$.

If $a, b \in H(x,y)$ are equivalent by a square β, then γ^a, γ^b have the common subdivision $\begin{pmatrix} -\beta^{-1} & \odot & \beta^{-1} \\ -1_a & \gamma & 1_a \\ -\beta & \odot & \beta \end{pmatrix}$ and so $\gamma^a = \gamma^b$. So, we obtain an operation of $\pi_1^h D$ on $\{\pi_2(D,x)\}_{x\in P}$. Similarly, there is an operation of $\pi_1^v D$ on $\{\pi_2(D,x)\}_{x\in P}$.

We now show how to obtain from a double groupoid two families of crossed modules.

We recall [5, 10, 12] that a crossed module (A,B,∂) consists of groups A, B, an operation of B on the right of the group A, written

$$(a,b)\mapsto a^b, \quad a\in A, \quad b\in B,$$

and a morphism $\partial: A\to B$ of groups. These must satisfy the conditions:

(i) $\partial(a^b) = b^{-1}\partial(a)b, \quad a\in A, \quad b\in B,$

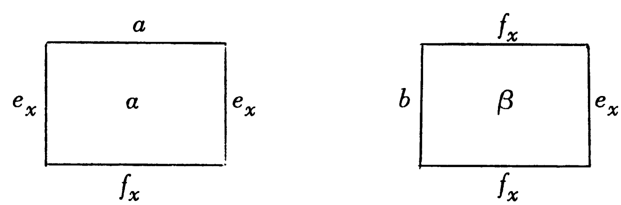
(ii) $a^{-1}a_1 a = a_1^{\partial(a)}, \quad a,\ a_1\in A.$

A *map* $(f,g):(A,B,\partial)\to(A',B',\partial')$ of crossed modules consists of morphisms $f: A\to A'$, $g: B\to B'$ of groups such that $g\partial = \partial' f$ and f is an operator morphism with respect to g, i.e.

if $a\in A$, $b\in B$, then $f(a^b) = f(a)^{g(b)}$.

So we have a category $\mathcal{C}$ of crossed modules.

Let D be a double groupoid, and let $x\in P$. We define the groups $\pi_2(D,H,x)$, $\pi_2(D,V,x)$ to be the sets of squares of D with bounding edges given respectively by



for some $a\in H$, $b\in V$ respectively, and with group structures induced from $+$, $\circ$, respectively. Clearly we have morphisms of groups

$$\epsilon: \pi_2(D,H,x)\to H\{x\}, \quad \alpha\mapsto \epsilon_0\alpha,$$

$$\partial: \pi_2(D,V,x)\to V\{x\}, \quad \beta\mapsto \partial_0\beta.$$

PROPOSITION 1. *If D is a double groupoid as above and x a point of D then we have crossed modules*

$$\gamma(D,x) = (\pi_2(D,H,x),\ H\{x\}, \epsilon),$$
$$\gamma'(D,x) = (\pi_2(D,V,x),\ V\{x\}, \partial).$$

PROOF. It is sufficient to define the operations and to verify the axioms (i) and (ii). Let $b \in H\{x\}$, $a \in \pi_2(D,H,x)$. We define

$$a^b = -1_b + a + 1_b .$$

It is trivial to verify that this is an operation and that condition (i) for a crossed module is satisfied. For the proof of condition (ii) we note that if $\epsilon(\beta) = b$, then a^b and $-\beta + a + \beta$ have the common subdivision

$$\begin{pmatrix} -1_b & a & 1_b \\ -\beta & \odot & \beta \end{pmatrix}$$

and so $a^b = -\beta + a + \beta$. A similar proof holds for $\gamma'(D,x)$.

If D is a pointed double groupoid (by which we mean a base point x is chosen), we abbreviate $\gamma(D,x)$ to $\gamma(D)$.

A *morphism* $f: D \to D'$ of double categories D, D' consists of four functions

$$C \to C',\quad H \to H',\quad V \to V',\quad P \to P'$$

commuting with the category structures. So there is a category of double categories. Similarly there is a category of double groupoids and their morphisms, and of pointed double groupoids (in which a base point is chosen for each double groupoid and morphisms preserve base point). Clearly γ defines a functor from this last category to the category of crossed modules.

