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COHERENT PROHOMOTOPY THEORY

by Timothy PORTER

The use of pro-objects in homotopy theory is now well established. The theory has been used successfully by Artin-Mazur [2] and Sullivan [29] in various geometric situations and, implicitly or explicitly, by various people working in «Shape Theory»; see for example the surveys of Edwards [10] or Mardesič [17].

The method has been essentially the same each time: the use of some functorial construction to pass from some geometric or topological situation to the category pro(H) of pro-objects in the homotopy category of CW-complexes or, equivalently, of Kan simplicial sets, then to use general results about this latter category to derive results in the original situation. This approach suffers from various disadvantages not withstanding the notable successes attributable to it. The main drawbacks are that the morphisms in pro(H) are difficult to handle and that no notion of limit or colimit seems to be functorial for this category. Some time ago, both Professor Wall at Liverpool and Gavin Wraith at Sussex suggested to me that a better approach would be to consider some notion of homotopy defined in the category pro(Kan) of pro-objects in the category of Kan complexes and simplicial maps and then to study the corresponding homotopy category. This paper tries to develop this theory far enough to give some possibly interesting applications in geometric and topological situations.

To see why this approach is worth attempting, one shoud note some of the implicit occurences of this theory, or of similar theories. Firstly, in the original paper of Christie [7], several different possible generalisations of continuous maps are considered. One is basically Borsuk's fundamental sequence definition [4]; another is Mardesič-Segal notion of maps of ANR-sequences associated to the spaces [18], and hence is essentially the pro(H)-method referred to above. However Christie also considered va-

rious other definitions, e.g. his strong homotopy of strong net-maps, which have yet to be fully exploited. This notion of a strong net-map required that the homotopies used in the net-maps were themselves linked by homotopies, i.e. in modern parlance, were coherent at level 1. In my own papers on obstruction theory and Čech homotopy [25], it was necessary to restrict to compact metric spaces to be able to use some homotopy limit construction and, for this purpose (in [25, II], Section 3, 3.7 Théorème), the homotopies linking the various maps in the towers had to be specified; essentially the towers were made into coherent diagrams in the sense of Vogt [30]. Finally in [24], in the use of homotopy limits, it was again necessary to restrict attention to coherent maps of coherent diagrams.

There would be one objection to this type of theory, namely that it might not be possible to use the usual Čech, Vietoris, etc, methods to give coherent objects instead of the non-coherent objects in pro(H) which are more usually studied. In an appendix to this paper it is shown that, with the possible exception of the Verdier hypercovering construction, all the usual topological constructions give functors to the subclass of coherent diagrams in pro(H) and the difficulty of geometric applications via the hypercovering method can be avoided using Lubkin's construction instead.

We make extensive use of the properties of simplicial sets using especially the categorical approach of Gabriel and Zisman [11] (for other treatments, see May [19], Curtis [8], Lamotke [14] and Quillen [29]). The notation used for the principal categories involved is as follows:

Kan = category of Kan complexes and simplicial maps,

Kan_0 = category of pointed connected Kan complexes and simplicial pointed maps,

K = extended homotopy category of simplicial sets as defined by Artin-Mazur [2], page 10.

K_0 = corresponding category of pointed connected objects.

NOTE ADDED IN PROOF. Since this paper was written, Edwards and Hastings have published a theory which is in many ways similar to that exposed here (cf. Čech and Steenrod homotopy theories with applications to Geometric Topology, *Lecture Notes in Math.* 542, Springer, 1976).

1. THE COHERENT PROCATEGORY OF KAN COMPLEXES.

The basics of the theory of coherent homotopy diagrams are to be found in the following sources: Boardman and Vogt [3], Vogt [32] and also some sections of Bousfield and Kan [5]. Although Vogt's detailed treatment has not been carried out for the case of diagrams of Kan complexes, it would seem unnecessary to give details of such a treatment here. Thus, as far as we will be concerned, a coherent diagram in the category Kan of Kan complexes (or in Kan_0, the corresponding pointed category) will consist of a small category I, an assignment to each i of ob(I) of a Kan complex $X(i)$ and, to each map $i \xrightarrow{a} j$ in I, a corresponding map $X(a)$ from $X(i)$ to $X(j)$ in such a way that, if

$$i_n \xrightarrow{a_n} i_{n-1} \longrightarrow \dots \longrightarrow i_1 \xrightarrow{a_1} i_0$$

is a «chain» of maps in I, then, in the corresponding «ordered n-simplex» diagram given by the maps

$$X(i_n) \longrightarrow X(i_{n-1}) \longrightarrow \dots \longrightarrow X(i_1) \longrightarrow X(i_0)$$

where the «vertices» are the $X(i)$ and the directed edges are the $X(a)$, each of the «2-simplexes» correspond to a homotopy between the composite of the two composable edges and the third edge; each «3-simplex» corresponds to a homotopy between the various composed homotopies given by its 2-dimensional faces and so on. (If any a_i is the identity the resulting homotopies are chosen to be identities as well.)

