On categories into which each concrete category can be embedded. II

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Given a contravariant functor $F$ from sets to sets, the category $S(F)$ has for objects pairs $(X, S)$, with $X$ a set and $S \subseteq FX$; morphisms are mappings $f: (X, S) \to (Y, T)$ such that $Ff(T) \subseteq S$. The paper characterizes those functors $F$ for which $S(F)$ is a universal category, i.e. every concrete category can be fully embedded into it. The characterization is very simple: $F$ must be nearly faithful, i.e. there must be a cardinal $\alpha$ such that for arbitrary mappings $f, g: X \to Y$ we have: if $f \neq g$, then either $Ff \neq Fg$ or $\text{card } f(X) < \alpha$, $\text{card } f(Y) < \alpha$.

The paper continues the author's previous characterization of covariant functors $F$ for which $S(F)$ (defined analogously) is binding. There are striking similarities between the two cases, yet the main result here has no analogy in the covariant case.

I

CONVENTIONS. Set denotes the category of sets and mappings.

The word *functor* will denote a contravariant set functor.

Let $e$ be a decomposition of a set $X$. Then the canonical mapping from $X$ to $X/e$ will be denoted by $e$, therefore the class of $e$ containing $x$ is denoted $e(x)$.

If $f: X \to Y$ is a mapping, then $\text{Ker } f$ is the canonical decomposition of $f$, i.e. $\text{Ker } f = \{ f^{-1}(y) \mid y \in \text{Im } f \}$.

The cardinal $\alpha$ is meant as the set of all ordinals with type less than $\alpha$; $\alpha^+$ denotes the cardinal successor of $\alpha$.

DEFINITION. A concrete category is called *universal* if every concrete category can be embedded into it.
THEOREM 1.1. The category $S(P^-)$ is universal.

PROOF. See [8].

NOTE. We recall the definition of the functor $P^-:

$$P^-(X) = \{ Z \mid Z \subseteq X \},$$

if $f: X \to Y$ then for every $Z \in P^Y$, $P^- f(Z) = f^{-1}(Z)$.

DEFINITION. A full embedding $\Psi$ from the concrete category $(K, U)$ to the concrete category $(L, V)$ is called strong if there exists a set functor $F: \text{Set} \to \text{Set}$ such that the diagram

$$
\begin{array}{ccc}
K & \xrightarrow{\Psi} & L \\
U \downarrow & & \downarrow V \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
$$

commutes.

DEFINITION. An object is rigid if it has no non-identical endomorphism.

Now we shall describe a "behaviour" of the functor $F$.

CONVENTION. Let $F$ be a functor. Then for a cardinal $\alpha$, $F^\alpha$ denotes the subfunctor of $F$ such that

$$F^\alpha_Y = \bigcup_{\text{card } Z \leq \alpha} \bigcup_{f \in Z^Y} \text{Im } Ff,$$

where $Z^Y$ is the set of all mappings from $Y$ to $Z$.

DEFINITION [4]. A cardinal $\alpha > 1$ is an unattainable cardinal of a functor $F$ if $F\alpha - F^{\alpha\alpha} \neq \emptyset$. Then put

$$F^\alpha_X = F^{\alpha+}_X - F^{\alpha}_X.$$

The class of all unattainable cardinals of $F$ is denoted by $\Lambda_F$.

THEOREM 1.2. Let $X$ be an infinite set such that there exists $\alpha \in \Lambda_F$, with $\alpha \leq \text{card } X$. Then $\text{card } FX \geq \text{card } 2^X$.

PROOF. See [4].

DEFINITION. Let $f, g: X \to Y$ be mappings onto. Then $f, g$ are diverse if there exists $Z \subseteq X$ such that either
A system $\mathcal{A}$ of mappings from $X$ to $Y$ is called diverse if arbitrary distinct mappings $f, g \in \mathcal{A}$ are diverse.

**Proposition 1.3.** If $\alpha$ is an unattainable cardinal of a functor $F$ and if $f, g : X \to \alpha$ are diverse, then

$$F f(F_\alpha \alpha) \cap F g(F_\alpha \alpha) = \emptyset.$$

**Proof.** See [4].

**Lemma 1.4.** Let $X$ be an infinite set. Then for every infinite cardinal $\alpha$ with $\alpha \leq \text{card } X$ there exists a diverse system $\mathcal{A}$ of mappings from $X$ to $\alpha$ such that $\text{card } 2^X$.

**Proof.** See [4].

**Definition.** We say that $f : X \to Y$ is coarser than $g : X \to Z$ if there exists $h : Z \to Y$ such that $h \circ g = f$.

**Proposition 1.5.** If $f : X \to Y$ then $\text{Im } F f = \bigcup \text{Im } F g$ where the union is taken over all $g : X \to \alpha$ coarser than $f$ and $\alpha \in A_F$.

**Proof.** See [7].

**Definition.** Let $F$ be a functor, $x \in FX$. Define

$$\mathcal{F}_F^X(x) = \{ e \mid e \text{ is a decomposition of } X, x \in \text{Im } F e \}.$$

Further we shall write

$$\| \mathcal{F}_F^X(x) \| = \min \{ \text{card } \text{Im } e \mid e \in \mathcal{F}_F^X(x) \}.$$

**Proposition 1.6.** Let $F$ be a functor; then $\alpha \in A_F$ iff there exists $x \in FX$ such that $\| \mathcal{F}_F^X(x) \| = \alpha$ for $\text{card } X \geq \alpha$. Further $\gamma \in F_\alpha Y$ iff $\| \mathcal{F}_F^Y(\gamma) \| = \alpha$.

**Proof.** Clearly $x \notin F_\alpha \alpha$. On the other hand $x \in \text{Im } F f$, where $f : X \to Y$ is onto and $\text{card } Y = \alpha$; therefore $x \in F_\alpha X$ and $\alpha \in A_F$. The rest is evident.

**Corollary 1.7.** If $\| \mathcal{F}_F^X(x) \|$ is finite, then there exists $e$ with

$$\mathcal{F}_F^X(x) = \{ e' \mid e \text{ is coarser than } e' \}.$$
PROOF. If \( e \neq e' \) and
\[
\text{card } \text{Im } e' = \text{card } \text{Im } e < \aleph_0,
\]
then \( e \) and \( e' \) are diverse and by Proposition 1.3 we get Corollary 1.7.

PROPOSITION 1.8. Let \( F \) be a functor, \( f : X \to Y \). Then for every \( y \in F Y \) it holds
\[
F^X_F(Ff(y)) \supset \{ e' \mid \text{there exists } e \in F^Y_F(y), \text{eo} f \text{ is coarser than } e' \}.
\]

PROOF is easy.

