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ET GEOMETRIE DIFFERENTIELLE

COHERENT PROHOMOTOPICAL ALGEBRA

by Timothy PORTER

Homotopical algebra à la Quillen [20] is an abstract form of homotopy theory designed to give a «non linear homological algebra» for use in more general categories than those in which the classical, linear version applied. In a classical situation, the category of chain complexes of right modules over some associative ring A , the two theories look very similar, but the «homotopical» theory is neatly axiomatised whilst the «homological» theory is less firmly fixed.

In the category **pro(Mod-A)** of pro-objects in the category **Mod-A** of right A-modules, there is a perfectly good homology theory since this category **pro(Mod-A)** is an abelian category with enough projectives. This homology theory is, in an obvious sense, an extension of the usual homology theory on **Mod-A** , however it is an extension of the homotopy theoretic structure on the corresponding category **C(Mod-A)**, of chain complexes in **Mod-A** .

In various papers [17, 18, 19] the possibility of extending homotopy theories from a category **C** to the corresponding procategory **pro(C)** has been considered in both the general abstract case and for specific application to the case where **C** is the category of Kan simplicial sets. In this application it was necessary to consider a «homotopical» structure on the category **pro(C(Mod-A))** - at least for A the ring of integers. It is this «prohomotopical algebra» that this paper considers.

The meaning of «coherent» in the title is analogous to its use by various algebraic topologists (see for example Vogt [26]), namely that diagrams commute up to homotopy and these homotopies satisfy «coherent» homotopy commutativity relations, thus the homotopies are «compatible».

The connection with Vogt's paper [26] is more than just the use

of one word ; Sections 4 to 7 deal with the algebraic analogue of the homotopy limits which are the subject of Vogt's paper. The construction of some of these algebraic homotopy limits is well known and even in the general case the two components of the construction are much used both in isolation and together, however the precise formulation in terms of the relevant homotopy structure seems to be unknown. What is a homotopy limit ? Briefly by defining a homotopy structure in $\mathbf{C(Mod\text{-}A)}$ and $\mathbf{pro(C(Mod\text{-}A))}$ one gets an inclusion of the corresponding homotopy categories,

$$\mathbf{Ho\,C(Mod\text{-}A)} \to \mathbf{Ho\,pro(C(Mod\text{-}A))},$$

where it exists holim is right adjoint to this functor.

1. PROCATEGORIES.

Although the definition and elementary properties of procategories are described in various sources (notably the original Séminaire Bourbaki notes of Grothendieck [10], as well as the Appendix to Artin and Mazur's notes [2] and some Seminar notes of Duskin [6]) the treatment is slightly different in each case and so we will summarize the terminology which will be used in this paper.

For convenience we will work within a universe U ; as usual, **Ens** will denote the category of U-small sets and all functions between them. All categories will be assumed to be U-categories, i.e. all *hom*-sets are U-small.

Let $\mathbf{C}$ be any category; then the Yoneda lemma gives an embedding

$$h : \mathbf{C}^{op} \to \mathbf{Hom(C, Ens)}.$$

Let $\mathbf{I}$ be a U-small category, i.e. the object class of $\mathbf{I}$ is a U-set. $\mathbf{I}$ is said to be *cofiltering* if it satisfies the following two conditions:

C1) For any objects i, j of $\mathbf{I}$, the *hom*-set $\mathbf{I}(i, j)$ is either empty or contains exactly one morphism.

C2) If i, j are objects of $\mathbf{I}$ there is a k in $\mathbf{I}$ and maps $k \to i$, $k \to j$.
(Some authors have considered a weaker form of C1, but in our situation this adds nothing and so we avoid it.)

If $\phi : \mathbf{J} \to \mathbf{I}$ is a functor of index categories we say ϕ is *cofinal* if for all i in $\mathbf{I}$, there is a j in $\mathbf{J}$ and a map in $\mathbf{I}$ from $\phi(j)$ to i.

If F is a functor from a U-small cofiltering index category to a category $\mathbf{C}$, then F is called a projective system. Any projective system determines a functor from $\mathbf{C}$ to $\mathbf{Ens}$ defined, for a projective system F, by

$$h_F(T) = \operatorname{colim}_{\mathbf{I}} h_{F(i)}(T) = \operatorname{colim}_{\mathbf{I}} \mathbf{C}(F(i), T).$$

A functor $K : \mathbf{C} \to \mathbf{Ens}$ is said to be *pro-representable* provided it is functorially isomorphic to a functor of the form h_F, for some projective system $F : \mathbf{I} \to \mathbf{C}$ indexed, as usual, by a cofiltering index category $\mathbf{I}$. The pro-representable functors form a full subcategory $\mathbf{PRO\text{-}REP(C)}$ of $\mathbf{Hom(C, Ens)}$. If $\mathbf{C}$ has finite projective limits, $\mathbf{PRO\text{-}REP(C)}$ corresponds to the category of all left exact functors on $\mathbf{C}$. This result has been proved at various times in varying forms by various people; a slightly weak form is attributed to Deligne and Lazard in [6] and, for the case we shall be interested in, namely where $\mathbf{C}$ is a U-small abelian category, the result may be found in Stauffer [24].

If $\mathbf{C}$ is a U-small abelian category, then all pro-representable functors from $\mathbf{C}$ to $\mathbf{Ens}$ factor through the forgetful functor from the category $\mathbf{Ab}$ of abelian groups to $\mathbf{Ens}$; so, in this case, there is an isomorphism of categories

$$\mathbf{PRO\text{-}REP(C)} \approx \mathbf{SEX(C, Ab)},$$

where $\mathbf{SEX(C, Ab)}$ is the category of all left exact additive functors from $\mathbf{C}$ to $\mathbf{Ab}$. The Yoneda embedding mentioned above gives an embedding

$$h : \mathbf{C}^{op} \to \mathbf{SEX(C, Ab)}$$

and, by taking $\mathbf{pro(C)}$ to be the category $\mathbf{SEX(C, Ab)}^{op}$, we get the canonical embedding $c : \mathbf{C} \to \mathbf{pro(C)}$. We may represent each object in $\mathbf{pro(C)}$ in at least one way as a projective limit, $\lim X$, of some projective system $X : \mathbf{I} \to \mathbf{C}$, considered as a projective system in $\mathbf{pro(C)}$. Every projective system in $\mathbf{C}$ admits a limit in $\mathbf{pro(C)}$ and every object of $\mathbf{pro(C)}$ is isomorphic to such a limit; thus $\mathbf{pro(C)}$ is a «cocompletion» of $\mathbf{C}$ (see Stauffer [24] for the corresponding completion construction).

The following proposition, proved in Duskin's notes [6], allows one to give an internal description of **pro(C)**.

Let $F: \mathbf{I} \to \mathbf{C}$ and $G: \mathbf{J} \to \mathbf{C}$ be projective systems in $\mathbf{C}$ indexed by $\mathbf{I}$ and $\mathbf{J}$ respectively, let h_F, h_G be the corresponding left exact functors; then there is a natural isomorphism

$$Hom_{\mathbf{pro(C)}}(h_G, h_F) \approx \lim_{\mathbf{I}} (\operatorname{colim}_{\mathbf{J}} \mathbf{C}(G(j), F(i))).$$

Thus this can be taken as an internal definition of the *hom*-set (or *hom*-group) in **pro(C)**.

We end this section with a brief résumé of some results which will be needed later.

1.1. Let $\phi: \mathbf{I} \to \mathbf{J}$ be a cofinal functor and $F: \mathbf{J} \to \mathbf{C}$ a projective system in $\mathbf{C}$ indexed by $\mathbf{J}$. Also let $F\phi: \mathbf{I} \to \mathbf{C}$ be the composite functor; then $h_F \approx h_{F\phi}$ in $\mathbf{Hom(C, Ens)}$, thus internally F and $F\phi$ are isomorphic in the category **pro(C)**.

1.2. Let $X: \mathbf{D} \to \mathbf{pro(C)}$ be a finite diagram in **pro(C)** and suppose that $\mathbf{D}$ contains no loops; then there is a prodiagram

$$X_{\mathbf{I}}: \mathbf{I} \to \mathbf{Hom(D,C)} \quad \text{such that} \quad X_{\mathbf{I}} \approx X \text{ in } \mathbf{Hom(D, pro(C))}.$$

As a special case of 1.2 we get

1.3. If $f: X \to Y$ is a map in **pro(C)**, there is a U-small cofiltering index category $\mathbf{I}$ and a projective system $f_{\mathbf{I}}: \mathbf{I} \to \mathbf{Ar(C)}$, where $\mathbf{Ar(C)}$ is the category of morphisms in $\mathbf{C}$, and an isomorphism $f_{\mathbf{I}} \approx f$ in $\mathbf{Ar(pro(C))}$.

In both 1.2 and 1.3 we say $X_{\mathbf{I}}$ (or $f_{\mathbf{I}}$) is obtained by *reindexing* X (or f).

Although the motivation for studying procategories comes from the application of the cocompleteness result, the internal description together with much use of 1.1, 1.2 and 1.3 is often more convenient to manipulate than the «dual of **SEX(C, Ab)**» definition.

2. THE HOMOTOPY STRUCTURE OF pro(C(Mod-A)).

We fix once and for all an associative ring A with identity, $1 \neq 0$.

Mod-A denotes the category of U-small unitary right A-modules and **C(Mod-A)** the corresponding category of chain complexes which will be assumed to be bounded below, i. e. K in **C(Mod-A)** satisfies the condition:

there is some $N<0$ such that $K_q = 0$ for $q<N$.

For brevity we refer to objects in a procategory **pro(C)** as being pro-objects in **C** ; with this terminology we shall refer to objects of **pro(Mod-A)** as being pro-A-modules («unitary» and «right» always being subsumed under the term «module») and objects of **pro(C(Mod-A))** as being *procomplexes* (again «bounded below» and «of A-modules», fixed throughout, will be left out).

The homotopy structure in **C(Mod-A)** is given by listing those maps which will be weak equivalences, fibrations and cofibrations (see Quillen [20] or, better for our purposes, Brown [4]). The weak equivalences are those chain maps which induce isomorphisms on homology; the fibrations are epimorphisms in **C(Mod-A)** and the cofibrations are those maps i such that: (i) i is a monomorphism;

(ii) $(\mathit{Coker}\, i)_q$ is projective in **Mod-A** for each q.

REMARK. For some purposes it would be better to consider chain complexes bounded above with weak equivalences as above, fibrations are epimorphisms with dimensionwise injective kernels and cofibrations are just monomorphisms. The difference between the two approaches corresponds to emphasizing projective resolutions rather than injective resolutions in a treatment of homological algebra.