Thus to the «2-simplex»

$$i_2 \xrightarrow{a_2} i_1 \xrightarrow{a_1} i_0 \quad \text{in I},$$

there corresponds a «2-simplex»

$$\begin{tikzcd} & X(i_1) \arrow[dr, "X(a_1)"] & \\ X(i_2) \arrow[ur, "X(a_2)"] \arrow[rr, "X(a_1 a_2)"'] & & X(i_0) \end{tikzcd}$$

and a specified homotopy

$$H : X(i_2)\times\Delta[1] \longrightarrow X(i_0)$$

between $X(a_1)X(a_2)$ and $X(a_1a_2)$. Similarly to

$$i_3 \xrightarrow{a_3} i_2 \xrightarrow{a_2} i_1 \xrightarrow{a_1} i_0$$

there corresponds a «tetrahedron»

$$\begin{tikzcd}
X(i_2) \arrow[drr, "X(a_1a_2)"] \arrow[ddr, "X(a_2)"] & & \\
& & X(i_0) \\
X(i_3) \arrow[uu, "X(a_3)"] \arrow[urr, "X(a_1a_2a_3)"] \arrow[r, "X(a_2a_3)"'] & X(i_1) \arrow[ur] &
\end{tikzcd}$$

with specified homotopies

$$H_0 : X(a_2)\,X(a_3) \overset{\sim}{\to} X(a_2\,a_3),\quad H_1 : X(a_1a_2)\,X(a_3) \overset{\sim}{\to} X(a_1a_2a_3),$$
$$H_2 : X(a_1)\,X(a_2a_3) \overset{\sim}{\to} X(a_1a_2a_3),\quad H_3 : X(a_1)\,X(a_2) \overset{\sim}{\to} X(a_1a_2)$$

and a specified map, a «higher homotopy»

$$H : X(i_3)\times\Delta[1]\times\Delta[1] \longrightarrow X(i_0)$$

which fills in the square

$$\begin{tikzcd}
X(a_1\,a_2)X(a_3) \arrow[d, "H_1"'] \arrow[dr, phantom, "H"] & X(a_1)X(a_2)X(a_3) \arrow[l, "H_3X(a_3)"'] \arrow[d, "X(a_1)H_0"] \\
X(a_1\,a_2\,a_3) \arrow[r, leftrightarrow, "H_2"'] & X(a_1)X(a_2a_3)
\end{tikzcd}$$

And so on in higher dimensions, see Boardman and Vogt [3] and Vogt [32] for details.

If we denote by L_n the category of which the objects are the integers $0, 1, \dots, n$ and in which $hom(i, j) = \emptyset$ if $j < i$ and contains a unique morphism otherwise, then a coherent map between I-indexed coherent

diagrams X and Y is a coherent $I \times L_{1}$-diagram which agrees with X on $I \times 0$ and Y on $I \times 1$.

The category of I-indexed coherent diagrams in Kan or Kan_0 is insufficient for our purposes; we need to have coherent morphisms between an I-indexed X and a J-indexed Y where I and J are not necessarily the same. There seems to be no very neat way to describe the intuitive definition given below but there is a very neat way to describe a homotopy category related to the construction.

Intuitively a *coherent pro-map* f between coherent pro-objects X and Y, as considered above, is a pro-map between X and Y as pro-objects (in the sense of Artin-Mazur, Appendix [2]) so that the resulting diagram, consisting of X and Y and all the linking maps, which represents f, has specified homotopies which make it coherent.

More precisely, a coherent pro-map $f:(X,I) \to (Y,J)$ is a collection (M, ϕ_I, ϕ_J, f_M), where M is a small cofiltering category,

$$\phi_I : M \to I \quad \text{and} \quad \phi_J : M \to J$$

are final functors and $f_M : X\phi_I \to Y\phi_J$ is a coherent map of M-indexed coherent diagrams.

The definition works because reindexing by M via ϕ_I and ϕ_J does not destroy coherence. The second definition corresponds to the intuitive one because any pro-map f in the usual sense can be re-indexed using Artin-Mazur [2], A.3.2, to give a map of M-indexed diagrams for some M associated with f ; re-indexing via cofinal functors gives a «coherent isomorphism» in an obvious sense and so f is intuitively coherent iff the re-indexed map is coherent in the M-indexed sense.