PROPOSITION 1.9. Let \( F \) be a functor, \( y \in F Y \). If for some \( \tilde{e} \in F^Y_F(y) \) and for some \( f : X \to Y \), \( \tilde{e} \circ f \) is onto, then
\[
F^X_F(Ff(y)) = \{ e' \mid \text{there exists } e \in F^Y_F(Y), \text{eo} f \text{ is coarser than } e' \}.
\]

PROOF. There exists a mapping \( h \) such that \( \tilde{e} \circ f \circ h = id \), then
\[
F \circ h \circ Ff(y) = F \circ \circ F \circ F \circ z = F \circ (z) = y,
\]
where \( z \in F(Y/\tilde{e}) \) with \( F \circ (z) = y \). Now by Proposition 1.8 we get Proposition 1.9.

DEFINITION. Let \( F \) be a functor. For \( x \in F X \) denote by \( e_x \) the finest decomposition which is coarser than each \( e \in F^X_F(x) \).

NOTE. If \( \alpha \) is a finite cardinal and \( x \in F_\alpha X \), then \( e_x \in F^X_F(x) \).

COROLLARY 1.10. Let \( F \) be a functor, \( \alpha \) a finite cardinal. If, for some \( f : X \to Y \) and for some \( y \in F_\alpha Y \) we have \( Ff(y) \in F_\alpha X \), then
\[
e_{Ff(y)} = \text{Ker}(e_{y} \circ f).
\]

PROOF is easy.

\[ \text{II} \]

**LEMMA 2.1.** The object \((6, V)\) is a rigid object of \( S(P^*) \), where
\[
V = \{ \{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \}, \{ 5 \}, \{ 0, 1 \}, \{ 0, 2 \}, \{ 0, 3 \}, \{ 0, 4 \},
\{ 1, 2 \}, \{ 1, 3 \}, \{ 1, 5 \}, \{ 2, 4 \}, \{ 2, 3, 4, 5 \}, \{ 1, 3, 4, 5 \},
\{ 1, 2, 4, 5 \}, \{ 1, 2, 3, 5 \}, \{ 0, 3, 4, 5 \}, \{ 0, 2, 4, 5 \}, \{ 0, 2, 3, 4 \},
\{ 0, 1, 3, 5 \}, \{ 1, 2, 3, 4, 5 \}, \{ 0, 2, 3, 4, 5 \}, \{ 0, 1, 3, 4, 5 \},
\{ 0, 1, 2, 4, 5 \}, \{ 0, 1, 2, 3, 5 \}, \{ 0, 1, 2, 3, 4 \} \}.
\]
PROOF. Since $\emptyset \not\in V$ and for every $i \in \mathbb{B}$, $\{i\} \in V$ we get, if $f: (\mathbb{B}, V) \rightarrow (\mathbb{B}, V)$ is a morphism of $S(P^*)$, then $f$ is a bijection. Therefore for every $\{i, j\} \in V$ we have

$$f^{-1}(\{i, j\}) = \{f^{-1}(i), f^{-1}(j)\} \in V.$$ 

Hence

$$\text{card}\{\{i, j\} \in V \mid j \in \mathbb{B} - \{i\}\} \leq \text{card}\{\{f^{-1}(i), j\} \in V \mid j \in \mathbb{B} - \{f^{-1}(i)\}\}$$

for every $i \in \mathbb{B}$ and thus we get for every $i \in \mathbb{B}$,

$$\text{card}\{\{i, j\} \in V \mid j \in \mathbb{B} - \{i\}\} = \text{card}\{\{f^{-1}(i), j\} \in V \mid j \in \mathbb{B} - \{f^{-1}(i)\}\}.$$ 

Hence

$$f^{-1}(5) = 5, \quad f^{-1}(2) = 2.$$ 

Now it is easy to verify that $f = \text{id}_\mathbb{B}$.

CONVENTION. An object of $S(\mathcal{F})$ will be called an $F$-space.

DEFINITION. Denote by $E(P^\ast)$ the full subcategory of $S(P^*)$ over those $(X, W)$ for which $Z \in W$ implies $X-Z \in W$ and $Z \not= \emptyset$.

PROPOSITION 2.2. There exists a strong embedding of $S(P^*)$ into $E(P^*)$.

PROOF. Let $(X, W)$ be a $P^*$-space. Define $\Psi(X, W) = (\mathbb{B} \wedge \mathbb{B}, W_S)$, where:

$$W_S = \{Z, X \wedge \mathbb{B} - Z \mid Z \in V, \text{card} Z < 3\} \cup$$

$$\cup\{0, 1, 2\} \cup Z, \{3, 4, 5\} \cup X-Z \mid Z \in W\};$$

for a given $f: (X_1, W_1) \rightarrow (X_2, W_2)$, $\Psi f = f \cup \text{id}_\mathbb{B}$. Clearly $\Psi$ is an embedding. We shall prove that it is also full. Let $f: \Psi(X_1, W_1) \rightarrow \Psi(X_2, W_2)$.

First we prove $f(X_1) \subseteq X_2$. Assume the contrary, i.e. $f(x) = i$ for some $x \in X_1$ and $i \in \mathbb{B}$. Then $f^{-1}(\{i\}) \in (W_1)_S$ and therefore either

$$f(\{0, 1, 2\}) = \{i\}, \quad \text{or} \quad f(\{3, 4, 5\}) = \{i\},$$

or $f((X_1 \wedge \mathbb{B}) - Z) = \{i\}$ for some $Z \in V$, $\text{card} Z < 3$.

Since $\emptyset \not\in (W_1)_S$ and $\{i\} \in (W_2)_S$ for every $i \in \mathbb{B}$, we have $6 \in \text{Im} f$ and therefore the last case is impossible. Further there exists

$$j \in \mathbb{B} \quad \text{such that} \quad \{i, j\} \in (W_2)_S$$

and so $f^{-1}(\{i, j\}) \in (W_1)_S$. Hence either $j \in f(X_1)$ or $f(X_1) = \{i\}$. In the former case we get again either
(we use the fact that \( \{ j \} \in (W_2)_S \) and so \( 6 \subset f^{-1}(\{ i, j \}) \); this is a contradiction. In the latter case

\[
f^{-1}(\{ i, j \}) = (X_1 \cup 6) - Z \quad \text{for some } Z \in V, \text{ card } Z < 3
\]

and therefore \( 6 \subset \text{Im} f \) and it is again a contradiction. Hence

\[
f(X_1) \subset X_2 \quad \text{and} \quad f(6) = 6.
\]

By Lemma 2.1 we have \( f/6 = \text{id}_6 \) and \( f/X_1 : (X_1, W_1) \to (X_2, W_2) \) is a morphism of \( S(P^-) \). Thus \( \Psi \) is a strong embedding.