We say a chain complex X is *fibrant* if the unique map $X \to 0$ is a fibration; thus all chain complexes are fibrant in the «bounded below» case. More limiting is the notion of cofibrant: X is *cofibrant* if the unique map $0 \to X$ is a cofibration, i. e. X is a cofibration if and only if X_q is a projective A-module for each dimension q.

The structure listed above (ignoring the cofibrations) almost defines a category of fibrant objects, in the sense of Brown [4], the only remaining structure is a «path space object». The exact construction of this is immaterial, but could be given by comparison with the corresponding structure

in the category of simplicial A-modules to which **C(Mod-A)**, or rather part of it, is isomorphic. (The construction is given in detail in Gabriel and Zisman [7].)

Dually the cofibrant objects form, of course, a category of cofibrant objects and the same comment as above goes through for this case. The definition of a category of cofibrant objects is essentially due to Brown [4]; one has just to dualise his axiom system. He does not explicitly give them but, on page 442, mentions the possibility briefly. The definition is however given explicitly in [18] to which we now turn for the following ideas.

We want to define weak equivalence, fibration and cofibration, in **pro(C(Mod-A))**. As noted in [18] any homotopy theory (à la Brown [4]) on a category **C** lifts to one on **pro(C)** and likewise for the dual theory. It should be noticed however that although the fibrations and cofibrations may interact nicely in **C**, they need not do so in **pro(C)**, i.e. Quillen's axiom (M1) may not hold (see Quillen [20], p. I.1.1).

Weak equivalence: A map $f: X \to Y$ in **pro(C(Mod-A))** is a *basic weak equivalence* if there is an indexing category **I** and a reindexing $f_I : X_I \to Y_I$ of f such that, for each i in **I**, $f_I(i): X_I(i) \to Y_I(i)$ is a weak equivalence in **C(Mod-A)**, or alternatively if f is an isomorphism. A weak equivalence is a composite of basic weak equivalences.

Fibration: As above with $f_I(i)$ a fibration in **C(Mod-A)**.

Cofibration: As above with «cofibration» replacing «weak equivalence».

The *path space objects in* **C(Mod-A)** are functorially constructed (one need only check the simplicial analogue) and so we can use this functoriality

$$(\ )^{\mathbf{I}}: \mathbf{C(Mod\text{-}A)} \to \mathbf{C(Mod\text{-}A)},$$

to define (functorially) a path space object $F^{\mathbf{I}} : \mathbf{I} \to \mathbf{C(Mod\text{-}A)}$ for each procomplex F.

If we denote by Σ the class of weak equivalences, we can form a category of fractions **pro(C(Mod-A))**$[\Sigma^{-1}]$, which we shall denote by: **Ho pro(C(Mod-A))**, for short. The set of maps from F to G in this categ-

ory is denoted by $[F, G]$.

The *shift functor*

$$\text{Shift}: \mathbf{C(Mod\text{-}A)} \to \mathbf{C(Mod\text{-}A)} \quad \text{given by} \quad \text{Shift}(\mathbf{O})_q = \mathbf{O}_{q+1}$$

acts as a loop space functor in $\mathbf{C(Mod\text{-}A)}$ and in $\mathbf{pro(C(Mod\text{-}A))}$. Its inverse is the suspension functor and there is clearly an adjointness

$$[\Sigma X, Y] \approx [X, \Omega Y]$$

(see Gabriel and Zisman [7], p. 102-106).

The main applicability of this category depends on the following:

PROPOSITION 2.1. *For any pro-*A*-module* M *there is a cofibrant procomplex* $F[M]$, *isomorphic to* $M_{(0)}$ *the procomplex with* M *in dimension zero and zeroes elsewhere.*

PROOF. Let U be the forgetful functor

$$U: \mathbf{Mod\text{-}A} \to \mathbf{Ens},$$

and F the free functor left adjoint to U. If we denote the composite FU by R, we get a procomplex $F[M]$ as follows:

$F[M]_0 = R(M)$.

Let $\delta: R \to 1$ be the natural map which comes from the adjointness: $F \to U$,

$F[M]_1 = R(\mathit{Ker}\,\delta(M))$, $\delta: F[M]_1 \to F[M]_0$ being $\delta(\mathit{Ker}\,\delta(M))$.

Assuming $F[M]_q$ is defined and also $\delta_q: F[M]_q \to F[M]_{q-1}$, then:

$$F[M]_{q+1} = R(\mathit{Ker}\,\delta_q) \quad \text{and} \quad \delta_{q+1} = \delta(\mathit{Ker}\,\delta_q).$$

Each $F[M]_{q+1}$ consists at each index of a free module, hence $F[M]$ is cofibrant. The complex $F[M](i)$ is clearly acyclic and $\delta(M): F[M]_0 \to M$ gives a morphism from $F[M]$ to $M_{(0)}$ in $\mathbf{pro(C(Mod\text{-}A))}$ which induces isomorphisms on the homology progroups. Thus $\delta(M)$ is a weak equivalence and the result follows.

Thus if we extend the embedding

$$(\ )_{(0)}: \mathbf{Mod\text{-}A} \to \mathbf{C(Mod\text{-}A)}: M \to M_{(0)}$$

to an embedding

$$\mathbf{pro\,(Mod\text{-}A)} \to \mathbf{pro\,(C\,(Mod\text{-}A))}$$

and consider each M in **pro(Mod-A)** as the corresponding $M_{(0)}$ in the category **pro(C(Mod-A))**, then each pro-A-module is weakly equivalent to a cofibrant object.

3. PROHOMOLOGY AND COHOMOLOGY.

Given any prochain complex C in **pro(C(Mod-A))**, one can define its n^{th} prohomology module $H_n(C)$ by the usual process of extending functors from a category to the corresponding procategory. There is the n^{th} homology functor

$$H_n : \mathbf{C\,(Mod\text{-}A)} \to \mathbf{Mod\,(A)}$$

and if one defines

$$H_n(C) : \mathbf{I} \to \mathbf{Mod\text{-}A}, \qquad \text{for } C : \mathbf{I} \to \mathbf{C\,(Mod\text{-}A)},$$

to be the composite, then we have for each index i :

$$H_n(C)(i) = H_n(C(i)).$$

If one wishes to add coefficients, say in an A-module M, one merely uses the classical homology $H_n(-;M)$ with coefficients in M and again forms the composite. A similar method works for cohomology functors but of course the resulting cohomology object is not a pro-A-module but an ind-A-module, that is an inductive system of A-modules (or alternatively an object of the completion of **Mod-A**). It is immediate that the homology profunctors:

$$H_n : \mathbf{pro\,(C\,(Mod\text{-}A))} \to \mathbf{pro\,(Mod\text{-}A)}$$

induce functors (which we will also denote by H_n)

$$H_n : \mathbf{Ho\,pro(C\,(Mod\text{-}A))} \to \mathbf{pro\,(Mod\text{-}A)}$$

and, by using the naturality of the universal coefficient sequences for homology and cohomology, it is easily shown that the other two cases follow, giving functors

$$H_n(-;M) : \mathbf{Ho\,pro(C\,(Mod\text{-}A))} \to \mathbf{pro\,(Mod\text{-}A)}$$

and

$$H^n(-;M) : \mathbf{Ho\,pro(C\,(Mod\text{-}A))}^{op} \to \mathbf{Ind\,(Mod\text{-}A)}.$$

The fact that $H^n(C;M)$ is an «indmodule» and not a promodule is inconvenient and it is better to have a cohomology object which is either in **pro(Mod-A)** or, as good, in **Mod-A** itself. One obvious way to arrive at a cohomology module would be to take the inductive limit of the cohomology «indmodule» and thus arrive at a Čech-style cohomology theory. We shall adopt another definition of cohomology more suited to the homotopical viewpoint, namely a «global» cohomology.

The *hom*-set in **Ho pro(C(Mod-A))** has a natural abelian group structure which agrees with that given by the loop-suspension group and cogroup structures in **Ho pro(C(Mod-A))**. Following Brown [4] we define

$$H^n(C;M) = [C, M_{(n)}] \quad \text{for} \quad n > 0,$$

where $M_{(n)}$ is the procomplex with M in dimension $-n$ and zero in all other dimensions.

If we recall the fact that the suspension functor

$$\Sigma : \mathbf{pro(C(Mod\text{-}A))} \to \mathbf{pro(C(Mod\text{-}A))} \quad \text{is given by} \quad \Sigma(C)_q = C_{q-1}$$

and that the loop space functor Ω is given by

$$\Omega(C)_q = C_{q+1},$$

the internal description of *Hom* in **pro(C(Mod-A))** shows that Σ and Ω are again adjoint, and it then follows by the Adjoint Functor Lemma of Brown [4], page 426, that

$$\begin{aligned} H^n(C;M) &= [C, M_{(n)}] = [C, \Omega^n M_{(0)}] = \\ &= [\Sigma^n C, M_{(0)}] = [\Sigma^n C, F[M]] \end{aligned}$$

where $F[M]$ is the cofibrant resolution of M given by 2.1 above.

The functor $H^n(C;M)$ is, of course, covariant in M and contravariant in C and, as one would expect, there is a long exact cohomology sequence. This result is proved in general by Brown [4] p. 432-433 and so we merely indicate how to translate from this situation to his general setting (see also [18]).

THEOREM 3.1. *Let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence in **pro(Mod-A)** *; then there is a long exact sequence, natural in* C *:*

$$\dots \longrightarrow H^n(C;L) \xrightarrow{\alpha*} H^n(C;M) \xrightarrow{\beta*} H^n(C;N) \xrightarrow{\partial} H^{n+1}(C;L).$$

PROOF. By the reindexing theorem, β can be represented by some

$$\beta_I : M_I \to N_I$$

where **I** is a small cofiltering category and β_I is an epimorphism in the category $(\text{Mod-A})^I$. Then $L_I \approx Ker_{\beta_I}$ is the fibre of the fibration β_I. Since all complexes are fibrant, the result follows from Brown's Proposition 4, [4], p. 432.

There is a corresponding result for a short exact sequence of cofibrant procomplexes; the exact statement of this result may be found in [18] 4.1, but we only need it in the particular case of cofibrant resolutions of modules.

THEOREM 3.2. *Let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence in **pro(Mod-A)***; then there is a long exact sequence for any pro-A-module* K *:*

$$H^n(N;K) \xrightarrow{\beta*} H^n(M;K) \xrightarrow{\alpha*} H^n(L;K) \xrightarrow{\partial} H^{n+1}(N;K).$$

PROOF. Careful use of the free-forgetful comonad construction of Proposition 2.1 allows one to replace

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

by an exact sequence of cofibrant procomplexes

$$0 \longrightarrow F[L] \longrightarrow F[M] \longrightarrow F[N] \longrightarrow 0$$

which is thus a cofibration sequence in the subcategory of cofibrant procomplexes. The two sequences are linked by the diagram hereafter, in which $\delta(L)$, $\delta(M)$ and $\delta(N)$ are weak equivalences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F[L] & \longrightarrow & F[M] & \longrightarrow & F[N] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta(L)} & & \downarrow{\scriptstyle \delta(M)} & & \downarrow{\scriptstyle \delta(N)} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Replacing K by $F[K]$, the result follows from the dual of Brown's result [4] p. 432, also mentioned in [18].