REMARK. It is important to remember that many different coherent diagrams may correspond to the same pro-object in the homotopy category; similarly for coherent pro-maps.

As Vogt remarks ([32], page 20) it is virtually impossible to define composition of these coherent maps in such a way as to ensure that this composition is associative; however following his treatment we can define

homotopy of coherent maps and hence a homotopy category.

Firstly we define when two coherent maps, which are I-indexed, are «simplicially homotopic»; $f, g: X \to Y$ are simplicially homotopic if there is a coherent $I \times L_2$-diagram α with

$$d^0(\alpha) = f, \quad d^1(\alpha) = g \quad \text{and} \quad d^2(\alpha) = s^0(X)$$

where the d^i and s^i are the obvious induced face and degeneracy operators.

If $f, g: X \to Y$, X is I-indexed, Y is J-indexed, f is the coherent pro-map $(M_f, \phi_I, \phi_J, f_M)$, and g the coherent pro-map $(M_g, \gamma_I, \gamma_J, g_M)$, then we can re-index both of f and g to get

$$M' = M'_f = M'_g, \quad \phi'_I = \gamma'_I, \quad \text{etc...},$$

and two coherent M'-indexed maps, $f'_{M'}$ and $g'_{M'}$ from $X\phi'_I$ to $Y\phi'_J$; we say the original maps were simplicially homotopic if f'_M and $g'_{M'}$ are simplicially homotopic as M'-indexed coherent maps.

As before, the definition does not depend on the choice of M_f, M_g or M', since the restriction to any new cofinal functors will give an equivalent simplicial homotopy. We can form up a homotopy category Copro(Kan) by this means. If we require a pointed version Copro(Kan_0), we merely have to ensure that all maps and homotopies in all the coherent diagrams are base-point preserving.

The categories Copro(Kan) and Copro(Kan_0) are the ones which will be studied in the sequel; however, although they are fairly «concrete» in the way they are defined, it is easier to define abstract homotopy categories which are equivalent to them and which are easier to study.

In the work of Boardman and Vogt [3], page 140-145 and again in Vogt [32], page 29-30, it is shown, for example, that the category of coherent I-indexed diagrams of well-pointed topological spaces and simplicial homotopy classes of maps is equivalent to a category of fractions of Top_0^I where a map $f: X \to Y$ in Top_0^I is formally inverted if, for all i in I, $f(i): X(i) \to Y(i)$ is a homotopy equivalence. An examination of this result shows that a very similar result holds for

Copro(Kan) and Copro(Kan_0);

they are equivalent to categories of fractions of pro(Kan) and pro(Kan_0).

Let Σ be the class of morphisms in pro(Kan) generated under composition by the class of isomorphisms and «level weak equivalences». A level weak equivalence is the image, under the canonical functor

$$Kan^I \to pro(Kan)$$

of some $f: X \to Y$ in Kan^I such that each $f(i): X(i) \to Y(i)$ is a homotopy equivalence where I is allowed to be any cofiltering small category.

Similarly one defines Σ_0 in pro(Kan_0). In either case it is fairly easy to check that Σ (resp. Σ_0) admits a calculus of fractions in the sense of Gabriel and Zisman [11] and so one may form the categories

$$pro(Kan)[\Sigma^{-1}] \quad \text{and} \quad pro(Kan_0)[\Sigma_0^{-1}]$$

which will be denoted by

$$Hopro(Kan) \quad \text{and} \quad Hopro(Kan_0)$$

respectively. If one thinks of pro(Kan) as being a class of categories of functors Kan^I, for different I, «glued» together by cofinality relations, it is then fairly obvious that one can extend the equivalences given by Boardman and Vogt [3] to one between Hopro(Kan) and Copro(Kan) and one between Hopro(Kan_0) and Copro(Kan_0). One of the main reasons for using Hopro(Kan) and Hopro(Kan_0) rather than

$$Copro(Kan) \quad \text{and} \quad Copro(Kan_0)$$

is that one has, almost immediately, the type of abstract homotopy theory discussed by Quillen [29] and Brown [6]. The details of this structure will appear elsewhere [26], but briefly we have the following definitions:

Basic fibrations: A map $f: X \to Y$ is a basic fibration if by reindexing one obtains a «level fibration» $f_I: X_I \to Y_I$, that is a fibration in Kan^I (or ${Kan_0}^I$) for some I.

Fibrations: A fibration is a composite of basic fibrations.

Basic cofibrations: As above for «basic fibration» but with «level fibration» changed to «level cofibration».