**Proposition 2.3.** There exists a full subcategory \( \mathcal{M} \) of \( S(P^-) \) such that:

1. If \((X, W) \in \mathcal{M}\) then \( \emptyset \notin W \), \( \emptyset \notin W \) and for every \( x \in X \) there exists \( Z \in W \) with \( x \in Z \);

2. If \( f, g : (X_1, W_1) \to (X_2, W_2) \) and \( (X_1, W_1), (X_2, W_2) \in \mathcal{M} \), then there exists \( Z \in W_2 \) with \( f^{-1}(Z) \neq g^{-1}(Z) \);

3. There exists a strong embedding from \( S(P^-) \) to \( \mathcal{M} \).

**Proof.** Define \( \Phi : S(P^-) \to S(P^-) \) as follows: \( \Phi(X, W) = (X \cup 6, W_D) \) with

\[
W_D = V \cup \{0, 1, 2\} \cup Z \mid Z \in W \cup \{3, 4, 5\} \cup Z \mid Z \subset X \};
\]

for a given \( f : (X_1, W_1) \to (X_2, W_2) \) put \( \Phi f = f \cup \text{id}_6 \). Evidently \( \Phi \) is an embedding. Now, we shall prove that, if

\[
f : (X_1 \cup 6, (W_1)_6) \to (X_2 \cup 6, (W_2)_D),
\]

then \( f(X_1) \subset X_2, f(6) \subset 6 \). For every \( i \in 6, \)

\[
\{i\} \in (W_2)_D \quad \text{and} \quad \emptyset \notin (W_1)_D,
\]

therefore \( 6 \subset \text{Im} f \). We assume that for some \( x \in X_1, f(x) = i \in 6 \). Then we have \( f^{-1}(\{i\}) \in (W_1)_D \) and hence either

\[
f(\{0, 1, 2\}) = \{i\} \quad \text{or} \quad f(\{3, 4, 5\}) = \{i\}.
\]

Further there exists \( j \in 6 \) such that \( \{i, j\} \in (W_2)_D \), and therefore

\[
f^{-1}(\{i, j\}) \in (W_1)_D.
\]

We get that \( f^{-1}(\{j\}) \cap X_1 \neq \emptyset \) but then either

\[
f(\{0, 1, 2\}) = \{j\} \quad \text{or} \quad f(\{3, 4, 5\}) = \{j\}
\]

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and hence \( 6 \subseteq f^{-1}(\{i, j\}) \) - a contradiction. Thus
\[
f(X_1) \subseteq X_2, \quad f(6) \subseteq 6.
\]
By Lemma 2.1, \( f/6 = id_6 \) and therefore \( f/X_1: (X_1, W_1) \rightarrow (X_2, W_2) \) is a morphism of \( S(\mathcal{P}^*) \). Put \( M = \Phi(S(\mathcal{P}^*)) \). Evidently \( \Phi: S(\mathcal{P}^*) \rightarrow M \) is a strong embedding and \( M \) has the required properties.

**NOTE.** The set functor carrying \( \Phi \) (or \( \Psi \)) is \( I \vee C_6 \) where \( I \) is the identity functor and \( C_6 \) is the constant functor to \( 6 \).

**COROLLARY 2.4.** There exists a full subcategory \( \mathcal{J} \) of \( E(\mathcal{P}^*) \) such that:

1. if \( (X, W) \in \mathcal{J} \), then \( W \neq \emptyset \);
2. if \( f, g: (X, W) \rightarrow (Y, S) \) and \( (X, W), (Y, S) \in \mathcal{J} \), then there exists \( Z \in S \) with \( f^{-1}(Z) \neq g^{-1}(Z) \);
3. there exists a strong embedding from \( S(\mathcal{P}^*) \) to \( \mathcal{J} \).

**PROOF** follows from Propositions 2.2 and 2.3.

**THEOREM 2.5.** If \( 2 \in A_F \), then there exists a strong embedding from \( S(\mathcal{P}^*) \) to \( S(F) \).

**PROOF.** Via Proposition 2.2 it suffices to prove that there exists a strong embedding from \( E(\mathcal{P}^*) \) to \( S(F) \). Define
\[
\Omega(X, W) = (X, W_F),
\]
where
\[
W_F = \{ x \in F_2 X \mid \text{there exists } Z \in S, Z \in e_x \};
\]
for a given \( f: (X, W) \rightarrow (Y, S) \), define \( \Omega f = f \). Clearly \( \Omega \) is an embedding; let us prove that it is full. Let \( f: (X, W_F) \rightarrow (Y, S_F) \) be a morphism of \( S(F) \). Then for every \( x \in S_F \) it holds:
\[
\text{there exist } Z_1, Z_2 \in S \text{ such that } \{ Z_1, Z_2 \} = e_x
\]
(see Corollary 1.7 and the definition of \( \Omega \)). On the other hand for every \( Z \in S \) there exists \( x \in S_F \) such that \( \{ Z, Y-Z \} = e_x \).

Now by Corollary 1.10, we get that
\[
\{ f^{-1}(Z), X-f^{-1}(Z) \} = e_{F f(x)}.
\]
Thus \( f^{-1}(Z) \in W \) and \( f : (X, W) \rightarrow (Y, S) \) is a morphism of \( S(P^1) \).

III

CONSTRUCTION 3.1. Let \( F \) be a functor such that \( a \in A_F \), where \( a > 1 \) is a finite cardinal. Then there exists an object \((X, V)\) of \( S(F)\) such that:

1° \( \text{card } X = a + 4 \);
2° \( V \subset F_a X \);
3° for every \( x \in X \) there exist \( y_1, y_2 \in V \) such that \( \{ x \} \in e_{y_1}, \{ x \} \notin e_{y_2} \);
4° if \( x \in V \), then \( F(e_x)(F_a a) \subset V \);
5° for \( x \in X \) denote by

\[ gr x = \text{card } \{ Z | \text{card } Z > 1, x \in Z, \text{there exists } y \in V, Z \subset e_y \} , \]

then \( gr x > 1 \) with at most one exception;
6° \( (X, V) \) is rigid;
7° if \( a \geq 3 \) then there exists \( x \in X \) such that \( gr x = 1 \) and if for some \( Z \subset X, x \in Z, gr x = 1 \) and \( Z \in e_y \) for some \( y \in V \) then \( \text{card } Z < 3 \).

We shall construct these objects by induction in \( a \). For \( a = 2 \), the object exists by Lemma 2.1 and Theorem 2.5.

We assume that for \( a < n \) the construction is performed and \( n \in A_F \).