Although many of the standard cohomology results from «classical» homological algebra carry over without appreciable change in the statement of the result, the methods of proof are often very different. We illustrate this point by reference to the form of the Universal Coefficient Theorem for Cohomology used in [19]. The main difficulty of the proof is at a point where, in the classical case, there is absolutely no trouble. We get rid of this difficulty in the following proposition.

PROPOSITION 3.3. *Let* C *in* **pro(C(Mod-A))** *satisfying the condition:*

There is a representation $C: \mathbf{I} \to \mathbf{C}(\mathbf{Mod}\text{-}A)$ *of* C *such that, for each* i *in* **I** *and* n,

$$\partial_n(i): C_n(i) \to C_{n-1}(i)$$

is the zero map.

Let F *be any pro-A-module and* $F_{(0)}$, *as before, the procomplex with* F *in dimension zero and zeroes elsewhere; then there is a natural isomorphism*

$$[C, F_{(0)}] \approx Hom_{\mathbf{pro(Mod\text{-}A)}}(C_0, F_{(0)}).$$

PROOF. First let it be remarked that the restriction on C is not the weakest condition which would make the proposition hold, however it is the form of the restriction which occurs naturally in the Universal Coefficient Theorem and hence we will not bother with the more general result. Secondly it should be noted that the proposition holds if F is replaced by any acyclic complex. The isomorphism

$$\Psi: [C, F_{(0)}] \to Hom_{\mathbf{pro(Mod\text{-}A)}}(C_0, F_{(0)})$$

is given by taking the induced maps on prohomology; explicitly if $f: C \to F_{(0)}$ is an actual map in **pro(C(Mod-A))**, then

$$f_*: H_*(C) \to H_*(F_{(0)})$$

is zero for $* \neq 0$ and

$$H_0(C) = C_0 , \quad H_0(F_{(0)}) = F ;$$

exactly the same occurs if f is a map in **Ho pro(C(Mod-A))** since then, using the notation of Brown [4] for the canonical functor

$$\gamma : \textbf{pro(C(Mod-A))} \to \textbf{Ho pro(C(Mod-A))},$$

f can be written as $f = \gamma(g)\gamma(t)^{-1}$, where t is a weak equivalence and hence induces an isomorphism on prohomology; we can thus define

$$\Psi(f) = g_0 \, t_0^{-1},$$

where g_0 and t_0 are the induced maps in dimension zero of g and t resp. We can define an inverse to Ψ as follows: If $f: C_0 \to F$ is a map in the category **pro(Mod-A)**, let $f_{(0)}$ be the map in **pro(C(Mod-A))** which is f in dimension zero and zero elsewhere, $f_{(0)}: C \to F_{(0)}$. We let

$$\phi(f) = \gamma(f_{(0)}) .$$

Clearly $\Psi\phi(f) = f$; it is less clear that

$$\phi\Psi(f) = f \quad \text{for } f \in [C, F_{(0)}] ;$$

this will be proven as soon as it is shown that the non-zero dimensions of any map in $[C, F_{(0)}]$ are irrelevant in as much as any «homotopy class» in $[C, F_{(0)}]$ can be represented by a map of the form $f_{(0)}$ as above. We rely on Brown's results on pages 424 and 425 of [4] to the effect that any homotopy class $f: C \to F_{(0)}$ can be written as

$$C \xrightarrow{\gamma(g)} G \xleftarrow{\gamma(t)} F_{(0)}$$

with $t \in \Sigma$ or alternatively as

$$C \xleftarrow{\gamma(s)} D \xrightarrow{\gamma(g)} F_{(0)}$$

with $s \in \Sigma$ (this comes from the fact that Σ admits a calculus of fractions, cf. Gabriel and Zisman [7], which is essentially that Σ satisfies a categorical Ore condition). The second of Brown's results is that if $f, g: X \to Y$ are maps in **pro(C(Mod-A))**, then $\gamma(f) = \gamma(g)$ if and only if there is a

weak equivalence $t: X' \to X$ equalising f and g up to the notion of homotopy coming from the path space object. Using these two results it suffices to prove the following lemma.

LEMMA 3.4. *Let* C, F *be as above and* $G \approx F$ *in* **pro(C(Mod-A))**. *If* f *is a map from* C *to* G *in* **pro(C(Mod-A))**, *then* $c(f) = c(\hat{f})$ *where* $\hat{f}$ *agrees with* f *in dimension zero and is zero elsewhere.*

PROOF. We construct a procomplex D with a weak equivalence $t: D \to C$ such that $ft = \hat{f}t$. Let E_n be the equaliser of

$$C_n \underset{0}{\overset{f_n}{\rightrightarrows}} G_n \qquad \text{for } n \neq 0,$$

and $E_0 = C_0$. We define $D_n = C_n \oplus E_{n-1} \oplus E_n$ and $t_n : D_n \to C_n$ by

$$t_n(c, e, e') = c + e'$$

regarding $E_n \subset C_n$; the n-th boundary operator $\partial_n : D_n \to D_{n-1}$, that is

$$\partial_n : C_n \oplus E_{n-1} \oplus E_n \to C_{n-1} \oplus E_{n-2} \oplus E_{n-1}$$

is given by the matrix operator

$$\partial_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and hence is zero on the first and third summands and maps the second summand identically to the third summand of D_{n-1}. It is easily checked that D is a procomplex and t is a weak equivalence.

The Lemma, and thus the Proposition, is proved. The way is now clear for a fairly classical proof of the Universal Coefficient Theorem for Cohomology in **pro(Mod-A)** where A is a principal ideal domain.

THEOREM 3.5. *Let* F *be in* **pro(Mod-A)** *for* A *a principal ideal domain and let* C *be a cofibrant object in* **pro(C(Mod-A))**; *then for each* $q > 0$, *there is a subgroup* $N_{q-1}(C;F)$ *of* $Ext^1(H_{q-1}(C), F)$ *and an exact sequence of abelian groups*

$$0 \to N_{q-1}(C;F) \to H^q(C;F) \xrightarrow{h} Hom(H_q(C), F) \to 0$$

where Hom and Ext^1 *are the usual bifunctors coming from the « global » classical homological algebra of* **pro(Mod-A)**. *Moreover h is the canonical « induced homomorphism » map.*

PROOF. First we can, if necessary, replace C, by reindexing, with a complex in which each $C_n(i)$ is free. We work with this « locally free » procomplex in the classical way; thus there is a short exact sequence

$$0 \longrightarrow Z \longrightarrow C \longrightarrow B \longrightarrow 0 \qquad (*)$$

in **pro(C(Mod-A))**, where

$$Z_q = Z_q(C) = Ker(\partial_q : C_q \to C_{q-1}), \quad B_q = B_{q-1}(C) = \partial_q C_q .$$

Since A is a principal ideal domain, Z and B are cofibrant being respectively subcomplexes of C and ΣC (the suspension of C). This implies that (*) is a cofibration sequence in the homotopy structure of **pro(C(Mod-A))**, hence there is a long exact sequence

$$H^{q-1}(Z;F) \xrightarrow{\delta} H^q(B;F) \to H^q(C;F) \to H^q(Z;F) \to \dots .$$

Since B and Z have trivial boundary operators,

$$H^q(Z;F) = Hom(Z_q(C),F), \quad H^q(B;F) = Hom(B_{q-1}(C),F),$$

by the previous proposition, and the naturality of the isomorphism constructed, there gives that the homomorphism $\gamma_q : B_q(C) \to Z_q(C)$ induces the link morphism of the long exact sequence

$$[Z, F_{(q)}] \to [B, F_{(q)}],$$

which is the same as

$$Hom(\gamma_q, 1) : Hom(Z_q(C), F) \to Hom(B_q(C), F);$$

in other words $\delta = Hom(\gamma_q, 1)$. Thus there is a natural short exact sequence

$$0 \to Coker(Hom(\gamma_{q-1}, 1)) \to H^q(C;F) \to Ker(Hom(\gamma_q, 1)) \to 0.$$

Looking at the short exact sequence

$$0 \longrightarrow B_q(C) \xrightarrow{\gamma_q} Z_q(C) \longrightarrow H_q(C) \longrightarrow 0$$

and applying the « classical » long exact *Hom - Ext* sequence gives

$$0 \to Hom(H_q(C), F) \to Hom(Z_q(C), F) \xrightarrow{Hom(\gamma_q, 1)}$$

$$\to Hom(B_q(C), F) \xrightarrow{\sigma} Ext^1(H_q(C), F) \to Ext^1(Z_q(C), F) \to \dots;$$

taking

$$N_q(C; F) = Im(\sigma : Hom(B_q(C), F) \to Ext^1(H_q(C), F))$$

gives us

$Coker(Hom(\gamma_q, 1)) \approx N_q(C; F)$ and $Ker(Hom(\gamma_q, 1)) \approx Hom(H_q(C), F)$,

which completes the proof except the statement about h.

This last statement is easily seen to follow from the commutativity of the diagram

$$\begin{tikzcd}
{[C, F_{(q)}]} \arrow[r] \arrow[d, "h"'] & {[Z, F_{(q)}]} \arrow[d, "\Psi"] \\
Hom(H_q(C), F) \arrow[r, dash] & Hom(Z_q, F)
\end{tikzcd}$$

since the isomorphism Ψ was exactly the «h» for Proposition 3.3.

REMARKS. 1. Unless $Z_q(C)$ is globally projective one cannot claim that

$$N_q(C; F) = Ext^1(H_q(C), F),$$

but in many of the applications this does not matter.

2. Theorem 3.5 was given without proof in [19] as Theorem 4.4. The first proof in something like this form appeared in [16], I.1.7.

4. HOMOTOPY LIMITS. FUNCTOR CATEGORIES.

As mentioned in the introduction, several of the homotopy limits we shall be considering are well known. As a first example we take the homotopy kernel.