Cofibrations: A cofibration is a composite of basic cofibrations.

Path space object: If X is in pro(Kan) then X^I is obtained by applying the usual path space functor

$$(\ )^I = \hom_{Kan}(\Delta[1],\ )$$

to X, index-wise. Similarly for pro(Kan_0).

Cylinder object: If $X: I \to Kan$, then $X \times I: I \to Kan$ is given by

$$X \times I(i) = X(i) \times I.$$

Similarly for pro(Kan_0).

The resulting structure is weaker than Brown's [6] but it still gives enough form for the applications considered here.

In the pointed case we get, for each X in Hopro(Kan_0), a loop object ΩX which has a natural group structure. ΩX is the fibre of

$$X^I \xrightarrow{(d_0, d_1)} X \times X$$

where the product is taken objectwise in pro(Kan_0). As proved in general by Brown [6], if $p: E \to B$ is a fibration in pro(Kan_0) with fibre F, then there is a natural right action $a: F \times \Omega B \to F$ of ΩB on F in Hopro(Kan_0).

Using the same notation as Quillen [27], we write

$$[X, Y] = \mathrm{Hom}_{Hopro(Kan_0)}(X, Y);$$

similarly for the unpointed case.

Again using Brown's paper [6] (and Porter [26]), we get the following: We will say that a sequence

$$F \to E \to B \quad \text{in} \quad Hopro(Kan_0)$$

is a fibration sequence if there is given an action

$$F \times \Omega B \xrightarrow{a} F \quad \text{in} \quad Hopro(Kan_0)$$

which, together with the sequence, is isomorphic to the image of a sequence and action defined by an actual fibration in pro(Kan_0), by the canonical functor pro(Kan_0) $\to$ Hopro(Kan_0).

PROPOSITION 1.1 (Brown [6], pages 432-433 and Porter [26]). *Given a*

fibration sequence

$$F \xrightarrow{i} E \xrightarrow{p} B, \quad F \times \Omega B \xrightarrow{a} F,$$

let i' *be the composite*

$$\Omega B \xrightarrow{(e,j)} F \times \Omega B \xrightarrow{a} F$$

where: $\Omega B \overset{e}{\to} F$ *is the unique map* $\Omega B \to e \to F$ *(for* e *the initial-terminal object and* $j : \Omega B \to \Omega B$ *in* Hopro(Kan_0) *is the group inverse for* ΩB. *Further let* a' *be the composite*

$$\Omega B \times \Omega E \xrightarrow{id \times \Omega p} \Omega B \times \Omega B \xrightarrow{m} \Omega B$$

*(*m *the multiplication). Then*

$$\Omega B \xrightarrow{i'} F \xrightarrow{i} E, \quad \Omega B \times \Omega E \xrightarrow{a'} \Omega B$$

is a fibration sequence and the sequence

$$\dots \to [A, \Omega^{q+1} B] \to [A, \Omega^q F] \to [A, \Omega^q E] \to [A, \Omega^q B] \to [A, \Omega^{q-1} F] \to ..$$

is exact (in the sense of [29], *Sect. 1.3.8) naturally in* A.

In fact the structure of Hopro(Kan_0) and pro(Kan_0) are strong enough to allow one to define local cofibrations, local cylinder objects and a suspension functor Σ in pro(Kan_0), all in the obvious way. One has also the usual adjointness $[\Sigma A, B] = [A, \Omega B]$. It is unfortunate that, although there is a homotopy theory «à la Brown» based on fibrations in pro(Kan_0) and a dual theory (see [26]) based on cofibrations, these two theories do not link together to form a pointed Quillen model category [29], since no axiom such as Quillen's (M1) seems to hold between the two classes of maps.

Finally a word on homotopy limits and colimits; these have been studied by Bousfield and Kan [5] and Vogt [32], and were used by me in [27] in the study of pro-simplicial sets. As far as we shall be concerned, the only property that we shall need is that Holim is right adjoint to the inclusion functor (constant functor)

$$J : K_0 \longrightarrow \mathrm{Hopro}(\mathrm{Kan}_0)$$

or, for that matter, into any homotopy category of diagrams, whilst Hocolim is left adjoint in the dual situation. Hence they are functorial in Hopro(Kan_0). Any other results that we use will be called for at need.