3. Special double groupoids.

By a *special double groupoid* we shall mean a double groupoid D as in Section 2 but with the extra condition that the horizontal and vertical category structures coincide. These double groupoids will, from now on, be our sole concern, and for these it is convenient to denote the sets of points, edges and squares by D_0, D_1 and D_2 respectively. The identities in D_1 will be written e_x, or simply e. The boundary maps $D_1 \to D_0$ will be written δ_0, δ_1.

By a *morphism* $f: D \to E$ of special double groupoids is meant a triple of functions

$$f_i: D_i \to E_i \quad (i = 0, 1, 2)$$

which commute with all three groupoid structures.

A *special connection* for a special double groupoid D will mean a connection (Γ, γ) with holonomy map γ equal to the identity $D_1 \to D_1$. Such a connection will be written simply Γ. A morphism $f: D \to E$ of special double groupoids with special connections Γ, Δ is said to *preserve the connection if* $f_2\Gamma = \Delta f_1$.

The category $\mathfrak{DG}$ has objects the pairs (D, Γ) of special double groupoid D with special connection, and arrows the morphisms of special double groupoids preserving the connection. The full sub-category of $\mathfrak{DG}$ on objects (D, Γ) such that D has only one point will be written $\mathfrak{DG}^!$. If (D, Γ) is an object of $\mathfrak{DG}^!$ then we have a crossed module $\gamma(D)$ by Proposition 1. Clearly γ extends to a functor γ from $\mathfrak{DG}^!$ to $\mathcal{C}$, the category of crossed modules. Our main result on double groupoids is :

THEOREM A. *The functor* $\gamma: \mathfrak{DG}^! \to \mathcal{C}$ *is an equivalence of categories.*

PROOF. We define an inverse $\xi: \mathcal{C} \to \mathfrak{DG}^!$ to γ.

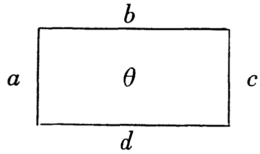
Let (A, B, ∂) be a crossed module. We define a special double groupoid with special connection $E = \xi(A, B, \partial)$ as follows. First, E_0 is to consist of a single point 1 say. Next $E_1 = B$, with its structure as a group (regarded as a groupoid with one vertex). The set E_2 of squares is to consist of quintuples

$$\theta = (\, a\, ;\, a \quad \begin{matrix} b \\ \\ d \end{matrix} \quad c\,)$$

such that $a \in A$, $a, b, c, d \in B$ and

(*) $$\partial(a) = a^{-1} b c d^{-1}.$$

The boundary operators on θ are given by the following diagram (there are 8 possible conventions for this boundary - the convention chosen is the most



convenient in terms of signs and the expressions for addition and composition of squares). The addition and composition of squares are given by

$$(1)\qquad (\alpha;\, a \overset{b}{\underset{d}{\quad}} c) + (\beta;\, c \overset{f}{\underset{h}{\quad}} g) = (\alpha\beta^{d^{-1}};\, a \overset{bf}{\underset{dh}{\quad}} g),$$

$$(2)\qquad (\alpha;\, a \overset{b}{\underset{d}{\quad}} c) \circ (\tau;\, j \overset{d}{\underset{i}{\quad}} h) = (\alpha^{j}\tau;\, aj \overset{b}{\underset{i}{\quad}} ch).$$

It is straightforward to check that these operations are well-defined, i.e. that with the above data

$$\partial(\alpha\beta^{d^{-1}}) = a^{-1} b f g h^{-1} d^{-1},$$

$$\partial(\alpha^{j}\tau) = j^{-1} a^{-1} b c h i^{-1}$$

(for which condition (i) of a crossed module is needed). It is also easy to check that each of these operations defines a groupoid structure on E_2 with the initial, final and zero maps for $+$ being respectively

$$\partial_0,\ \partial_1,\ a \mapsto (1;\, a \overset{1}{\underset{1}{\quad}} a)$$

and for $\circ$ being

$$\epsilon_0,\ \epsilon_1,\ a \mapsto (1;\, 1 \overset{a}{\underset{a}{\quad}} 1).$$

The verification of the interchange law requires condition (ii) for a crossed module, but again is routine and is left to the reader.