Let \( G \) be a functor with \( n - 1 \in A_G \). Let \((X', V')\) be a \( G \)-space fulfilling the conditions 1-7 for \( n - 1 \). We assume that \( a \notin X' \) and put \( X = X' \cup \{ a \} \). We choose an arbitrary decomposition \( \tilde{e} \) of \( X \) in \( n \) classes such that

\[ \text{card } \tilde{e} (a) = 2 \text{ and if } gr x = 1 \text{ for some } x \in X', \text{ then } x \in e(a). \]

Put

\[ V = \bigcup F e(F_a X / e) \cup F \tilde{e}(F_a X / \tilde{e}) \]

where the union is taken over all \( e \) such that \( \{ a \} \in e \) and the restriction of \( e \) to \( X' \) is equal to \( e_x \) for some \( x \in V' \). Let \( f : (X, V) \rightarrow (X, V) \) be a morphism of \( S(F) \); then \( f \) is a bijection by Corollary 1.10 and Condition 3 for \((X', V')\). Further \( gr a = 1 \) and for \( x \in X - \{ a \} \), \( gr x > 1 \). Hence \( f(a) = a \).

Clearly \( f : (X', V') \rightarrow (X', V') \) is a morphism of \( S(G) \) and hence \( f = id_X \).

The other required properties are easy to verify.
PROPOSITION 3.2. If $\alpha > 2$ is a finite cardinal and $\alpha \in \mathcal{A}_F$, then there exists a $F$-space $(X, V)$ and $x_0 \in X$ such that the following conditions hold:

1. $(X, V)$ is rigid, $V \subset \mathcal{F}_\alpha X$;
2. for every $\gamma \in V$, $\text{card }\gamma_i(E_{1-i}) \leq \alpha - 1$ for $i = 0, 1$, where $E_i$ is the class of $\epsilon \gamma_i$ containing $x_0$;
3. for every $x \in X$ there exist $\gamma_1, \gamma_2 \in V$ such that $\{x\} \in \epsilon \gamma_1 \setminus \epsilon \gamma_2$.

PROOF. Since $\alpha > 2$ we can choose by Construction 3.1 the $F$-space $(X, V)$ fulfilling Conditions 1-7. Therefore there exists $x \in X$ with $gr x = 1$. Put $x = x_0$. Clearly $((X, V), x_0)$ fulfills Conditions 1-3 from Proposition 3.2.

LEMMA 3.3. Let $\alpha$ be an infinite cardinal. Then for every set $X$ such that $\text{card }X = \alpha$ and every subsets $X_1, X_2, X_3$ of $X$ such that $X_1 \cap X_2 = X_2 \cap X_3 = \emptyset$, $\text{card }X_1 = \text{card }X_2 = \alpha$ and every mapping $f: X_2 \to X_3$ onto, there exists a diverse system $\mathcal{A}$ of mappings $g: X \to \alpha$ such that $\text{card }\mathcal{A} = \text{card }2^X$ and every $g \in \mathcal{A}$ fulfills:

1. for every $i \in \alpha$, $g^{-1}(\{i\}) \cap X_j \neq \emptyset$ for $j = 1, 2$;
2. there exists no non-constant mapping $h$ coarser than $g$ with $h(x) = h(f(x))$ for every $x \in X_2$.

PROOF. If there exists $Z \subset X_3$ such that $\text{card }Z < \alpha$ and $\text{card }f^{-1}(Z) = \alpha$, then put $Y = X_1 - Z$. By Lemma 1.4 there exists a diverse system $\mathcal{B}$ of mappings from $Y$ to $\alpha$ with $\text{card }\mathcal{B} = \text{card }2^X$. Now, for every $h \in \mathcal{B}$ we choose $g_h: X \to \alpha$ such that $g_h / Y = h$, $\text{card }g_h(Z) = 1 = \text{card }g_h(X_2 - f^{-1}(Z))$ and, for $i \in \alpha$, $g_h^{-1}(\{i\}) \cap f^{-1}(Z) \neq \emptyset$.

If there exists no $Z \subset X_3$ with this property, then we choose a decomposition $\{Z_1, Z_2\}$ of $X_2$ such that $\text{card }Z_1 = \text{card }Z_2 = \text{card }X_2 - f(Z_1) = \alpha$.

By Lemma 1.4 there exists a diverse system $\mathcal{B}$ of mappings from $Z_1$ to $\alpha$ with $\text{card }\mathcal{B} = \text{card }2^X$. Now, for every $h \in \mathcal{B}$ we choose $g_h: X \to \alpha$ such that
and for every $i \in \alpha,$

$$g_{i}^{-1}(\{i\}) \cap f(Z_{2}) \neq \emptyset, \quad g_{i}^{-1}(\{i\}) \cap X_{1} \neq \emptyset.$$  

Then $\mathcal{A} = \{g_{h} \mid h \in \mathcal{B}\}$ has the required properties.

**CONDITION A.** An $F$-space $(X, V)$ fulfills the condition $A$ if for arbitrary subsets $X_1, X_2, X_3$ of $X$ such that

$$\text{card } X_1 = \text{card } X_2 = \text{card } X \quad \text{and} \quad X_1 \cap X_2 = X_2 \cap X_3 = \emptyset$$

and for arbitrary mapping $f: X_2 \to X_3$ onto there exists $y \in V$ such that

- a) for every $e' \in F^X_F(y)$ there exists $e \in F^X_F(y)$ coarser than $e'$, such that $e(x) \cap X_i \neq \emptyset$ for every $x \in X$ and $i = 1, 2$;

- b) there exists $e \in F^X_F(y)$ such that for every $e' \in F^X_F(y)$ we have
  1° $e'^\cap * e \in F^X_F(y)$ and
  2° a mapping $h$ from $X$ is constant whenever

$$h(x) = h(f(x)) \quad \text{for every } x \in X_2$$

and $h$ is coarser than $e$ ($e'^\cap * e$ denotes a co-intersection of $e'$ and $e$).

**PROPOSITION 3.4.** Let $\alpha \in \Lambda_{F}$ be an infinite cardinal such that there exists $x \in F_{\alpha} \alpha$ with non-trivial $e_{x}$. Then there exists an $F$-space $(X, V)$ and a $x_{0}$ of $X$ such that:

- a) $(X, V)$ is rigid;

- b) $\text{card } X = \alpha, \quad V \subset F_{\alpha} X$;

- c) for every $a \in X$ there exists $y_{a} \in V$ such that $e_{y_{a}}(a) \neq e_{y_{a}}(x_{0})$;

- d) $(X, V)$ fulfills condition $A$.