Suppose $f : X \to X'$ is a map in **C(Mod-A)**; the homotopy kernel of f is a chain complex K, with map $K \xrightarrow{i} X$ such that $\gamma(fi) = \gamma(0)$ and, given any (K', j) with the same property, there is a unique homotopy class k of maps (i.e. a map in **HoC(Mod-A)**) from K' to K such that

$$\gamma(j) = \gamma(i)k.$$

The complex K is easily constructed ; it is the Γf constructed in Gabriel and Zisman, p. 104 - 105 [7] ; briefly one forms the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots & (-2) \\
\cdots \arrow[r] & X'_1 \arrow[u] \arrow[r, "-d^{X'}_1"] & X'_0 \arrow[u] \arrow[r, "-d^{X'}_0"] & X'_{-1} \arrow[u] \arrow[r] & \cdots & (-1) \\
\cdots \arrow[r] & X_1 \arrow[u, "f_1"] \arrow[r, "d^X_1"] & X_0 \arrow[u, "f_0"'] \arrow[r, "d^X_0"] & X_{-1} \arrow[u, "f_{-1}"'] \arrow[r] & \cdots & (0) \\
\cdots \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots & (1) \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \\
 & (1) & (0) & (-1) & &
\end{tikzcd}$$

and from this one obtains a chain complex by the usual means of forming the corresponding total complex ; thus

$$(\Gamma f)_n = X_n \oplus X'_{n+1} \quad \text{and} \quad \partial_n : (\Gamma f)_n \to (\Gamma f)_{n-1}$$

is given by the matrix

$$\partial_n = \begin{pmatrix} d^X_n & f_n \\ 0 & -d^{X'}_{n+1} \end{pmatrix} ;$$

$i : \Gamma f \to K$ is the projection onto the second factor.

This use of a double complex followed by taking the total complex is absolutely typical as will be seen later.

Although our eventual purpose is to try to construct a homotopy limit functor

$$\text{holim} : \mathbf{Ho\,pro(C(Mod\text{-}A))} \to \mathbf{Ho\,C(Mod\text{-}A)}$$

right adjoint to the constant projective system functor

$$\bar{J} : \mathbf{Ho\,C(Mod\text{-}A)} \to \mathbf{Ho\,pro(C(Mod\text{-}A))},$$

not only is it convenient to consider more general homotopy limits first, but also for many applications outside the range of this paper these constructions are more important.

Let $\mathbf{I}$ be any small category and $\mathbf{Hom(I, C(Mod\text{-}A))}$ the category of functors from $\mathbf{I}$ to $\mathbf{C(Mod\text{-}A)}$. We define a homotopy structure in the category $\mathbf{Hom(I, C(Mod\text{-}A))}$ in very nearly the same way as was used, in Section 2, for $\mathbf{pro(C(Mod\text{-}A))}$ (cf. [18]):

- A map $f : X \to Y$ in $\mathbf{Hom(I, C(Mod\text{-}A))}$ is a weak equivalence if, for every object i in $\mathbf{I}$, $f(i) : X(i) \to Y(i)$ is a weak equivalence in the category $\mathbf{C(Mod\text{-}A)}$;
- f is a fibration if each $f(i)$ is an epimorphism, and is a cofibration if, for each i and integer n, $(\mathit{Coker}\, f(i))_n$ is a projective A-module;
- the path space functor on $\mathbf{C(Mod\text{-}A)}$ extends to give the path space object in $\mathbf{Hom(I, C(Mod\text{-}A))}$.

There is an inclusion functor

$$E_{\mathbf{I}} : \mathbf{C(Mod\text{-}A)} \to \mathbf{Hom(I, C(Mod\text{-}A))}$$

where, for each chain complex K, $E_{\mathbf{I}}(K)(i) = K$ for all i. The weak equivalences of $\mathbf{C(Mod\text{-}A)}$ are preserved by $E_{\mathbf{I}}$ and so $E_{\mathbf{I}}$ induces a functor

$$E_{\mathbf{I}} : \mathbf{Ho\,C(Mod\text{-}A)} \to \mathbf{Ho\,Hom(I, C(Mod\text{-}A))};$$

we want to try to produce a right adjoint to this functor which will be the homotopy limit, $\mathrm{holim}_{\mathbf{I}}$ for this case. In general, this right adjoint does not exist; the construction we outline below works as long as there are finitely many objects in $\mathbf{I}$; if this is not so, the construction may lead to a chain complex which is not bounded below. To avoid this difficulty we restrict our attention to diagrams of positive complexes or, for virtually no extra work, diagrams which are uniformly bounded below, i.e. there is an integer n such that $X_q = 0$ for $q < n$. We also rephrase the definition of $\mathrm{holim}_{\mathbf{I}}$ slightly, as follows:

Given a diagram of complexes $X : \mathbf{I} \to \mathbf{C(Mod\text{-}A)}$, $\mathrm{holim}_{\mathbf{I}} X$ is that complex, if it exists, which represents the functor

$$[\, E_I(-), X] : \mathbf{Ho\,C(Mod\text{-}A)} \to \mathbf{Ens}\,.$$

Thus there is a natural isomorphism of functors

$$[-, \mathrm{holim}_I X] \approx [\, E_I(-), X]$$

and hence a canonical homotopy class

$$\mu : E_I(\mathrm{holim}_I X) \to X \quad \text{in} \quad \mathbf{Ho\,Hom(I, C(Mod\text{-}A))}\,.$$

If we prove the existence of $\mathrm{holim}_I X$ for arbitrary $\mathbf{I}$ and X a diagram of positive complexes, then this will show that $\mathrm{holim}_I X$ always exists for $\mathbf{I}$ with finitely many objects and also that $\mathrm{holim}_I X$ exists for arbitrary $\mathbf{I}$ and X an $\mathbf{I}$-diagram which is uniformly bounded below, since, if X is uniformly bounded below (i. e.

$$X_q = 0 \quad \text{for} \quad q < n\ ),$$

then $\Sigma^n X : \mathbf{I} \to \mathbf{C(Mod\text{-}A)}$ is a positive complex and there is a chain of natural isomorphisms :

$$[\, E_I(-), X] \approx [\, E_I(-), \Omega^n \Sigma^n X] \approx [\, \Sigma^n E_I(-), \Sigma^n X] \approx$$
$$\approx [\, E_I(\Sigma^n -), \Sigma^n X] \approx [\, (\Sigma^n -), \mathrm{holim}_I \Sigma^n X] \approx [-, \Omega^n \,\mathrm{holim}_I \Sigma^n X]\ ;$$

thus $\mathrm{holim}_I X$ is the same as $\Omega^n \mathrm{holim}_I \Sigma^n X$.

Henceforth we restrict our attention to $\mathbf{C^+(Mod\text{-}A)}$, the subcategory of positive complexes and the corresponding functor categories.

Before we outline the construction we must recall some facts about double complexes, the basic reference being Hilton and Stammbach [11], Chapter V.

A double complex $B = \{B_{p,q}\}$ is a family of A-modules indexed by $Z \times Z$, together with families of maps

$$\{\partial'_{p,q} : B_{p,q} \to B_{p-1,\, q}\} \quad \text{and} \quad \{\partial''_{p,q} : B_{p,q} \to B_{p,\, q-1}\}$$

such that

$$\partial'\partial' = 0, \quad \partial''\partial'' = 0 \quad \text{and} \quad \partial''\partial' + \partial'\partial'' = 0\,.$$

Associated to each double complex B are two «total» complexes (see [11], p. 167); we choose to use the second form which is as follows :

$$(Tot\,\mathrm{B})_n = \prod_{p+q=n} \mathrm{B}_{p,q}$$

and, if $b = \{b_{p,q}\} \epsilon (Tot\,\mathrm{B})_n$, then ∂b is given by

$$(\partial b)_{p,q} = \partial' b_{p+1,q} + \partial'' b_{p,q+1}.$$

One of the principal examples we will be adapting is the chain complex of homomorphisms from a complex D to another complex E ; in this case ([11], p. 168-169) $\mathrm{B}_{p,q} = Hom_{\mathbf{A}}(\mathrm{D}_{-p}, \mathrm{E}_q)$,

$$(\partial' f)d = (-1)^{p+q+1} f(\partial d) \quad \text{for } d \epsilon \mathrm{D}_{p+1},\ f\colon \mathrm{D}_{-p} \to \mathrm{E}_q$$

and

$$(\partial'' f)(d) = \partial(f d) \qquad \text{for } d \epsilon \mathrm{D}_{-p},\ f\colon \mathrm{D}_{-p} \to \mathrm{E}_q.$$

Thus the associated total complex $Tot\,\mathrm{B} = Hom_{\mathbf{A}}(\mathrm{D}, \mathrm{E})$ has differential

$$(\partial^H f)_{p,q} = (-1)^{p+q} f_{p+1,q}\partial + \partial f_{p,q+1}$$

where

$$f = \{f_{p,q}\} \text{ and } f_{p,q}\colon \mathrm{D}_{-p} \to \mathrm{E}_q.$$

We will need only the positive dimensions, so restrict to $p+q = n > 0$.

If we let D and E be objects in $\mathbf{Hom}(\mathbf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}\mathbf{A}))$ and

$$\mathrm{B}_{p,q} = Hom(\mathrm{D}_{-p}, \mathrm{E}_q)$$

given by the *hom*-group in this functor category, we get a total complex $Tot\,\mathrm{B}$ such that

$$(Tot\,\mathrm{B})_n = \prod_{p+q=n} \mathrm{B}_{p,q} = \prod_{p \epsilon \mathbf{Z}} Hom(\mathrm{D}_p, \mathrm{E}_{p+n}),\quad n > 0,$$

and ∂^H as above. We write $\mathcal{H}om_{\mathbf{A}}(\mathrm{D}, \mathrm{E})$ for $Tot\,\mathrm{B}$.

The homotopy limit of a diagram $\mathrm{X}\colon \mathbf{I} \to \mathbf{C}^+(\mathbf{Mod}\text{-}\mathbf{A})$ is defined as $\mathcal{H}om_{\mathbf{A}}(\mathrm{A}(\mathbf{I}/\_), \mathrm{X})$, where $\mathrm{A}(\mathbf{I}/\_)$ is the diagram of complexes described below (cf. Bousfield and Kan [3], Chapter XI):

For each n, let $\mathbf{I}_n$ be the set of all sequences

$$u = (i_0 \xleftarrow{a_1} i_1 \leftarrow \dots \leftarrow i_{n-1} \xleftarrow{a_n} i_n)$$

of maps in $\mathbf{I}$ (consisting of n composable maps). Define maps

$$d_j : \mathbf{I}_n \to \mathbf{I}_{n-1} \quad \text{for } j = 0, \dots, n,$$

by

$$d_0(u) = (i_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_n} i_n),$$

$$d_i(u) = (i_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_j \alpha_{j+1}} \dots \xleftarrow{\alpha_n} i_n), \quad 0 < j < n,$$

$$d_n(u) = (i_0 \xleftarrow{\alpha_0} \dots\dots \xleftarrow{\alpha_{n-1}} i_{n-1}).$$

For any i in $\mathbf{I}$, $A(\mathbf{I}/_i)_n$ is the free A-module generated by all pairs $<\alpha, u>$ where

$$u = (i_0 \leftarrow \dots \leftarrow i_n) \in \mathbf{I} \quad \text{and} \quad \alpha : i_0 \to i.$$

Defining

$$d_0(<\alpha, u>) = <\alpha\alpha_1, d_0(u)> \quad \text{and} \quad d_i(<\alpha, u>) = <\alpha, d_i(u)>$$

for $i > 0$, we can put

$$\partial(<\alpha, u>) = \sum_{i=0}^{n} (-1)^j d_j(<\alpha, u>)$$

and extend linearly to get a chain complex $A(\mathbf{I}/_i)$.