The special connection $\Gamma : E_1 \to E_2$ for E is given by

$$\Gamma(a) = (1;\, a \overset{a}{\underset{1}{\quad}} 1).$$

The verification of the transport law is trivial.

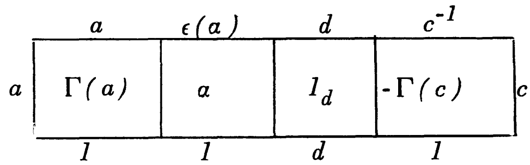
This completes the description of $E = \xi(A, B, \partial)$ and it is clear that

ξ extends to a functor $\xi : \mathcal{C} \to \mathfrak{D}\mathfrak{G}^!$. It is immediate that $\gamma\xi : \mathcal{C} \to \mathcal{C}$ is naturally equivalent to the identity. We now prove that $\xi\gamma$ is naturally equivalent to the identity.

Let (D, Γ) be an object of $\mathfrak{D}\mathfrak{G}^!$. Let $E = \xi\gamma(D)$. Then $E_0 = D_0$, $E_1 = D_1$. We define $\eta : E \to D$ to be the identity on E_0 and E_1 and on E_2 by

$$\eta\left(a;\ a \begin{matrix} b \\ \\ d \end{matrix}\ c\right) = \Gamma(a) + a + 1_d - \Gamma(c)$$

(for $\epsilon(a) = a^{-1} b c d^{-1}$) as shown in the diagram



which clearly has the correct bounding edges. Clearly η is a bijection: $E_2 \to D_2$, so to prove η is an isomorphism it suffices to prove that η preserves $+, \circ$ and connections.

For $+$ we have, by the definition of $\beta^{d^{-1}}$ and using the notation of equation (1):

$$\eta\left(a\beta^{d^{-1}};\ a \begin{matrix} bf \\ \\ dh \end{matrix}\ g\right) = \Gamma(a) + a\beta^{d^{-1}} + 1_{dh} - \Gamma(g)$$

$$= \Gamma(a) + a + 1_d - \Gamma(c) + \Gamma(c) + \beta + 1_h - \Gamma(g)$$

$$= \eta\left(a;\ a \begin{matrix} b \\ \\ d \end{matrix}\ c\right) + \eta\left(\beta;\ c \begin{matrix} f \\ \\ h \end{matrix}\ g\right).$$

For $\circ$ we have, using the notation of equation (2),

$$(3) \qquad \eta\left(a^j\tau;\ aj \begin{matrix} b \\ \\ i \end{matrix}\ ch\right) = \Gamma(aj) + a^j\tau + 1_i - \Gamma(ch),$$

while on the other hand

$$(4) \qquad \eta\left(a;\ a \begin{matrix} b \\ \\ d \end{matrix}\ c\right) \circ \eta\left(\tau;\ j \begin{matrix} d \\ \\ i \end{matrix}\ h\right) =$$

$$= (\Gamma(a) + a + 1_d - \Gamma(c)) \circ (\Gamma(j) + \tau + 1_i - \Gamma(h)).$$

The equality of (3) and (4) follows from the fact that by the transport law the right hand sides of both (3) and (4) have the common subdivision

$$\begin{pmatrix} \Gamma(a) & 1_j & -1_j & a & 1_j & 1_{\epsilon(\tau)} & 1_i & -1_h & -\Gamma(c) \\ 0_j & \Gamma(j) & -1_j & \odot & 1_j & \tau & 1_i & \Gamma(h) & 0_h \end{pmatrix}.$$

Finally η preserves the connection since

$$\eta(\odot;\, a \ \begin{smallmatrix} a \\ \\ 1 \end{smallmatrix} \ 1) = \Gamma(a) + \odot + 1_1 - \Gamma(1) = \Gamma(a).$$

Since the naturality of η in the category $\mathcal{DG}^!$ is clear, we have now proved that η is a natural equivalence from $\xi\gamma$ to the identity functor.

4. Retractions.

In the theory of groupoids a key role is played by the retractions ([8], 47,92 and [2], 6.7.3 and 8.1.5). The object of this section is to set up similar results on double groupoids to those for groupoids (as in the last section, double groupoid here will mean special double groupoid with special connection).