**PROOF.** We choose a set $X$ with $\text{card } X = \alpha$ and choose $x_0 \in X$. For every $a$ we choose a bijection $f_{a}: X \to \alpha$ such that $e_{x}(f_{a}(a)) \neq e_{x}(f_{a}(x_{0}))$. Then

$$e_{Ff_{a}(x)}(a) \neq e_{Ff_{a}(x)}(x_{0}).$$

Put

$$\mathcal{B}_{0} = \{Ff_{a}(x) \mid a \in X \setminus \{x_{0}\}\}.$$  

Now, we choose bijections
\[ \Psi_1 : \text{card} 2^X \to \{ f : X \to X \mid f \neq \text{id}_X \}, \]
\[ \Psi_2 : \text{card} 2^X \to \{ (X_1, X_2, X_3, f) \mid \text{card} X_1 = \text{card} X_2 = \alpha, \]
\[ X_1 \cap X_2 = X_2 \cap X_3 = \emptyset, f : X_2 \to X_3 \text{ is onto} \}. \]

For \( i \in \text{card} 2^X \) denote
\[ C_i = \{ y \in F_a X \mid F(\Psi_1(i))(y) \neq y \}. \]

As an application of Lemma 1.4 we get that \( \text{card} C_i = \text{card} 2^X \). Further for \( \Psi_2(i) = (X_1, X_2, X_3, f) \) denote
\[ D_i = \{ y \in F_a X \mid \text{there exists } e \in \mathcal{F}_P^X(y) \text{ with} \]
\[ 1^o \text{ for every } x \in X, e(x) \cap X_j \neq \emptyset \text{ for } j = 1, 2, \]
\[ 2^o \text{ for every } e' \in \mathcal{F}_P^X(y), e' \cap e \in \mathcal{F}_P^X(y), \]
\[ 3^o \text{ there exists no non-constant mapping from } X \]
\[ \text{coarser than } e, \text{ with } h(x) = h(f(x)) \text{ for every } \]
\[ x \in X_2 \}. \]

If we construct the system \( \mathcal{A} \) from Lemma 3.3 for \( (X_1, X_2, X_3, f) = \Psi_2(i) \),

then for every \( g \in \mathcal{A} \) we have \( F g(x) \in D_i \) and therefore \( \text{card} D_i = \text{card} 2^X \).

Now we shall construct, by induction on \( i \in \text{card} 2^X \), sets \( B_i, C_i \) such that:
\[ \text{card} C_i < \text{card} 2^X, \quad B_i \subset C_i \cap F_a X \text{ for every } i. \]

Put \( C_0 = B_0 \). We assume that we have the sets \( B_i, C_i \) for \( i < j \). If \( j \) is a

limit ordinal, put
\[ B_j = \bigcup_{i<j} B_i, \quad C_j = \bigcup_{i<j} C_i. \]

If \( j = k + 1 \) then
\[ a) \text{ we choose } x_k^1 \in C_k \setminus C_k, \text{ such that } F(\Psi_1(k))(x_k^1) \notin C_k, \]
\[ b) \text{ we choose } x_k^2 \in D_k \setminus (C_k \cup \{ x_k^1, F(\Psi_1(k))(x_k^1) \}). \]

Put
\[ B_j = B_k \cup \{ x_k^1, x_k^2 \}, \quad C_j = C_k \cup \{ x_k^1, F(\Psi_1(k))(x_k^1), x_k^2 \}. \]

Evidently
\[ \text{card} C_j < \text{card} 2^X \text{ and } B_j \subset C_j \cap F_a X. \]

Put \( V = \bigcup B_j \) where the union is taken over all \( j \in \text{card} 2^X \). The \( F \)-space
\( (X, V) \) has the required properties.
LEMMA 3.5. Let $(X, V)$ be a rigid $F$-space. If $g_1 : Y_1 \to X$, $g_2 : Y_2 \to X$ are onto, then every mapping $h : Y_1 \to Y_2$ such that $F h (F g_2 (V)) \subseteq F g_1 (V)$ fulfills $g_2 \circ h = g_1$.

PROOF. Assume the contrary, i.e., $g_2 \circ h \neq g_1$. Then there exists $f : X \to Y_1$ such that

$$g_1 \circ f = id_X \quad \text{but} \quad g_2 \circ h \circ f \neq id_X.$$

Further it is clear to verify that $g_2 \circ h \circ f$ is an $F$-morphism of $(X, V)$ - a contradiction.

CONSTRUCTION 3.6. Let $\emptyset = ((X, V), x_0)$ be a couple where $(X, V)$ is an $F$-space, $\text{card} X > 1$ and $x_0 \in X$. For every set $Y$ and every $Z \subseteq Y$ define $g_Z : U \to X$ where $U = (Y \times (X \setminus \{ x_0 \})) \cup \{ x_0 \}$ as follows

$$g_Z (x_0) = x_0, \quad g_Z (y, x) = x_0 \quad \text{if} \quad y \in Y \setminus Z, \quad x \in X \setminus \{ x_0 \},$$

$$g_Z (y, x) = x \quad \text{if} \quad y \in Z, \quad x \in X \setminus \{ x_0 \}.$$

Define a functor $\Sigma_{\emptyset} : S (P^*) \to S (F)$:

$$\Sigma_{\emptyset}(Y, W) = (U, \bigcup_{Z \subseteq W} F g_Z (V)),$$

and for $f : (Y_1, W_1) \to (Y_2, W_2)$ put

$$\Sigma_{\emptyset} f = (f \times id_{X \setminus \{ x_0 \}}) \cup id_{\{ x_0 \}}.$$

Clearly $\Sigma_{\emptyset}$ is faithful and if $\Sigma_{\emptyset}$ is full, then $\Sigma_{\emptyset}$ is a strong embedding.

NOTE. If $Z_1, Z_2$ are distinct subsets of $Y$, then $g_{Z_1}$ and $g_{Z_2}$ are diverse.

LEMMA 3.7. Let $\emptyset = ((X, V), x_0)$ and $Z \subseteq Y$. If $y \in V$ fulfills:

let $e \in \mathcal{F}_F^X (y)$ such that for every $e' \in \mathcal{F}_F^X (y)$, $e' \cap * e \in \mathcal{F}_F^X (y)$,

then $F g_Z (y)$ fulfills:

for every $e \in \mathcal{F}_F^U (F g_Z (y))$, $e \cap \text{Ker} (e \circ g_Z) \in \mathcal{F}_F^U (F g_Z (y))$.

PROOF follows from Proposition 1.9.

LEMMA 3.8. Let $n$ be a finite unattainable cardinal of $F$. Let an $F$-space $(X, V)$ and $x_0 \in X$ fulfill the conditions 1-3 from Proposition 3.2. If

$$g : \Sigma_{\emptyset} (Y_1, W_1) \to \Sigma_{\emptyset} (Y_2, W_2)$$
is an $S(F)$-morphism and $\emptyset \not\in W_1$, then for every $Z_2 \in W_2$, $Z_2 \neq \emptyset$, there exists $Z_1 \in W_1$ such that $F(g_{Z_2} \circ g)(V) \subset Fg_{Z_1}(V)$.