2. HOMOTOPY AND HOMOLOGY OF COHERENT DIAGRAMS.

Let $X \xrightarrow{f} Y$ be a map in Hopro(Kan_0). Because Y^I exists we can form a homotopy limit Γf defined by the diagram

$$\begin{tikzcd} \Gamma f \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and, by the general methods of Gabriel and Zisman [11], Chapter V, we can construct for any A in Hopro(Kan_0) an exact sequence

$$\to [A,\Omega^n \Gamma f] \to [A,\Omega^n X] \to [A,\Omega^n Y] \to [A,\Omega^{n-1}\Gamma f] \to$$
$$\to [A,\Omega^{n-1}X] \to \ldots \to [A,X] \to [A,Y],$$

where all but the last three terms are groups with group structure given by the cogroup structure of Ω in Hopro(Kan_0).

In particular if we take

$$A = S^1 \sim \Delta[1]/\dot{\Delta}[1]$$

to be the simplicial circle considered, via the functor

$$J : K_0 \longrightarrow \text{Hopro}(Kan_0)$$

as an object in Hopro(Kan_0), we have, by reason of the adjointness relation $[\Sigma A, B] = [A, \Omega B]$ and the identification of $\Sigma^{n-1} S^1$ as S^n, the simplicial n-sphere, a long exact sequence of homotopy groups :

$$\to \bar{\pi}_n(\Gamma f) \to \bar{\pi}_n(X) \xrightarrow{f_*} \bar{\pi}_n(Y) \to \bar{\pi}_{n-1}(\Gamma f) \to \ldots$$
$$\ldots \to \bar{\pi}_2(Y) \to \bar{\pi}_1(\Gamma f) \to \bar{\pi}_1(X) \xrightarrow{f_*} \bar{\pi}_1(Y).$$

NOTE. By taking $A = \dot{\Delta}[1]$ one can obtain the same sequence with three extra terms

$$\to \bar{\pi}_0(\Gamma f) \to \bar{\pi}_0(X) \to \bar{\pi}_0(Y),$$

where $\pi_0(X)$ is the «set of connected components of X», in some sense.

Using the adjonction (see Vogt [30])

$$[J(A),X] = [A,\bar{X}]_{K_0}, \quad \text{where } \bar{X} = \operatorname{holim} X,$$

we can identify $\bar{\pi}_i(X)$ with $\pi_i(\bar{X})$. This suggests that a possible choice of homology theory for Hopro(Kan₀) would be that given by

$$\bar{H}_n(X;G) = H_n(\bar{X};G),$$

for any coefficients G; however of more use to us are the homotopy and homology progroups obtained by applying the functors

$$\pi_n : \mathrm{Kan}_0 \to \mathrm{Gp}, \quad H_n : \mathrm{Kan} \to \mathrm{Ab} = \text{abelian groups},$$

objectwise to give functors

$$\pi_n : \mathrm{pro}(\mathrm{Kan}_0) \to \mathrm{pro}(\mathrm{Gp}), \quad \mathrm{H}_n : \mathrm{pro}(\mathrm{Kan}) \to \mathrm{pro}(\mathrm{Ab})$$

and then noting that each f in Σ_0 is sent to an isomorphism in pro(Gp) by each π_n, so there is a unique induced functor

$$\bar{\pi}_n : \mathrm{Hopro}(\mathrm{Kan}_0) \to \mathrm{pro}(\mathrm{Gp}).$$

Similarly for H_n, there is an induced functor

$$\bar{\mathrm{H}}_n : \mathrm{Hopro}(\mathrm{Kan}_0) \to \mathrm{pro}(\mathrm{Gp}).$$

The connection between $\pi_n(X)$ and $\bar{\pi}_n(X)$ is given by the Bousfield-Kan spectral sequence [5], page 281, the E_2-term of which is given by

$$E_2^{s,t} = \lim{}^s \pi_t(X), \quad 0 \leqslant s < t.$$

This spectral sequence converges to $\bar{\pi}_n(X)$ given suitable conditions: again see [5]. We have already studied some of the properties of these homotopy groups and progroups in the paper [24], Section 3, and we continue this study here in Section 5.

3. COHOMOLOGY OF COHERENT DIAGRAMS.

In [25, I] we gave a definition of a «weak» cohomology theory for a category of diagrams and, although it worked well enough for the purposes of that paper, it was to say the least somewhat inelegant to handle; with the extra categorical structure available to us here, we shall define in a

fairly elegant manner a cohomology theory which is very easy to handle and avoids most of the complications of that introduced before.

Firstly some notation; we let

S Ab = category of simplicial abelian groups,

C Ab = category of complexes of abelian groups,

C^+ Ab = subcategory of C Ab consisting of those complexes which are zero in negative dimensions.