If $\beta : i \to j$ is a map in $\mathbf{I}$, then $A(\mathbf{I}/_\beta) : A(\mathbf{I}/_i) \to A(\mathbf{I}/_j)$ is given by extending the map $<\alpha, u> \to <\beta\alpha, u>$ linearly. This gives a diagram of chain complexes.

We define $\mathrm{holim}_{\mathbf{I}} X$, for any diagram $X : \mathbf{I} \to \mathbf{C}(\mathbf{Mod}\text{-}A)$, to be

$$\mathcal{H}om_A(A(\mathbf{I}/\_), X).$$

If X is uniformly bounded, then $\mathrm{holim}_{\mathbf{I}} X$ is a «bounded below» complex and hence is in $\mathbf{C}(\mathbf{Mod}\text{-}A)$; if not then $\mathrm{holim}_{\mathbf{I}} X$ will not be bounded below, although it still may be «weakly equivalent» to a complex in $\mathbf{C}(\mathbf{Mod}\text{-}A)$.

If $f_{p,q} : A(\mathbf{I}/\_)_{-p} \to X_q$ is a morphism in $\mathbf{pro}(\mathbf{Mod}\text{-}A)$, then the value that $f_{p,q}(i_0)$ takes on the generator $<Id_{i_0}, u>$ for

$$u = (i_0 \leftarrow i_1 \leftarrow \dots \leftarrow i_p)$$

completely determines the value of $f_{p,q}(i)$ on $<a, u>$ for any $a : i_0 \to i$ in $\mathbf{I}$; moreover the generator $<Id_{i_0}, u>$ of $A(\mathbf{I}/_{i_0})_{-p}$ is not the image of anything else in $A(\mathbf{I}/\_)_{-p}$ (unless there is an isomorphism between i_0 and some other i in $\mathbf{I}$). Thus we get an isomorphism

$$B_{p,q} = \mathcal{H}om_{A}(A(\mathbf{I}/\_)_{-p}, X_q) \approx \prod_{u \in \mathbf{I}_p} X_q(i_0).$$

REMARKS. 1. This construction is, in fact, well known although not usually approached by this route. It occurs, as far as the author knows, for the first time in a series of short papers by Roos [21, 22 and 23]. An account of the applications of this «Roos complex», at least when $\mathbf{I}$ is a directed category, may be found in Jensen [12], Section 4. (The dual construction has been used by André [1] and Deheuvels [5] and would lead here to a homotopy colimit construction.)

2. It is helpful to compare this construction with the «cosimplicial replacement» method of Bousfield and Kan [3]. Their method is, of course, the non-additive, and therefore more difficult, version of this type of construction.

The first description of $\operatorname{holim}_{\mathbf{I}} X$ as $\mathcal{H}om_{A}(A(\mathbf{I}/\_), X)$ indicates that $\operatorname{holim}_{\mathbf{I}}$ is functorial in X as far as morphisms in $\mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$ are concerned. We have as yet not shown that it is the «homotopy limit» we required. The first step on the way to proving that is the following proposition.

PROPOSITION 4.1. *There is a natural isomorphism*

$$\mathcal{H}om(C \otimes_{A} A(\mathbf{I}/\_), X) \approx \mathcal{H}om(C, \operatorname{holim}_{\mathbf{I}} X),$$

where naturality in X *is restricted to diagrams of complexes which are uniformly bounded below.*

PROOF. First it should be mentioned that, for C in $\mathbf{C}^{+}(\mathbf{Mod}\text{-}A)$, the diagram $C \otimes_{A} A(\mathbf{I}/\_)$ is defined by, in dimension n and for i an object of $\mathbf{I}$:

$$(C \otimes_{A} A(\mathbf{I}/\_))_n(i) = (C \otimes_{A} A(\mathbf{I}/_i))_n = \bigoplus_{p+q=n} C_p \otimes_{A} A(\mathbf{I}/_i)_q,$$

$$\partial^{\otimes}(c \otimes a) = \partial c \otimes a + (-1)^p c \otimes \partial a, \quad \text{for } c \in C_p, \quad a \in A(\mathbf{I}/_i)_q.$$

The classical proof that $\otimes$ and Hom are adjoint can easily be adapted to give an isomorphism

$$\mathrm{Hom}(C_p \otimes_A A(I/\_)_q, X_r) \approx \mathrm{Hom}(C_p, \mathrm{Hom}(A(I/\_)_q, X_r)),$$

where the use of Hom in each case should be obvious. This isomorphism is natural and so we get a graded module isomorphism

$$\mathcal{H}om(C \otimes_A A(I/\_), X) \approx \mathcal{H}om(C, \mathcal{H}om(A(I/\_), X)).$$

(This works because we used the $\oplus$-total complex with $\otimes$ and the Π-total complex with $\mathcal{H}om$.)

It remains to check the differentials, but this is exactly as in the classical case (Hilton and Stammbach [11], p. 169-170).

The next step is to show that holim_I preserves weak equivalences.

PROPOSITION 4.2. *Let* $f: X \to Y$ *be a weak equivalence in*

$$\mathbf{Hom}(\mathbf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}\mathbf{A}));$$

then $\mathrm{holim}_I f: \mathrm{holim}_I X \to \mathrm{holim}_I Y$ *is also a weak equivalence.*

PROOF. The double complex for X is

$$B_{p,q}(X) = \begin{cases} \prod_{u \in I_{-p}} X(i_0) & \text{if } p \leqslant 0, \ q \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

The filtration of $Tot B(X)$ given by

$$F^p\, Tot B(X) = \prod_{r \leqslant p} B_{p,q}(X)$$

satisfies

$$\mathrm{colim}_p F^p\, Tot B(X) = Tot B(X)$$

and also

$$\lim_p (Tot B(X) / F^p\, Tot B(X)) = Tot B(X),$$

and similarly for $B(Y)$. The corresponding E_1-term of the spectral sequence related to this filtration is

$$E_1^{p,q}(X) = H^p(H_{q-p}(B(X), \partial''), \partial')$$

(using the notation of Hilton and Stammbach [11]) and the same for Y. Thus f induces an isomorphism

$$E_I^{p,q}(f): E_I^{p,q}(X) \to E_I^{p,q}(Y)$$

and this shows that

$$f_*: H(Tot\,B(X)) \to H(Tot\,B(Y))$$

is an isomorphism.

PROPOSITION 4.3. *If* $\not{f}$: C → D *is a weak equivalence in* $\mathbf{C}^+(\mathbf{Mod}\text{-}A)$, *then*

$$\not{f} \otimes_A A(\mathsf{I}/\_): C \otimes_A A(\mathsf{I}/\_) \to D \otimes_A A(\mathsf{I}/\_)$$

is a weak equivalence in $\mathbf{Hom}(\mathsf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$.

PROOF. This follows immediately from the Kunneth spectral sequence (see for example MacLane [13], Chapter XII, 12) and the comparison Theorem for spectral sequences.

THEOREM 4.4. *The functors*

$$? \otimes A(\mathsf{I}/\_): \mathbf{C}^+(\mathbf{Mod}\text{-}A) \to \mathbf{Hom}(\mathsf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$$

and

$$\mathrm{holim}_{\mathsf{I}}: \mathbf{Hom}(\mathsf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}A)) \to \mathbf{C}^+(\mathbf{Mod}\text{-}A)$$

induce functors (which will be denoted by the same symbols) on the corresponding homotopy categories. Moreover $? \otimes A(\mathsf{I}/\_) \dashv \mathrm{holim}_{\mathsf{I}}$ *as functors on the homotopy categories.*

PROOF. The first part follows from 4.2 and 4.3 and the second part is an immediate consequence of the first, together with the adjoint functor Lemma of Brown [4], p. 426.

COROLLARY 4.5. *The functors*

$$E_{\mathsf{I}}: \mathbf{Ho}\,\mathbf{C}^+(\mathbf{Mod}\text{-}A) \to \mathbf{Ho}\,\mathbf{Hom}(\mathsf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$$

and

$$\mathrm{holim}_{\mathsf{I}}: \mathbf{Ho}\,\mathbf{Hom}(\mathsf{I}, \mathbf{C}^+(\mathbf{Mod}\text{-}A)) \to \mathbf{Ho}\,\mathbf{C}^+(\mathbf{Mod}\text{-}A)$$

are adjoint.

PROOF. It suffices to show that $E_{\mathsf{I}}(\text{-})$ and $? \otimes A(\mathsf{I}/\_)$ are naturally isomor-

phic functors on $\mathbf{Ho\,C^{+}(Mod\text{-}A)}$. First we adapt a result of Bousfield and Kan, which is stated in [3], p. 293, for the simplicial analogue of this situation :

«$A(\mathbf{I}/_{-})$ is naturally weakly equivalent to $E_{\mathbf{I}}(A)$.»
This follows since, by [3] p. 293, the chain complex $A(\mathbf{I}/_{i})$ is an acyclic resolution of A for each i.

The weak equivalence is given locally by

$$1 \longrightarrow Id_i : \qquad A \longrightarrow A(\mathbf{I}/_{i}),$$

and on tensoring with C, we get $C \otimes_A A \to C \otimes_A A(\mathbf{I}/_{i})$ is a weak equivalence. Thus the corresponding morphism

$$E_{\mathbf{I}}(C) = C \otimes_A E_{\mathbf{I}}(A) \to C \otimes_A A(\mathbf{I}/_{-})$$

is a natural weak equivalence, which gives

$$[E_{\mathbf{I}}(C), X] \approx [C \otimes_A A(\mathbf{I}/_{-}), X] \approx [C, \mathrm{holim}_{\mathbf{I}} X]$$

and which proves the Corollary.

If we let $\mathbf{U.B.B(I, C(Mod\text{-}A))}$ denote the full subcategory of the category $\mathbf{Hom(I, C(Mod\text{-}A))}$, consisting of those diagrams which are uniformly bounded below, the suspension and loop space functors will, as shown earlier, give the following generalization of 4.5.