PROOF. Assume the contrary, i.e., there exist $y_0, y_1 \in Fg_{Z_2}(V)$ such that:

$$Fg(y_0) \subseteq Fg_{Z_0}(V) \quad \text{and} \quad Fg(y_1) \subseteq Fg_{Z_1}(V)$$

where $Z_0 \neq Z_1$.

We can assume that there exists $v \in Z_0 - Z_1$. Put $Fg(y_i) = z_i$ for $i = 0, 1$.

Then

$$\text{card } g_{Z_0}(\{v\} \times (X - \{x_0\})) > n - 1 \quad \text{and} \quad \{v\} \times (X - \{x_0\}) \subset g_{Z_1}(x_0).$$

By Corollary 1.10 we get that

$$\text{card } e_t(\tilde{g}_{Z_2}(\{v\} \times (X - \{x_0\}))) > n - 1$$

and

$$e_t(x_0) \supseteq g_{Z_2}(\{v\} \times (X - \{x_0\}))$$

where $t_0, t_1 \in V$ such that $Fg_{Z_2}(t_i) = y_i$ for $i = 0, 1$; but this contradicts the Condition 2 from Proposition 3.2.

LEMMA 3.9. Let $\alpha$ be an infinite unattainable cardinal of $F$. Let $(X, V)$ and $x_0 \in X$ fulfill the conditions a-d from Proposition 3.4. If

$$g: \Sigma_0(Y_1, W_1) \to \Sigma_0(Y_2, W_2)$$

is an $S(F)$-morphism and $\emptyset \not\in W_1$, then for every $Z_2 \in W_2$, $Z_2 \neq \emptyset$ there exists $Z_1 \in W_1$ such that $F(g_{Z_2} \circ g)(V) \subset Fg_{Z_1}(V)$.

PROOF. Assume the contrary, i.e., there exist $y_0, y_1 \in Fg_{Z_2}(V)$ such that:

$$Fg(y_i) \subseteq Fg_{Z_i}(V) \quad \text{for} \quad i = 0, 1,$$

where $Z_0 \neq Z_1$. We can assume that $v \in Z_0 - Z_1$ and $w \in Z_1$. Put

$$U_i = (Y_i \times (X - \{x_0\})) \lor \{x_0\} \quad \text{for} \quad i = 1, 2.$$

By Proposition 1.8 and Lemma 3.7 there exists $e_i \in Fg_{Z_i}(V)$ such that $e_i$ is coarser than

$$\text{Ker } g_{Z_2} \subset g \cap \text{Ker } g_{Z_i} \quad \text{for} \quad i = 0, 1.$$

Therefore we get that
\[
\text{card}(g_{Z_2} \circ g(\{v\} \times (X - \{x_0\}))) - g_{Z_2} \circ g(\{w\} \times (X - \{x_0\})) = \\
= \text{card}(g_{Z_2} \circ g(\{v\} \times (X - \{x_0\}))) - g_{Z_2} \circ g(\{v\} \times (X - \{x_0\}))) = \alpha.
\]

Put
\[
X_1 = g_{Z_2} \circ g(\{v\} \times (X - \{x_0\})),
\]
\[
X_2 = g_{Z_2} \circ g(\{w\} \times (X - \{x_0\})),
\]
\[
X_3 = \{g_{Z_2} \circ g((v, x)) \mid g_{Z_2} \circ g((w, x)) \in X_2\},
\]
\[
f(g_{Z_2} \circ g((v, x))) = g_{Z_2} \circ g((v, x)).
\]

Since \(v \notin Z_1\) we have \(X_3 \cap X_2 = \emptyset\). Clearly \(X_1 \cap X_2 = \emptyset\) and \(f\) is onto. Therefore there exists \(t \in V\) from Condition A for \((X_1, X_2, X_3, f)\). Denote:
\[
y_3 = F g_{Z_2}(t), \quad z_3 = F g(y_3), \quad z_3 \in F g_{Z_3}(V).
\]

By a, Condition A we have \(v, w \in Z_3\). Further there is \(e_1 \in \mathcal{F}_F^U(z_3)\) coarser than \(\text{Ker} g_{Z_3}\) and \(\text{Ker}(e_0 \circ g_{Z_2} \circ g)\), where \(e_0 \in \mathcal{F}_F(t)\) from b of Condition A. Hence there exists a mapping \(p\) such that \(e_1 = p \circ e_0 \circ g_{Z_2} \circ g\). Since \(e_1\) is coarser than \(\text{Ker} g_{Z_3}\) we get that
\[
p \circ e_0(x) = p \circ e_0(f(x)) \quad \text{for every} \quad x \in X_2
\]
- and thus \(p \circ e_0\) is constant and so is \(e_1\). This contradicts
\[
z_3 \in F g_{Z_3}(V) \subset F g_{Z_3}(F_{\alpha} X) \subset F_{\alpha} U_1.
\]

**THEOREM 3.10.** Let \(\alpha > 2\) be an unattainable cardinal of \(F\). Then there exists a strong embedding from \(S(P^-)\) to \(S(F)\) whenever there exists \(x \in F_{\alpha} X\) such that \(e_x\) is non-trivial.

**PROOF.** Let \((X, V)\) be an \(F\)-space and \(x_0 \in X\) fulfilling the conditions 1-3 from Proposition 3.2 if \(\alpha\) is finite, or the conditions a-d from Proposition 3.4 if \(\alpha\) is infinite. We shall restrict the functor \(\Sigma_0\) to the category \(\mathcal{W}\), where \(\mathcal{W} = ((X, V), x_0)\). By Lemmas 3.8 and 3.9, if
\[
g : \Sigma_0(Y_1, W_1) \to \Sigma_0(Y_2, W_2)
\]
is an \(S(F)\)-morphism, then for every \(Z_2 \in W_2\) there exists
\[
Z_1 \in W_1 \quad \text{such that} \quad F(g_{Z_2} \circ g)(V) \subset F g_{Z_1}(V)
\]
and then by Lemma 3.5 \( g_{Z_2} \circ g = g_{Z_1} \). Since \( W_2 \) is a cover of \( Y_2 \), we get that

\[
g(Y_1 \times \{ a \}) \subseteq Y_3 \times \{ a \} \quad \text{for every } a \in X-\{ x_0 \} \quad \text{and } g(x_0) = x_0 .
\]

For every \( a \in X-\{ x_0 \} \), define \( g_a : Y_1 \rightarrow Y_2 \) as follows:

\[
g_a(y_1) = y_2 \quad \text{iff} \quad g((y_1, a)) = (y_2, a)
\]

Then \( g_a : (Y_1, W_1) \rightarrow (Y_2, W_2) \) is an \( S(P^-) \)-morphism for every \( a \in X-\{ x_0 \} \).