Let G be a double groupoid. Then (as in Section 1) the set $\pi_0 G$ is the set of components of the groupoid (G_1, G_0). We say that G *is connected* if $\pi_0 G$ is empty or has one element. We say a subdouble groupoid G' of G *is representative in* G if G' meets each component of G ; we say G' *is full in* G if the groupoid (G'_1, G'_0) is full in (G_1, G_0) and also the set of squares with given boundary edges in G' is the same for G' as for G.

A double groupoid T is called a *tree double groupoid* if for each x, y in T_0 there is exactly one edge a in T_1 with

$$\delta_0 a = x, \quad \delta_1 a = y,$$

and for each quadruple (x, y, z, w) of points of T_0 there is exactly one a in T_2 with

$$\partial_0 \epsilon_0 a = x, \quad \partial_1 \epsilon_0 a = y, \quad \partial_0 \epsilon_1 a = z, \quad \partial_1 \epsilon_1 a = w.$$

For example if X is any set, there is a tree double groupoid $T(X)$ in which

$$T(X)_0 = X, \quad T(X)_1 = X \times X \quad \text{with } \delta_0(x,y) = x, \quad \delta_1(x,y) = y,$$

and

$$T(X)_2 = \{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mid x, y, z, w \in X \}$$

with the boundaries of $\alpha = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ given by

$$\partial_0 \alpha = (x, z), \quad \epsilon_0 \alpha = (x, y), \quad \partial_1 \alpha = (y, w), \quad \epsilon_1 \alpha = (z, w).$$

The groupoid structures and connection are then uniquely defined so as to make $T(X)$ a double groupoid.

THEOREM B. *Let G' be a full, representative subdouble groupoid of the double groupoid G. Then*

(i) there is a retraction $r: G \to G'$.

(ii) if $f: G \to H$ is a morphism of double groupoids such that f_0 is injective, then there is a full, representative subdouble groupoid H' of H and a pushout square

$$\begin{array}{ccc} G & \xrightarrow{\ r\ } & G' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ H & \xrightarrow[\ s\ ]{} & H' \end{array}$$

in which f' is the restriction of f and s is a retraction.

(iii) if G'_0 is a singleton, then there is an isomorphism $g: G \to G' \times T$, where T is a tree double groupoid.

PROOF. (i) For each $x \in G_0$ choose an edge θx in G_1 such that

$$\delta_0 \theta x \in G'_0 \quad \text{and} \quad \delta_1 \theta x = x;$$

this is possible since G' is representative in G. Define

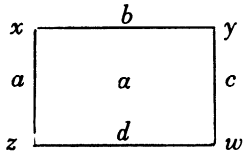
$$r_0 : G_0 \to G'_0 \quad \text{by} \quad r_0(x) = \delta_0 \theta x.$$

Since G'_1 is full in G_1, for any edge a in G_1 the edge

$$r_1(a) = \theta\delta_0(a).a.(\theta\delta_1(a))^{-1}$$

belongs to G'_1. So we have $r_1: G_1 \to G'_1$ defined, and r_1 is the usual retraction defined for groupoids.

Now let $\alpha \in G_2$ have boundaries given by the diagram



We define $r_2(\alpha)$ to be

$$\begin{bmatrix} \Gamma\theta(x) & 1_b & -\Gamma\theta(y) \\ 0_a & \alpha & 0_c \\ \Gamma^{-1}\theta(z) & 1_d & -\Gamma^{-1}\theta(w) \end{bmatrix}$$

(it is convenient to denote $(\Gamma(b))^{-1}$ by $\Gamma^{-1}(b)$ or $\Gamma^{-1}b$).