COROLLARY 4.6. *The functors*

$$E_{\mathbf{I}} : \mathbf{Ho\,C(Mod\text{-}A)} \to \mathbf{Ho\,U.B.B(I, C(Mod\text{-}A))}$$

and

$$\mathrm{holim}_{\mathbf{I}} : \mathbf{Ho\,U.B.B(I, C(Mod\text{-}A))} \to \mathbf{Ho\,C(Mod\text{-}A)}$$

are adjoint.

5. HOMOTOPY LIMITS. AN EXAMPLE.

We started the previous chapter with an example, from Gabriel and Zisman [7], and in this section it would seem useful to compare the general construction with theirs. First however it is necessary to alter their construction slightly due to a difference of convention. On page 103 of their book,

they define the complex of homomorphisms between two chain complexes but their differential is not the same as that which we have used above ; this has the effect of replacing, in the diagram which represents the double complex, the differentials in X' by their negative counterparts. The convention we are using alters, instead, the signs of alternate f's again leading to a double complex. Thus their diagram of page 104, which was reproduced in Chapter 4, is to be replaced by the following :

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots & (-2) \\
\cdots \arrow[r] & X_1' \arrow[r, "d_1^{X'}"] \arrow[u] & X_0' \arrow[r, "d_0^{X'}"] \arrow[u] & X_{-1}' \arrow[r] \arrow[u] & \cdots & (-1) \\
\cdots \arrow[r] & X_1 \arrow[r, "d_1^{X}"] \arrow[u, "f_1"] & X_0 \arrow[r, "d_0^{X}"] \arrow[u, "-f_0"'] & X_{-1} \arrow[r] \arrow[u, "f_{-1}"'] & \cdots & (0) \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots & (1) \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \\
 & (1) & (0) & (-1) & &
\end{tikzcd}$$

Thus the resulting total complex has $(\Gamma f)_n = X_n \oplus X'_{n+1}$ as before but the differential $\partial_n : (\Gamma f)_n \to (\Gamma f)_{n-1}$ is now given by the matrix

$$\partial_n = \begin{pmatrix} d_n^{\Psi} & (-1)^{n+1} f_n \\ 0 & d_n^{X'} \end{pmatrix} .$$

It is easy to check that the result is weakly equivalent to that given by Gabriel and Zisman's definition.

The description of the homotopy kernel of $f: X \to X'$ as a homotopy

limit is as the $\mathrm{holim}_{\mathbf{I}}$ of the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

and hence $\mathbf{I}$ is the category represented by the diagram

$$\begin{array}{ccc} & & 3 \\ & & \downarrow{\scriptstyle\beta} \\ 1 & \xrightarrow{\alpha} & 2 \end{array}$$

with, of course, identity maps at the objects as well. A simple calculation shows that the relevant double complex $\{B_{p,q}\}$ is given by

$$B_{0,q} = \underset{1}{X_q} \oplus \underset{2}{X'_q},$$

$$B_{-1,q} = \underset{11}{X_q} \oplus \underset{22}{X'_q} \oplus \underset{21}{X'_q} \oplus \underset{23}{X'_q},$$

$$B_{-2,q} = \underset{111}{X_q} \oplus \underset{222}{X'_q} \oplus \underset{221}{X'_q} \oplus \underset{211}{X'_q} \oplus \underset{223}{\Psi'_q} \oplus \underset{233}{X'_q},$$

and so on where the symbols on the line below refer to the element of $\mathbf{I}_{-p}$ which generates the corresponding factor. Thus 211 is the element of $\mathbf{I}_2$ given by

$$1 \xrightarrow{Id} 1 \xrightarrow{\alpha} 2.$$

Keeping this labelling we can write

$$\partial_{0,q}(a,b) = (0,0,(-1)^{q+1}(f_q(a)-b),(-1)^q b),$$

$$\partial'_{1,q}(a,b,c,d) = ((-1)^q a,(-1)^q b,(-1)^q b,(-1)^q f(a),(-1)^q b,0),$$

and so on. (The general form is quite easy to write down in this way but it is easier to write the general form, for a diagram $X : \mathbf{I} \to \mathbf{C}^+(\mathbf{Mod}\text{-}A)$, as

$$(\partial'_{-p,q} a)(a_1, \dots, a_{p+1}) =$$

$$X(a_1)(a(a_2,\dots,a_{p+1})) + \sum_{i=2}^{p+1}(-1)^i a(a_1,\dots,\hat{a}_i,\dots,a_{n+1}),$$

where for convenience of printing we have used the function notation for indices and have denoted

$$u = (i_0 \xleftarrow{\alpha_1} i_2 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_p} i_p) \in \mathbf{I}_p$$

by a list of maps $(\alpha_1, \dots, \alpha_p)$.)

It is very easily checked that the homology groups for fixed q give

$$H_{0,q} \approx Ker f_q ,$$
$$H_{-1,q} \approx X'_q / Im f_q = Coker f_q ,$$
$$H_{-p,q} \approx 0 \quad \text{for} \quad p \geqslant 2 \quad \text{or} \quad p < 0 .$$

These are the same as for the Gabriel and Zisman construction - fixed q being the q-th column.

If we denote their double complex by $\{ C_{p,q} \}$, we get

$$C_{p,q} = 0 \quad \text{if} \quad p \neq 0 \quad \text{or} \quad -1,$$
$$C_{0,q} = X_q , \qquad C_{-1,q} = X'_q ,$$
$$\partial'_{0,q} : C_{0,q} \to C_{1,q} \quad \text{is} \quad (-1)^{q+1} f_q$$

and there is an obvious map

$$\phi = \{ C_{p,q} \xrightarrow{\phi_{p,q}} B_{p,q} \}$$

given by

$$X_q \to X_q \oplus X'_q \quad \text{as first factor,}$$
$$X'_q \to X_q \oplus X'_q \oplus X'_q \oplus X'_q \quad \text{as third factor,}$$

which induces an isomorphism in homology. Either by using a simple spectral sequence argument or direct computation, one obtains a weak equivalence from their total complex to our $\mathrm{holim}_{\mathbf{I}}$ complex.

It immediately is noticeable that the greater part of the double complex which we constructed can be thrown away. An amended construction enables one to simplify the final form at the expense of earlier complication. When we defined $A(\mathbf{I}/_{-})$, we formed the diagram of free complexes constructed, in the usual manner, from the simplicial sets $\mathbf{I}/_i$, used by Bousfield and Kan

[3], Chapter XI, Section 2. As a result we included all the degenerate simplices of the $I/_i$ as generators of $A(I/_i)$, however those parts of $A(I/_i)$ which come from degenerate generators can contribute nothing to the final homotopy limit.

Thus the final homotopy kernel used only the *21* copy of X'_q in dimension *-1* and the *1* copy of X_q in dimension zero and hence only the *2* copy of X'_q was indexed by a non-degenerate simplex and yet this contributed nothing directly to the homotopy limit - this copy of X'_q was in fact used to get rid of the degenerate X'_q further down the p-dimensions of the double complex.

This complication can be eliminated if we replace the free complex $A(I/_i)$ by a weakly equivalent one, $N(I/_i)$, constructed as follows :

Using the simplicial face maps, that is the d_i's introduced for our definition of the boundary operator in $A(I/_i)$, we let

$$N(I/_i)_n = \bigcap_{j=1}^{n} Ker\, d_j\,, \quad \text{where } d_j : A(I/_i)_n \to A(I/_i)_{n-1}\,.$$

This complex is easily shown to be weakly equivalent to $A(I/_i)$ and hence

$$\mathcal{H}om(N(I/_-),X) \quad \text{and} \quad \mathcal{H}om(A(I/_-),X)$$

are weakly equivalent.

The complex corresponding to $\mathcal{H}om(N(I/_-),X)$ is given by

$$B'_{p,q}(X) = \prod_{u \in I'_{-p}} X(i_0),$$

with the same convention as before but where I'_{-p} is the subset of I_{-p} of those u of the form

$$u = (\, i_0 \xleftarrow{\alpha_1} i_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_p} i_p\,),$$

where all the α_i's are non-identity maps. Thus, in looking at $\mathrm{holim}_I\, X$, one can throw away those parts generated by u's in $I_{-p} - I'_{-p}$. In calculations of homotopy limits, this is clearly an important consideration.

6. HOMOTOPY LIMITS. EXTENSION TO Ho pro(C(Mod-A)).

The method of extending holim_I to a functor

$$\mathrm{holim}: \mathbf{Ho\,pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A)) \to \mathbf{Ho}\,\mathbf{C}^+(\mathbf{Mod}\text{-}A)$$

is basically simple. If $X : \mathbf{I} \to \mathbf{C}^+(\mathbf{Mod}\text{-}A)$ is a positive prochain complex then define $\mathrm{holim}\,X$ to be $\mathrm{holim}_{\mathbf{I}}\,X$; the only difficulty is how to check holim defined in this way is a functor. To do this it suffices to show that if $\phi : \mathbf{J} \to \mathbf{I}$ is a cofinal functor, then there is an induced weak equivalence

$$\phi^* : \mathrm{holim}_{\mathbf{J}}\,X\phi \to \mathrm{holim}_{\mathbf{I}}\,X .$$

This result is, as so much of the above, proved for the case of simplicial sets in Bousfield and Kan [3], Chapter XI, Section 9. With the previous results, which were algebraic analogues of their simplicial results, it has been worth giving separated, if linking, constructions and proofs as these are easier to understand than the simplicial constructions and proofs to which they correspond ; however with the result needed here, their «cofinality Theorem» of page 217, a proof of the algebraic analogue can be constructed merely by using the Π-total complex construction in place of their cosimplicial total complex construction. The algebraic result is no simpler to prove and their proof shows clearly why the algebraic form is true, thus we refer the reader to [3], p. 316-320, instead of needlessly duplicating the proof.

With this result the rest is easy. Given any positive prochain complexes

$$X : \mathbf{I} \to \mathbf{C}^+(\mathbf{Mod}\text{-}A) \quad \text{and} \quad Y : \mathbf{J} \to \mathbf{C}^+(\mathbf{Mod}\text{-}A)$$

and a map $f : X \to Y$ in $\mathbf{pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A))$, there is a small cofiltering category $\mathbf{M}_f$ with cofinal functors

$$\phi : \mathbf{M}_f \to \mathbf{I}, \quad \psi : \mathbf{M}_f \to \mathbf{J}$$

and a map

$$\bar{f} : X\phi \to Y\psi \quad \text{in} \quad \mathbf{Ho\,Hom}(\mathbf{M}_f, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$$

such that $\bar{f}$ and f are isomorphic in $\mathbf{pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A))$. Define

$$\mathrm{holim}\,f : \mathrm{holim}\,X \to \mathrm{holim}\,Y$$

by the diagram

$$\begin{tikzcd}
\mathrm{holim}\,X \arrow[r, dashed, "\mathrm{holim}\,f"] \arrow[d, "\approx"'] & \mathrm{holim}\,Y \arrow[d, "\approx"] \\
\mathrm{holim}_{\mathbf{M}_f} X\phi \arrow[r, "\mathrm{holim}_{\mathbf{M}_f} \bar{f}"] & \mathrm{holim}_{\mathbf{M}_f} Y\psi
\end{tikzcd}$$

as a map in $\mathbf{Ho}\,\mathbf{C}^+(\mathbf{Mod}\text{-}A)$, the vertical arrows being weak equivalences since ϕ and ψ are cofinal. Clearly choosing a different $(M, \bar{f})$ does not change holim f.