Further for every \( a, b \in X-\{ x_0 \} \) and every \( Z \in \mathcal{W} \),

\[
g_a^{-1}(Z) = g_b^{-1}(Z)
\]

Properties of \( \mathcal{M} \) imply that \( g_a = g_b \) for every \( a, b \in X-\{ x_0 \} \), thus \( \Sigma \bigcup \) is a strong embedding from \( \mathcal{M} \) to \( S(F) \). By Proposition 2.3, we obtain the Theorem.

**IV**

**DEFINITION [9].** We say that a colimit of a diagram \( D : \mathcal{D} \rightarrow \mathcal{K} \) is absolute if every covariant functor \( F : \mathcal{K} \rightarrow \mathcal{L} \) preserves it.

**LEMMA 4.1.** Let

\[
f_i : A \rightarrow B_i, \quad g_i : B_i \rightarrow C, \quad i = 1, 2,
\]

be morphisms of the category \( \mathcal{K} \) and let

\[
h_1 : B_1 \rightarrow A, \quad h_2 : C \rightarrow B_2
\]

be morphisms of \( \mathcal{K} \) such that

\[
g_1 \circ f_1 = g_2 \circ f_2, \quad f_2 \circ h_1 = h_2 \circ g_1, \quad f_1 \circ h_1 = id_{B_1}, \quad g_2 \circ h_2 = id_C.
\]

Then the push-out of \( f_i : A \rightarrow B_i \), \( i = 1, 2, \) is absolute.

**PROOF.** See [10].

**LEMMA 4.2.** Let \( f : X \rightarrow Y, \quad g : X \rightarrow Z \) be mappings onto such that there exists exactly one \( z \in Z \) with \( \text{card } g^{-1}(z) > 1 \). Then the push-out of \( f, g \) is absolute.

**PROOF.** Let \( h_1 : Y \rightarrow V, \quad h_2 : Z \rightarrow V \) be this push-out. Choose \( k_1 : Y \rightarrow X \)
such that \( f \circ k_1 = \text{id}_Y \) and
\[
k_1(y) \in g^{-1}(\{g(z)\}) \quad \text{whenever} \quad f^{-1}(\{y\}) \cap g^{-1}(\{g(z)\}) \neq \emptyset.
\]
Further we choose \( k_2 : V \to Z \) such that
\[
h_2 \circ k_2 = \text{id}_Y \quad \text{and} \quad k_2 \circ h_2 \circ g(z) = g(z)
\]
and
\[
k_2(v) = g \circ k_1(h_1^{-1}(v)) \quad \text{for} \quad v \in V - \{h_2 g(z)\}.
\]
It is easy to verify that the definition of \( k_2 \) is correct and \( g \circ k_1 = k_2 \circ h_1 \).

Now, Lemma 4.2 follows from Lemma 4.1.

**DEFINITION.** A decomposition \( e \) is called *finite* if every class of \( e \) is finite and \( e \) has only a finite number of non-singleton classes.

**COROLLARY 4.3.** Let \( F \) be a functor, \( x \in FX \). If \( e_x = \{X\} \), then every finite decomposition is an element of \( \mathcal{F}_F^X(x) \).

**PROOF.** If \( e \) is a finite decomposition, then \( e \) is a co-intersection of decompositions \( e_i, \ i = 1, 2, \ldots, n \) such that every \( e_i \) has only one non-singleton class. If \( e_i \in \mathcal{F}_F^X(x) \) then by induction we get from Lemma 4.2 that \( e \in \mathcal{F}_F^X(x) \). Further every decomposition \( e_i \) is a co-intersection of
\[
e_i^j, \ j = 1, 2, \ldots, m,
\]
where every decomposition \( e_i^j \) has only one non-singleton class and every class of \( e_i^j \) has at most two points. Now, by induction we get from Lemma 4.2 that
\[
e_i^j \in \mathcal{F}_F^X(x) \quad \text{whenever} \quad e_i^j \in \mathcal{F}_F^X(x).
\]
Since \( e_x = \{X\} \), it is easy to verify by Lemma 4.2 that every \( e_i^j \in \mathcal{F}_F^X(x) \).

We recall the definition of the union and the co-union.

**DEFINITION.** Let \( f : Y \to X, \ g : Z \to X \) be monomorphisms. The monomorphism \( h : V \to X \) is called a union of \( f, g \) (we shall write \( f \cup g = h \)) if there exist
\[
f_1 : Y \to V, \ g_1 : Z \to V \quad \text{such that} \quad h \circ f_1 = f, \ h \circ g_1 = g,
\]
and for every \( h' : V' \to X \) for which there exist
\[
f_2 : Y \to V', \ g_2 : Z \to V' \quad \text{such that} \quad h' \circ f_2 = f, \ h' \circ g_2 = g.
\]
there exists

\[ h_1 : V \rightarrow V' \quad \text{with} \quad h = h' \circ h_1. \]

The dual notion is a **co-union** (we shall write \( h = f \cup^* g \) if \( h \) is a co-union of \( f, g \)).

The covariant set functor \( F \) **preserves finite unions** if for arbitrary one-to-one mappings \( f : Y \rightarrow X, \ g : Z \rightarrow X \) we have

\[ F f \cup F g = F(f \cup g). \]

\( F \) **preserves unions with a finite set** if for arbitrary one-to-one mappings with \( Z \) finite,

we have \( F f \cup F g = F(f \cup g) \).

The contravariant set functor \( F \) **dualizes finite co-unions** if for arbitrary mappings \( f : X \rightarrow Y, \ g : X \rightarrow Z \) onto, we have

\[ F f \cup F g = F(f \cup^* g) ; \]

\( F \) **dualizes co-unions with a finite decomposition** if for arbitrary mappings \( f : X \rightarrow Y, \ g : X \rightarrow Z \) onto, where \( \text{Ker} g \) is a finite decomposition, we have

\[ F f \cup F g = F(f \cup^* g) . \]

**DEFINITION.** A set functor \( F \) (covariant or contravariant) is said to be **nearly faithful** if there exists a cardinal \( \alpha \) such that, for arbitrary mappings \( f \neq g : X \rightarrow Y, \ F f = F g \) implies that

\[ \text{card} f(X) < \alpha \quad \text{and} \quad \text{card} g(X) < \alpha . \]

**MAIN THEOREM 4.4.** Let \( F \) be a contravariant set functor. Then \( S(F) \) is a universal category if and only if \( F \) is nearly faithful.

To prove the Main Theorem we shall first prove a detailed characterization Theorem analogous to the covariant case (see below). Notice that the (covariant) identity functor \( I \) is faithful but \( S(I) \) is far from universal.

First we recall that a permutation with only one 2-cycle is called a transposition.