It is straightforward to check that r_2 preserves $+$ and $\circ$. To prove that r_2 preserves connections we have to prove that

$$r_2\Gamma(a) = \Gamma(\theta x.\, a.(\theta y)^{-1}) \quad \text{if} \quad \delta_0 a = x, \quad \delta_1 a = y.$$

Now

$$r_2\Gamma(a) = \begin{bmatrix} \Gamma\theta(x) & 1_a & -\Gamma\theta(y) \\ 0_a & \Gamma a & \odot \\ \Gamma^{-1}\theta(y) & \odot & -\Gamma^{-1}\theta(y) \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma(\theta(x).a) & -\Gamma\theta(y) \\ \Gamma^{-1}\theta(y) & -\Gamma^{-1}\theta(y) \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma(\theta(x).a) & -\Gamma\theta(y) \\ (0_{\theta(y)})^{-1} & (0_{\theta(y)})^{-1} \end{bmatrix} .$$

However it is a simple consequence of the transport law that

$$\Gamma(b^{-1}) = (-\Gamma b) \circ (0_b)^{-1} .$$

It follows that $r_2\Gamma(a) = \Gamma(\theta x.\, a.(\theta y)^{-1})$. (That r_2 preserves Γ also follows from Proposition 1 of [3].)

(ii) Let H' be the full subdouble groupoid of H on

$$f(G_0') \cup (H_0 \setminus f(G_0)).$$

Then we have a commutative diagram

$$\begin{array}{ccc} G' & \xrightarrow{\;i\;} & G \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ H' & \xrightarrow[\;j\;]{} & H \end{array}$$

in which i, j are inclusions and f' is the restriction of f.

We choose edges θx, $x \in G_0$, and so $r : G \to G'$ as in (i). For each y in H_0, let

$$\phi(y) = f_1\theta(x) \quad \text{if} \quad y = f(x)$$

(in which case x is unique), and otherwise let $\phi(y) = e_y$. Then $\phi f_0 = f_1\theta$ and the edges $\phi(y)$, $y \in H_0$ with the connection Δ for H determine a retraction $s : H \to H'$. We prove (*) is commutative. If $x \in G_0$, then

$$sf(x) = \delta_0\,\phi f(x) = \delta_0 f\theta(x) = f\delta_0\,\theta(x) = f'r(x).$$

If $a : x \to y$ in G_1, then

$$sf(a) = (\phi f(x)).f(a).(\phi f(y))^{-1} =$$
$$= f(\theta(x).a.\theta(y)^{-1}) = fr(a).$$

Finally, if $a \in G_2$, then $sf(a) = fr(a)$ is immediate from the definition and the fact that f preserves the connection.

To prove (*) a pushout, suppose given a commutative diagram of double groupoids

$$\begin{array}{ccc} G & \xrightarrow{\;r\;} & G' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ H & \xrightarrow[\;v\;]{} & K \end{array}$$

If there is a morphism $w : H' \to K$ such that $ws = v$, then $w = wsj = vj$, so there is at most one such w. On the other hand, let $w = vj$. Then

$$wf' = vjf' = vfi = uri = u,$$

so we need verify only $ws = v$. If $y \in H_1$, then $\phi(y)$, and hence also $v\phi(y)$, is an identity for $y \neq f(x)$ for any x, while if $y = f(x)$, then

$$v\phi(y) = v\phi f(x) = vf\theta(x) = ur\theta(x) = u(e_{rx})$$

which is again an identity. It follows that $ws = vjs = v$ on H_1. Finally on H_2 the relation $ws = v$ follows from the fact that if $y = f(x)$ then

$$w\Delta\phi(y) = w\Delta f\theta(x) = wf\Gamma\theta(x) = vf\Gamma\theta(x)$$
$$= ur\Gamma\theta(x) = u\Gamma r\theta(x) = u\Gamma e_{rx} = u(\odot) = \odot,$$

and so from the definition of s we obtain that, on H_2, $ws = vjs = v$. This completes the proof of (ii).

(iii) Let $G_0' = \{x_0\}$. Let $T = T(G_0)$ be the tree double groupoid defined above. Then there is a unique morphism $f: G \to T$ of double groupoids such that f_0 is the identity. Let $r: G \to G'$ be the retraction defined as in (ii) by choices of edges $\theta(x)$ with

$$\delta_0\theta(x) = x_0, \quad \delta_1\theta(x) = x.$$

Define $g: G \to G' \times T$ to have components r and f respectively. Clearly g_0 is a bijection, and it is a standard fact about groupoids that g_1 is a bijection. But g_2 is also a bijection, since it has inverse

$$\left(\beta, \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \mapsto \begin{bmatrix} \Gamma\theta'(x) & 1_{b'} & -\Gamma\theta'(y) \\ 0_{a'} & \beta & 0_{c'} \\ \Gamma^{-1}\theta'(z) & 1_{d'} & -\Gamma^{-1}\theta'(w) \end{bmatrix}$$

where $\theta'(x) = \theta(x)^{-1}$ and β has

$$\partial_0\beta = a', \quad \epsilon_0\beta = b', \quad \partial_1\beta = c', \quad \epsilon_1\beta = d'.$$

Therefore g is an isomorphism.