If $f : X \to Y$ is a basic weak equivalence in $\mathbf{pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A))$, then there is a reindexing $(M, \bar{f})$ where $\bar{f}$ is a weak equivalence in the category $\mathbf{Hom}(M, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$ and hence $\operatorname{holim}_M \bar{f}$ is a weak equivalence. With the map holim f defined, as above, «up to homotopy», clearly holim f is a weak equivalence and holim thus induces a functor from $\mathbf{Ho}\,\mathbf{pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A))$ to $\mathbf{Ho}\,\mathbf{C}^+(\mathbf{Mod}\text{-}A)$.

If $J : \mathbf{C}^+(\mathbf{Mod}\text{-}A) \to \mathbf{pro}(\mathbf{C}^+(\mathbf{Mod}\text{-}A))$ is the «constant» functor and K a positive complex, then any f in $[J(K), X]$ can be represented by a reindexed $\bar{f} : E_{M_f}(K) \to \bar{X}$ in some $\mathbf{Ho}\,\mathbf{Hom}(M_f, \mathbf{C}^+(\mathbf{Mod}\text{-}A))$, where $\bar{X}$ is weakly equivalent to X. $\bar{f}$ factors uniquely through $\operatorname{holim} \bar{X}$ and, since $\operatorname{holim} \bar{X}$ and holim X are weakly equivalent, this shows that

$$[J(K), X] \approx [K, \operatorname{holim} X].$$

Using the suspension-loop construction as before we can extend holim to the full subcategory of $\mathbf{pro}(\mathbf{C}(\mathbf{Mod}\text{-}A))$ given by the pro-objects which are cofinally bounded below; this would seem the largest subcategory on which it can be guaranteed that this construction works.

7. HOMOTOPY LIMITS. ELEMENTARY PROPERTIES AND APPLICATIONS.

One of the chief disadvantages of the inverse lim as a functor from $\mathbf{pro}(\mathbf{Mod}\text{-}A)$ is that it is not exact, only left exact. It is therefore natural to ask a similar question about holim, the only problem is to give an idea of what one means by «exact» in $\mathbf{Ho}\,\mathbf{pro}(\mathbf{C}(\mathbf{Mod}\text{-}A))$ and $\mathbf{Ho}\,\mathbf{C}(\mathbf{Mod}\text{-}A)$. To avoid this thorny problem, we replace «preserving exact sequences» by «preserving fibration sequences» in the sense of the abstract homotopy theory in the two categories. This use of fibration sequences is the natural form in which to ask the question as in $\mathbf{C}(\mathbf{Mod}\text{-}A)$ the fibration sequences are exact sequences.

A sequence in $\mathbf{pro}(\mathbf{C}(\mathbf{Mod}\text{-}A))$

$$0 \to L \to M \to N \to 0$$

is a fibration sequence in **pro(C(Mod-A))** if it is an exact sequence, in the usual sense, in that category.

A sequence of maps

$$0 \longrightarrow L \xrightarrow{\sigma} M \xrightarrow{\tau} N \longrightarrow 0$$

in **Ho pro(C(Mod-A))** (or **Ho C(Mod-A)**) is a fibration sequence if there is a fibration sequence

$$0 \longrightarrow L' \xrightarrow{s} M' \xrightarrow{t} N' \longrightarrow 0$$

in **pro(C(Mod-A))** (or **C(Mod-A)**) and a commutative diagram in the corresponding homotopy category

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{\gamma(s)} & M' & \xrightarrow{\gamma(t)} & N' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & L & \xrightarrow{\sigma} & M & \xrightarrow{\tau} & N & \longrightarrow & 0 \end{array}$$

where λ, μ, ν are weak equivalences and $\gamma : \mathbf{C} \to \mathbf{Ho\,C}$, as usual, denotes the canonical projection functor.

Since holim is a right adjoint as a functor from **pro C$^+$(Mod-A)** to **Ho C$^+$(Mod-A)**, it is «left exact» and the second description in terms of the product indicates that it is also «right exact» in as much as it sends fibrations (i.e. epimorphisms locally) to fibrations in **Ho C$^+$(Mod-A)**. It is then an easy matter to extend to **Ho pro(C$^+$(Mod-A))** and to see that holim sends fibration sequences to fibration sequences.

This property of holim is useful in the 1st application we shall study.

7.1. The derived functors of lim.

Let $C : \mathbf{I} \to \mathbf{Mod\text{-}A}$ be a pro-A-module and $C_{(q)} : \mathbf{I} \to \mathbf{C}^+(\mathbf{Mod\text{-}A})$ the corresponding procomplex with C in dimension q and zeroes elsewhere. The double complex for $C_{(q)}$ is

$$B_{r,s}(C_{(q)}) = \prod_{u \in \mathbf{I}'_{-r}} C_{(q)s}(i_0)$$

and hence

$$B_{r,s}(C_{(q)}) = 0 \quad \text{if} \quad s \neq q$$

and for $s = q$,

$$B_{r,q}(C_{(q)}) = \prod_{u \in I'_{-r}} C(i_0)$$

which is independent of q. It is well known that the complex $B_{*,q}(C_{(q)})$ has (co)homology

$$H^n(B_{*,q}(C_{(q)})) \approx \lim^{(n)} C$$

where, as usual, $\lim^{(n)}$ denotes the n^{th} derived functor of the limit functor (see for instance Jensen [12] or Roos [21]).

The total complex in this case is very easy to describe

$$(\operatorname{holim} C_{(q)})_n = \prod_{u \in I_{q-n}} C(i_0)$$

and so the homology of $\bar{C}_{(q)} = \operatorname{holim} C_{(q)}$ is given by

$$H_n(\bar{C}_{(q)}) = \lim^{(q-n)} C.$$

The fact that the derived functors of lim depend only on the isomorphism class in **pro(Mod-A)** and not on the particular indexing category **I** used is well known. It is, for example, stated by Verdier in his Note [25]; Duskin, in his Seminar Notes [6], expands Verdier's discussion and proves that $\lim^{(n)} C$ depends only on the pro-object C, not on the particular representation $C : \mathbf{I} \to \mathbf{Mod}\text{-}A$ used in the calculation of $\lim^{(n)} C$. Using some of the ideas of those notes we can shorten his proof considerably by applying the identification of the properties of $\operatorname{holim} C_{(q)}$ made above.

PROPOSITION 7.1.1. *If* $C : \mathbf{I} \to \mathbf{Mod}\text{-}A$ *is a pro-object in* **Mod-A** *and* $\phi : \mathbf{J} \to \mathbf{I}$ *a cofinal functor, then the canonical morphism*

$$\lim^{(n)} C \to \lim^{(n)} C\phi$$

is an isomorphism for all $n \geqslant 0$.

PROOF. We know that

$$\operatorname{holim} C_{(q)} \to \operatorname{holim} C_{(q)}\phi$$

is a weak equivalence, hence the zeroth homologies of each are isomorphic. However

$$H_0(\operatorname{holim} C_{(q)}) \approx \lim{}^{(q)} C$$

and

$$H_0(\operatorname{holim} C_{(q)}\phi) \approx \lim{}^{(q)} C\phi\ .$$

We follow Duskin in using this to prove that pro-modules which are isomorphic to zero have zero limit and all derived functors are zero on them.

PROPOSITION 7.1.2. *If* $C : \mathbf{I} \to \textbf{Mod-A}$ *is a pro-object in* **Mod-A**, *which is isomorphic to zero, then* $\lim^{(n)} C = 0$ *for all* $n \geqslant 0$.

PROOF. $C \approx 0$ if and only if for each i in $\mathbf{I}$ there is a

$$f(i) \in \mathbf{I} \quad \text{with} \quad C(f(i) \to i) = 0,$$

since this is precisely the condition that the two maps

$$\mathrm{Id}_C : C \to C \quad \text{and} \quad 0 : C \to C$$

are equal in

$$\lim\nolimits_{\mathbf{I}} (\operatorname{colim}_{\mathbf{I}} Hom(C(i), C(i'))).$$

Now let $\mathbf{I}^2$ be the category

$$\mathbf{I}^2 = \{ (j,i) \mid Hom(j,i) \neq \emptyset,\ i, j \in \mathbf{I} \}$$

and define a functor $\operatorname{Im} C : \mathbf{I}^2 \to \textbf{Mod-A}$ by

$$\operatorname{Im} C(i,j) = Im\ C(j \to i)\ .$$

The maps in $\mathbf{I}^2$ are of the form $(i,j) \to (i',j')$ if there is a diagram in $\mathbf{I}$:

$$\begin{array}{ccc} j & \longrightarrow & j' \\ \downarrow & & \downarrow \\ i & \longrightarrow & i' \end{array}$$

(recall that in the definition of cofiltering that is being used here,

$$Hom(i,i') = \emptyset \quad \text{or is a singleton}).$$

If $(i,j) \to (i',j')$ in $\mathbf{I}^2$, then $\operatorname{Im} C(i,j) \to \operatorname{Im} C(i',j')$ is given by

$$Im(C(j \to i)) \ \to\ Im(C(j \to i')) \subset Im(C(j' \to i'))$$

where the first map is composition with $i \to i'$.

The diagonal $\Delta : \mathbf{I} \to \mathbf{I}^2$ is cofinal and $\operatorname{Im} C\,\Delta = C$ so

$$\lim^{(n)} \operatorname{Im} C \approx \lim^{(n)} C .$$

However the subcategory

$$\mathbf{K} = \{ (i,j) \mid C(j \to i) = 0 \}$$

is also cofinal and $\operatorname{Im} C$ restricted to $\mathbf{K}$ is the zero pro-object with all

$$\operatorname{Im} C(i,j) = 0 \ ; \quad \text{thus} \quad \lim^{(n)} \operatorname{Im} C \approx 0$$

and the result follows.