**THEOREM 4.5.** For a contravariant functor \( F \) the following conditions are
equivalent:

1. $S(F)$ is universal;
2. there exists a strong embedding from $S(P^*)$ to $S(F)$;
3. $S(F)$ has more than $\text{card } 2^{2^F\emptyset} + \text{card } 2^{F^1}$ non-isomorphic rigid spaces;
4. there exists a rigid $F$-space $(X, V)$ with $\text{card } X > 1$;
5. $F$ does not dualize co-unions with finite decomposition;
6. there exists a set $X$ and $x \in FX$ such that $e_x$ is non-trivial;
7. there exists a set $X$ and a transposition $t: X \to X$ such that $Ft \neq F\text{id}_X$;
8. there exists a cardinal $\alpha$ such that for every set $X$ with $\text{card } X \geq \alpha$ and every transposition $t: X \to X$ it holds $Ft \neq F\text{id}_X$.

PROOF. We recall that $6 \implies 2$ follows from Theorems 2.5 and 3.10. The implication $2 \implies 1$ follows from Theorem 1.1. The implications

$$1 \implies 3 \implies 4$$

are evident. Further $5 \implies 6$ follows from Corollary 4.3 and Proposition 1.5. The implication $8 \implies 7$ is obvious and so is

$$\text{non } 8 \implies \text{non } 4 \; \text{- thus } \; 4 \implies 8.$$ 

Therefore the theorem will be proved as soon as we show that $7 \implies 6$. Let $t: X \to X$ be a transposition such that $Ft \neq F\text{id}_X$, therefore there exists $x \in FX$ such that $Ft(x) \neq x$. Denote $a, b$ distinct points of $X$ such that

$$t(a) = b, \; t(b) = a.$$ 

If $e_x = \{X\}$, then there exists $e \in F^X_F(x)$ with $e(a) = \{a, b\}$ and

$$e(y) = \{y\} \; \text{for } y \in X - \{a, b\}.$$ 

Then $e = e_0 t$ and thus $Ft_F e = Fe$ - hence $Ft(x) = x$, because $x$ is in $\text{Im } F e$ - a contradiction.

$6 \implies 5$. Let $x \in FX$ such that $e_x$ is non-trivial. By Proposition 1.9 we can suppose that there exists $a \in X$ such that $\{a\} \in e_x$. We choose $b \in X$ such
that $e_x(a) \neq e_x(b)$. Let

$$e_1 = \{ X - \{ a \}, \{ a \} \}, \quad e_2 = \{ \{ a, b \} \cup \{ x \} \mid x \in X - \{ a, b \} \}.$$ 

We have that $x \notin \text{Im} F e_1 \cup \text{Im} F e_2$. On the other hand $e_2$ is a finite decomposition and $e_1 \cup^* e_2 = \text{id}_X$.

**PROOF OF MAIN THEOREM.** If $F$ is nearly faithful, then $F$ fulfills the condition 8 of Theorem 4.5 and thus $S(F)$ is universal. If $S(F)$ is universal, then $F$ fulfills the condition 7 of Theorem 4.5 and by [5] it is nearly faithful.

We recall the analogous results on covariant set functors. Here, instead of universality, those $F$ are characterized for which $S(F)$ is binding. (This means that the category of graphs is fully embeddable in $S(F)$ and, assuming the non-existence of too many non-measurable cardinals, it is the same as universality, see [3].) Let us remark that via Theorem 4.5, $S(F)$ is universal iff it is binding, for contravariant $F$.

For a covariant set functor $F$, denote for $x \in FX$,

$$\mathcal{F}_F^X(x) = \{ Z \subseteq X \mid x \in \text{Im} Fi, \ i : Z \to X \text{ is the inclusion} \}.$$ 

It is well-known (see [11]) that either $\mathcal{F}_F^X(x)$ is a filter or

$$\mathcal{F}_F^X(x) \cup \{ \emptyset \} = \exp Z = \{ Z \mid Z \subseteq X \}.$$ 

**THEOREM 4.6.** For a covariant set functor $F$ the following conditions are equivalent:

1. $S(F)$ is binding;
2. there exists a strong embedding from the category of graphs to $S(F)$;
3. $S(F)$ has more than

$$\text{card } 2^F \emptyset + (\text{card } 2^{F1} \cdot \text{card } 2^{2^{F1}})$$

non-isomorphic rigid spaces;
4. there exists a rigid $F$-space $(X, V)$ such that $\text{card } X > \text{card } 2^{F1}$;
5. $F$ does not preserve unions with a finite set;
6. there exists a set $X$ and $x \in FX$ such that $\mathcal{F}_F^X(x)$ is not an ultra-filter and $\cap Z \neq \emptyset$ where the intersection is taken over all $Z \subseteq \mathcal{F}_F^X(x)$;
7. there exists a set $X$, a transposition $t : X \to X$ and a mapping $p : X \to X$
such that $p(y) = y$ iff $t(y) \neq y$ and there exists $x \in FX$ with

$$Ft(x) \neq x \neq Fp(x);$$

$\exists^a$ there exists a cardinal $\alpha$ such that for every set $X$, $\text{card} X > \alpha$ and every transposition $t: X \to X$ and every mapping $p: X \to X$ such that $p(y) = y$ iff $t(y) \neq y$, there exists $x \in FX$ with $Ft(x) \neq x$, $Fp(x) \neq x$.

**COROLLARY 4.7.** In the finite set theory, $S(F)$ is a universal category if and only if $F$ is a non-constant functor, i.e. $S(F)$ is universal iff $F$ does not dualize co-unions.

Again, the situation for covariant functors was described in [6].

**THEOREM 4.8.** In the finite set theory, $S(F)$ is a universal category if and only if $F$ is not naturally equivalent to $(I \times C_M) \vee C_N$ for some $M, N$ (we recall that $C_M$ is the constant functor to $M$ and $I$ is the identity functor), i.e. $S(F)$ is universal iff $F$ does not preserve unions.

**EXAMPLE 4.9** (A non-constant functor which is not nearly faithful). Denote by $\beta$ the usual set functor, assigning to a set $X$ the set $\beta X$ of all ultrafilters on $X$, and to a mapping $f$ the mapping $\beta f$ which sends an ultrafilter $\mathcal{F}$ to the ultrafilter with base

$$\{ f(Z) | Z \in \mathcal{F} \}.$$ 

Let $\tilde{\beta}$ be the factor-functor of $\beta$ with $\mathcal{F}, \mathcal{G} \in \beta X$ merged iff either $\mathcal{F} = \mathcal{G}$ or $\mathcal{F}$ and $\mathcal{G}$ are fixed (i.e. $\cap Z \neq \emptyset$, where the intersection is taken over all $Z \in \mathcal{F}$). Then, clearly, $\tilde{\beta}$ merges transpositions and so does the (non-constant) functor $F = P^* \circ \tilde{\beta}$. 
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