COROLLARY. *If G is a connected double groupoid and x is a point of G, then there is an isomorphism $G \cong G\{x\} \times T$ where $G\{x\}$ is the full sub-double groupoid of G with point set $\{x\}$, and T is a tree double groupoid.*

As an interesting application of this corollary, we deduce properties of a «rotation» τ which interchanges the horizontal and vertical compositions of squares in a double groupoid.

Let G be a double groupoid, with connection Γ. The *rotation* τ associated with Γ is the function $\tau: G_2 \to G_2$ such that if $\alpha \in G_2$ then the edges of α and $\tau(\alpha)$ are related by

$$\begin{array}{ccc} & b & \\ a & \boxed{\ \alpha\ } & c \\ & d & \end{array} \qquad\qquad \begin{array}{ccc} & a^{-1} & \\ d & \boxed{\ \tau(\alpha)\ } & b \\ & c^{-1} & \end{array}$$

and $\tau(\alpha)$ is defined to be

$$\tau(\alpha) = \begin{bmatrix} -1 & -\Gamma^{-1}(b^{-1}) & 0_b \\ -\Gamma(a) & \alpha & \Gamma^{-1}(c^{-1}) \\ 0_d & \Gamma(d) & -1_c \end{bmatrix}.$$

THEOREM C. *The rotation τ satisfies:*

(i) $\tau(\alpha+\beta) = \tau(\alpha) \circ \tau(\beta)$ *whenever* $\alpha+\beta$ *is defined,*

(ii) $\tau(\alpha \circ \gamma) = \tau(\gamma) + \tau(\alpha)$ *whenever* $\alpha \circ \gamma$ *is defined,*

(iii) $\tau^2(\alpha) = -\alpha^{-1}$,

(iv) $\tau^4 = id$,

(v) τ *is a bijection.*

PROOF. Clearly (iv) is a consequence of (iii) which also implies that τ^2 is a bijection; (v) follows easily.

For the proofs of (i), (ii) and (iii) we use Theorem A and B (iii), which imply that it is sufficient to prove the theorem for the case of a double groupoid $G = \xi(A, B, \partial)$ determined by a crossed module (for the theorem is clearly true for tree double groupoids, is true for a product if true for each factor, and is true in one groupoid if true in an isomorphic one).

However, in the notation of the proof of Theorem A, a straightforward calculation shows that:

$$\text{if } \theta = \left(a\,;\, a \begin{array}{c} b \\ \\ d \end{array} c\right), \text{ then } \tau(\theta) = \left(a^d;\, d \begin{array}{c} a^{-1} \\ \\ c^{-1} \end{array} b\right).$$

From this it is easy to deduce (i) and (ii) . Note also that

$$\tau^2(\theta) = (a^{d(c^{-1})};\ c^{-1} \begin{matrix} d^{-1} \\ \\ b^{-1} \end{matrix} a^{-1}).$$

But by the definitions of + and $\circ$,

$$(-\theta)^{-1} = (a^{d(c^{-1})};\ c^{-1} \begin{matrix} d^{-1} \\ \\ b^{-1} \end{matrix} a^{-1}).$$

This completes the proof of Theorem C.

REMARKS. 1° We have been able to find direct proofs of (i) and (ii) of Theorem C using only the laws for double groupoids. A similar proof for (iii) has been found by P. J. Higgins ; this proof involves an «anti-clockwise» rotation σ , and checking that

$$\sigma\tau = \tau\sigma = 1, \quad \tau(a)+\sigma(-a) = 0.$$

2° The methods of this paper can be applied to $\rho(X, Y, Z)$, the homotopy double groupoid of a triple described in [3], to prove the existence of homotopies between various maps

$$(I^2, \dot{I}^2, \ddot{I}^2) \to (X, Y, Z).$$

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