There is an equivalence of categories $N : SAb \to C^+Ab$ given by the normalized chain complex functor (see May [19]). This equivalence extends to one

$$N : \mathrm{pro}(SAb) \to \mathrm{pro}(C^+Ab),$$

and, if F is a pro-abelian group, we will denote by $F_{(q)}$ the complex consisting of F in dimension q and the zero group everywhere else. Via this equivalence there is an object $K(F, q)$ in pro(SAb) such that we have $NK(F, q) \cong F_{(q)}$; $K(F, q)$ is to be used as an Eilenberg-Mac Lane object in our construction.

The category pro(SAb) inherits a notion of coherent homotopy equivalence from pro(Kan$_0$) and the «free-forgetful» adjunction

$$\mathrm{pro}(\mathrm{Kan}_0) \underset{Z}{\overset{U}{\leftrightarrows}} \mathrm{pro}(SAb)$$

preserves these equivalences; that is to say, if f is in Σ_0, Z(f) is in the corresponding Σ_{ab} and, if $g \in \Sigma_{ab}$, its underlying map U(g) is in Σ_0.

By a result of Quillen's [29] or, in a sightly differing form, of Brown's [6] (Adjoint functor Lemma, p. 426), the induced transformations:

$$\mathrm{Hopro}(\mathrm{Kan}_0) \underset{\mathrm{Ho}Z}{\overset{\mathrm{Ho}U}{\leftrightarrows}} \mathrm{Hopro}(SAb) = \mathrm{pro}(SAb)[\Sigma_{ab}^{-1}]$$

are also adjoint.

Likewise in pro(CAb) we can define an abstract homotopy theory as in [26] by inverting weak equivalences where a basic weak equivalence is a level map f_I such that, for each i, $f_I(i)$ induces isomorphisms on all

homology groups.

We can form Hopro(C Ab) and Hopro(C^+ Ab) and moreover N induces an equivalence

$$N : \mathrm{Hopro}(S\,Ab) \to \mathrm{Hopro}(C^+ Ab)$$

Noticing that the inclusion of pro(C^+ Ab) in pro(C Ab) induces an embedding

$$\mathrm{Hopro}(C^+ Ab) \to \mathrm{Hopro}(C\,Ab),$$

we can define, for any X in Hopro(Kan_0), its q^{th} cohomology group with coefficients in the pro-abelian group F to be

$$H^q(X;F) = [NZ(X), F_{(q)}],$$

where [,] on the right is used to denote the Hom-set in Hopro(C Ab).

This construction is derived from that in Brown ([6] page 425-428) where a similar method is used to define sheaf-cohomology; his discussion extends, via the observation above, to show that this cohomology theory is representable as follows :

$$\begin{aligned} H^q(X;F) &= [NZ(X), F_{(q)}] && \text{in Hopro}(C\,Ab), \\ &= [NZ(X), F_{(q)}] && \text{in Hopro}(C^+ Ab), \\ &= [Z(X), K(F,q)] && \text{in Hopro}(S\,Ab), \\ &= [X, K(F,q)] && \text{in Hopro}(Kan_0); \end{aligned}$$

hence we have at one and the same time a geometric and an algebraic method of handling this cohomology theory. This avoids many of the complications encountered in [25] .

4. ELEMENTARY PROPERTIES OF THE COHOMOLOGY THEORY.

Many of the elementary properties of the functors $H^q(\,;F)$ follow from the representation

$$H^q(\,;F) = [\ , K(F,q)] .$$

Thus, if we have a subobject A of X in pro(Kan_0), we can form a «local» cofibration sequence

$$A \xrightarrow{i} X \longrightarrow C_i$$

where C_i is the homotopy colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & X \\ e & \end{tikzcd}$$

and hence is the «mapping cone» of i. Defining, in the usual way,

$$H^q(X,A;F) \text{ to be } H^q(C_i;F),$$

we can get a long exact sequence

$$\to H^q(X,A;F) \to H^q(X;F) \to H^q(A,F) \to H^{q+1}(X,A;F) \to \dots$$

The exactness is due to the fact that any object in pro(Kan_0) is cofibrant and hence the dual of Brown's Theorem [6] works (see also [26]).

Since, for a family $\{X_\lambda\}$ of objects in Hopro(Kan_0), their smash product $\vee_\lambda X_\lambda$ is the coproduct in Hopro(Kan_0) one has immediately the result corresponding to Lemma 1.8 of [25, I]:

$$\begin{aligned} H^q(\vee_\lambda X_\lambda;F) &\simeq [\vee_\lambda X_\lambda, K(F,q)] \\ &\simeq \prod_\lambda [X_\lambda, K(F,q)] \quad \text{since } \vee \text{ is the coproduct,} \\ &\simeq \prod_\lambda H^q(X_\lambda;F). \end{aligned}$$

For certain other of the properties which we will need it is more convenient to use the algebraic description.