Finally we prove that if C and D are isomorphic in **pro(Mod-A)** then

$$\lim^{(n)} C \approx \lim^{(n)} D \quad \text{for all} \quad n \geqslant 0 .$$

If $f : C \to D$ is a morphism then it can be realised in some functor category as $f_I : C_I \to D_I$ without influencing

$$\lim^{(n)} f : \lim^{(n)} C \to \lim^{(n)} D .$$

If f is an isomorphism in **pro(Mod-A)**, then so is f_I and hence $Ker\, f_I$ and $Coker\, f_I$ are zero objects in **pro(Mod-A)**. If we look at the exact sequences

$$0 \longrightarrow Ker\, f_I \longrightarrow C_I \xrightarrow{f_I} Im\, f_I \longrightarrow 0$$

and

$$0 \longrightarrow Im\, f_I \xrightarrow{f_I} D_I \longrightarrow Coker\, f_I \longrightarrow 0$$

and apply the connected sequence of derived functors of lim to both, it is immediate that

$$\lim^{(n)} f : \lim^{(n)} C_I \to \lim^{(n)} D_I$$

is an isomorphism for each n. Thus we have

PROPOSITION 7.1.3. *For each* $n \geqslant 0$, $\lim^{(n)}$ *sends isomorphic pro-objects to isomorphic modules.*

REMARK. Some of the results above suggest that it might be worth trying to define a homotopy limit starting with the category of all chain complexes instead of just the bounded below complexes. Although the resulting homotopy category would not be as rich in structure as that developed earlier in this paper, there would seem to be no place in the construction of holimits at

which this extra structure has been used in an essential way. Thus it seems that, with the restriction on only considering $n \geqslant 0$ in the complex of homomorphisms of Chapter 4 lifted, the holim of $C_{(0)}$ would have homology recording all the $\lim^{(n)} C$ and not just a finite number of them.

7.2. Holimits within Ho pro(C(Mod-A)).

We have defined holim as a functor on **Ho pro(C(Mod-A))** and also, for individual small categories **I**, on **Ho Hom(I, C(Mod-A))**. We can combine these to give homotopy limits of coherent diagrams of procomplexes.

If we have a small category **I** then we can form the functor category **Hom(I, C(Mod-A))**, and taking «pro» of this gives us the category of pro-**I**-diagrams in **C(Mod-A)**. There is an obvious structure in

$$\mathbf{pro}(\mathbf{Hom}(\mathbf{I}, \mathbf{C}(\mathbf{Mod}\text{-}A)))$$

lifted from that in **Hom(I, C(Mod-A))** and it is natural to expect that the homotopy limit

$$\mathrm{holim}_{\mathbf{I}} : \mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A)) \to \mathbf{C}^{+}(\mathbf{Mod}\text{-}A),$$

will extend to

$$\mathrm{holim}_{\mathbf{I}} : \mathbf{pro}(\mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))) \to \mathbf{pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$$

as follows : If

$$X : \mathbf{J} \to \mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$$

represents a pro-**I**-diagram of positive complexes, then we can use the functor

$$\mathrm{holim}_{\mathbf{I}} : \mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A)) \to \mathbf{C}^{+}(\mathbf{Mod}\text{-}A)$$

to give a pro-positive complex

$$\mathrm{holim}_{\mathbf{I}} X : \mathbf{J} \to \mathbf{C}^{+}(\mathbf{Mod}\text{-}A).$$

This pro-object does not depend on the representation of X since if $\phi : \mathbf{J}' \to \mathbf{J}$ is cofinal, then

$$(\mathrm{holim}_{\mathbf{I}} X)\phi = \mathrm{holim}_{\mathbf{I}}(X\phi);$$

so $\mathrm{holim}_{\mathbf{I}} X$ is a well defined pro-object in **C⁺(Mod-A)**.

If X and Y are weakly equivalent pro-**I**-diagrams of positive complexes

then it is almost immediate that $\mathrm{holim}_{\mathbf{I}} X$ and $\mathrm{holim}_{\mathbf{I}} Y$ are weakly equivalent pro-positive complexes, hence $\mathrm{holim}_{\mathbf{I}}$ extends to a functor

$$\mathrm{holim}_{\mathbf{I}} : \mathbf{Ho\,pro}(\mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))) \to \mathbf{Ho\,pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A)).$$

The use of $\mathbf{pro\,Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$ does give holimits, but it must be said that this is only for $\mathbf{I}$-diagrams with a high level of coherence. The more useful situation would be to look for holimits for objects in

$$\mathbf{Hom}(\mathbf{I}, \mathbf{pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A)))$$

but as yet there would seem to be no way of constructing $\mathrm{holim}_{\mathbf{I}}$ without trying to use the category $\mathbf{pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$ in place of $\mathbf{C}^{+}(\mathbf{Mod}\text{-}A)$ in all the proofs of Section 4 and I don't even know if that will work. There is, however, an easier way out if the set of objects of $\mathbf{I}$ is finite and $\mathbf{I}$ contains no loop, in which case given any

$$X : \mathbf{I} \to \mathbf{pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$$

we can use Artin and Mazur's uniform approximation result [2] to obtain a small cofiltering category $\mathbf{J}$ and a pro-$\mathbf{I}$-diagram

$$X' : \mathbf{J} \to \mathbf{Hom}(\mathbf{I}, \mathbf{C}^{+}(\mathbf{Mod}\text{-}A))$$

such that the $\mathbf{I}$-diagram given by

$$X''(i) : \mathbf{J} \to \mathbf{C}^{+}(\mathbf{Mod}\text{-}A) : [X''(i)](j) = [X'(j)](i)$$

is isomorphic to X in $\mathbf{Hom}(\mathbf{I}, \mathbf{pro}(\mathbf{C}^{+}(\mathbf{Mod}\text{-}A)))$. Thus for such diagrams as X we can define $\mathrm{holim}_{\mathbf{I}} X$ to be $\mathrm{holim}_{\mathbf{I}} X'$. Of course, this does not depend on the choice of X' and $\mathbf{J}$.

In particular this enables one to define the homotopy kernel of a map or more generally the homotopy equaliser of a pair of maps in $\mathbf{pro}(\mathbf{C}(\mathbf{Mod}\text{-}A))$, since the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is obviously able to be replaced by reindexing by

$$\begin{tikzcd} & 0 \arrow[d] \\ X_I \arrow[r, "f_I"] & Y_I \end{tikzcd}$$

and forming $\Gamma f_I(i)$ for each i in I gives the required homotopy kernel. (There is no difficulty over the use of $\mathbf{C}$ instead of $\mathbf{C}^+$ as each individual $X_I(i) \xrightarrow{f_I(i)} Y_I(i)$ has a holimit.)

It is to be hoped that the homotopy kernel construction in the category **pro(C(Mod-A))** will give rise to a «Whitehead-type» Theorem, i.e. a result which will state that if X and Y are «reasonably nice» and $f: X \to Y$ induces an isomorphism of the pro-homology graded modules

$$f_* : H_*(X) \to H_*(Y),$$

then f is a weak equivalence. The results for simplicial sets proved in [19] suggest some result may be possible.

8. «PROHOMOTOPICAL DIMENSION».

In this last section it is intended to put forward a tentative definition of «prohomotopical dimension». The author is not convinced that this is the «right» definition or that it is a suitable notion to study in depth, but the definition does seem to convey some information about the promodules and is related to the cohomology theory in the usual way.

It is worth noting that there already exist results related to the «classical» homological dimensions of promodules. The most striking feature of these results is that the cardinality of the indexing category determines to some extent the dimension of the promodule ; for this and other results of a similar nature the reader is referred to the following sources :

Jensen [12], Mitchell [14], Osofsky [15] and Goblot [8, 9].

The *right prohomotopical dimension of a pro-A-module* M is the smallest integer n such that $H^m(M;-)$ is the zero functor for all $m > n$. Similarly the *left prohomotopical dimension of* M is the smallest integer n such that $H^m(-,M)$ is the zero functor for all $m > n$.

These definitions are the obvious analogues of the projective and injective dimensions of right modules. In fact, restricted completely to the category **C(Mod-A)**, this is precisely what they are. It is not at first obvious that anything new has been introduced via these definitions; however this is easily seen by looking at the simple results given by the universal coefficient Theorem of Section 3.

Let A be a principal ideal domain and M a pro-A-module. Putting $C = F[M]$ in Theorem 3.5 gives

$$H_0(C) = M, \quad H_r(C) = 0 \text{ for } r > 0,$$

and hence that

$$H^0(M;F) = Hom(M,F),$$
$$H^1(M;F) = N_0(M;F) \subset Ext^1(M,F),$$
$$H^r(M;F) = 0 \text{ for } r > 1,$$

for any module F. Although $N_1(M;F)$ is defined using an explicit cofibrant C weakly equivalent to M, its appearance as $H^1(M;F)$ should convince one that it does not depend on that resolution.

This contrasts with the existence of a pro-abelian group M with

$$Ext^i(c(Z);M) \neq 0 \text{ for all integers } i$$

as constructed by Jensen [12]. Note also that replacing M by F and vice versa shows that

$$H^0(-;M) = Hom(-,M), \quad H^1(-;M) = N_1(-;M),$$
$$H^r(-;M) = 0 \quad \text{for } r > 1,$$

so both right and left prohomotopical dimensions are less than 2. If M is cofibrant, for example if $M(i)$ is free for each index i, then we can take the resolution $F[M]$ to be $0 \to M \to 0$ and so $B_0 = 0$. In this case

$$N_1(M;F) = Im(Hom(B_0,F) \to Ext^1(M,F)) = 0.$$

Taking $A = Z$ again and examining the case where $M = c(Z)$ and F is the famous «dyadic solenoid» indexed by the natural numbers with

$F(n) = Z$ for each n,

and $p_n^{n+1} : F(n+1) \to F(n)$ sends 1 to 2, we get

$$N_0(M, F) = 0 \quad \text{but} \quad Ext^1(M, F) = \lim{}^{(1)} F \neq 0 .$$

(A detailed calculation of $\lim^{(1)} F$ is to be found in Roos [23] .)

This example shows very well the difference between cohomological dimension and prohomotopical dimension. It seems likely that if $M : \mathbf{I} \to \mathbf{Mod\text{-}A}$ is a promodule, then the right prohomotopical dimension of M (r. proh. d. M) is bounded above by the *lim sup* of the projective dimensions of the $M(i)$, however I have not been able to prove this as yet. The problem is not a simple one since taking N to be a right A-module of maximal projective dimension and letting $\mathbf{I}$ be any small cofiltering category we can construct a pro-A-module M indexed by $\mathbf{I}$ with each $M(i) = N$ and for each morphism α in $\mathbf{I}$, $M(\alpha)$ being the zero map. It is easily shown that

$$\text{r. proh. d. } M = \text{l. proh. d. } M = 0$$

but

$$\mathit{lim\,sup}\ \text{p. d. } M(i) = \text{p. d. } N ,$$

which, depending on the ring, could be arbitrarily large.

It is hoped to pursue this problem and other related problems in a sequel to this paper. It is also hoped to investigate the existence and form of homotopy colimits within future notes.

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