PROPOSITION 4.1. *If*

$$(1) \qquad 0 \to A \to B \to C \to 0$$

is an exact sequence in pro(C Ab) *such that there is some cofiltering, small category* I *and an exact sequence*

$$0 \to A_I \to B_I \to C_I \to 0,$$

obtained by reindexing (1) and in which each $A_{I,q}$, $B_{I,q}$ *and* $C_{I,q}$ *are projective, then for any* G *in* pro(Ab) *there is a long exact sequence*

$$\to [C,G_{(q)}] \to [B,G_{(q)}] \to [A,G_{(q)}] \to [C,G_{(q+1)}] \to \dots$$

PROOF. It is necessary to describe the (algebraic) homotopy theory in pro(C Ab) in more detail. The usual homotopy theory in C Ab has for cofibrations those monomorphisms which have dimensionwise projective cokernels; hence, if we «lift» these cofibrations in pro(C Ab), in the manner of [26], we get cofibrations defined «locally», in the same way as we handled the maps in Σ and defined a notion of fibration in pro(Kan_0). With this definition, the above sequence is a cofibration sequence in the full subcategory of pro(C Ab) consisting of cofibrant objects. The exactness of the sequence for a pro-free abelian group G would then follow from the dual of Brown's argument [6] (see also [26]), on noticing that the «loop space» functor Ω and the «shift» functor

$$\text{Shift} : \text{pro}(C\,Ab) \to \text{pro}(C\,Ab)$$

defined by

$$\text{Shift}(C)_q = C_{q-1} \text{ for } C \text{ in } \text{pro}(C\,Ab),$$

are the same. If G is not «locally free» and hence cofibrant, we must employ a different method. Since the free resolution given by the free-forgetful adjoint pair Sets $\rightleftarrows$ Ab is functorial, we can construct a pro-chain complex R such that for each index, i say, $R(i)$ is the canonical free resolution of $G(i)$. There is a natural map $R \to G_{(0)}$ in pro(C Ab), and this map induces an isomorphism on homology; hence it is in Σ_C. Since R is cofibrant and R is weakly equivalent to $G_{(0)}$, there is an isomorphism

$$[\ ,R] \simeq [\ ,G_{(0)}].$$

The result follows from the exactness of the R-sequence on repeated application of the loop space / shift functor. (This is a special case of Theorem 3.4 of [26].)

Now let X be an object in pro(Kan_0) indexed by a small category I. If for each i in I, one has $X(i) = X_1(i) \cup X_2(i)$, where X_1 and X_2 are subobjects of X in pro(Kan_0), then we shall write $X = X_1 \cup X_2$, and

$$A = X_1 \cap X_2, \text{ where } A(i) = X_1(i) \cap X_2(i).$$

(X, X_1, X_2) is a proper triad and there is a form of the Mayer-Vietoris se-

quence, as in [25, I], 2.5.

THEOREM 4.2. *If* (X, X_1, X_2) *is a proper triad in* pro(Kan$_0$) *there is a long exact sequence*

$$\dots \to H^q(X;F) \xrightarrow{j^*} H^q(X_1;F) \oplus H^q(X_2;F) \xrightarrow{i^*} H^q(A;F) \to H^{q+1}(X;F)$$

where

$$j^*(u) = (j_1^*(u), j_2^*(u)) \quad \text{and} \quad i^*(u_1 + u_2) = i_1^*(u_1) - i_2^*(u_2)$$

$(i_1 : A \to X_1,\ i_2 : A \to X_2,\ j_1 : X_1 \to X$ *and* $j_2 : X_2 \to X$ *are the inclusions*).

PROOF. There is an exact sequence

$$0 \to C(X_1 \cap X_2) \xrightarrow{C(i)} C(X_1) \oplus C(X_2) \to C(X) \to 0$$

(where we have written $C(X) = NZ(X)$ for convenience). This satisfies the conditions of Proposition 4.1.

COROLLARY 4.3. *If* (X, X_1, X_2) *is a proper triad in* pro(Kan$_0$), *then, for any*

$$u_1 \in H^q(X_1;F),\quad u_2 \in H^q(X_2;F) \quad \text{such that} \quad i_1^*(u_1) = i_2^*(u_2),$$

there is a v *in* $H^q(X;F)$ *such that*

$$v/X_1 = j_1^*(v) = u_1, \quad v/X_2 = j_2^*(v) = u_2.$$

Moreover if $A \sim e$, *then* v *is unique.*

PROOF. Since

$$i_1^*(u_1) = i_2^*(u_2), \quad i^*(u_1, u_2) = 0,$$

which gives the existence of v such that $j^*(v) = (u_1, u_2)$ by 4.2. If $A \sim e$, $H^{q+1}(A;F) = 0$ so j^* is a monomorphism and v is unique.

If $f \in [C, D]$ is a map in Hopro(C Ab), it induces a map

$$f_* : H_*(C) \to H_*(D) \quad \text{in} \quad \text{pro}(Ab);$$

in particular if $D = F_{(q)}$ we get a non-zero map

$$f_q : H_q(C) \to F \quad \text{in} \quad \text{pro}(Ab).$$

Thus there is a homomorphism

$$h : H^q(C;F) \to Hom_{\text{pro}(Ab)}(H_q(C), F).$$

THEOREM 4.4. *Let* F *be in* pro(Ab) *and* C *in* pro(CAb) *be such that by reindexing one can get a complex* C_I *which is dimensionwise free at each* i *in* I. *Then for each* $q > 0$ *there is a subgroup* $N_q(C;F)$ *of the group* $Ext^1(H_q(C), F)$ *and an exact sequence of abelian groups*

$$0 \to N_q(C;F) \to H^q(C;F) \to Hom(H_q(C), F) \to 0$$

where $Ext^1(C,D)$ *denotes the* Ext^1*-functor in the category* pro(Ab).

The proof is classical and is essentially that given in [25, I], I.7. A full proof is given in [27].

The form in which we will use these results is slightly different to that given above but is an easy extension of these definitions and theorems. In fact, let $f: X \to Y$ be a morphism in Hopro(Kan_0); then, by reindexing, one can consider it to be an object in Hopro(Ar(Kan_0)) where Ar(Kan_0) is the category of morphisms in the category Kan_0. If we use the Hilton-Eckmann definition of homotopy groups of a map, adapted for use in Kan_0 (see Hilton [12]), we get pro-groups $\pi_i(f)$, $H_i(f)$ and, for F in pro(Ab), cohomology groups $H^*(f;F)$ ($H^q(f;F)$ is defined either by means of a mapping cylinder construction at the algebraic level or by $[f, p_{F,q}]$ in the category Hopro(Ar(Kan_0)), where $p_{F,q}$ is the fibration from $\Gamma K(F,q)$ to $K(F,q)$, where $\Gamma(\ )$ denotes the path space functor as usual in the category Hopro(Kan_0). Thus ΓX is Γf for $f: e \to X$ the unique map.

5. HOMOTOPY DECOMPOSITION OF MAPS IN pro(Kan_0).

In [12] Hilton gives a neat functorial way of decomposing maps in the category of pointed CW-complexes; because of the functoriality of his method it is exploitable in more general situations. If we were only interested in obtaining homotopy decompositions of objects in pro(Kan_0), this would be given us already by the functorial Postnikov decomposition obtained via the functors $cosk_n$, for natural numbers n; however, as we intend to use the homotopy decomposition for an obstruction theory in this context, we have need for the greater generality.

REMARK. The following discussion follows almost exactly that given, for

a simpler situation, in [25, III].

We will say that a pointed pro-simplicial set X is 1-connected if $\pi_1(X) = 0$.

Suppose $f: X \to Y$ is a morphism in $\text{pro}(\text{Kan}_0)$. There is a Hurewicz homomorphism $\phi_n : \pi_n(f) \to H_n(f)$, where $\pi_n(f)$, $H_n(f)$ are to be interpreted as in the Remark at the end of Section 4, and a long exact sequence of homotopy progroups for f ; if f is a composite $f = gh$,

$$\dots \to \pi_r(h) \xrightarrow{(1,g)_*} \pi_r(f) \xrightarrow{(h,1)_*} \pi_r(g) \xrightarrow{\partial} \pi_{r-1}(h) \to \dots$$

One also has the cohomology groups $H^q(g;G)$ for G in pro(Ab):

$$H^q(f;G) = [f, p_{G,q}], \text{ where } p_{G,q}: K(G,q)^I \to K(G,q)$$

and the [,] is used to denote the hom-set in $\text{Hopro}(\text{Kan}_{0,\text{pairs}})$.

$\text{Kan}_{0,\text{pairs}}$ is the category of maps in Kan_0 and $\text{Hopro}(\text{Kan}_{0,\text{pairs}})$ is obtained from $\text{pro}(\text{Kan}_{0,\text{pairs}})$ by inverting the maps f which are isomorphic, by reindexing, to an f_I such that, for each i in I and

$$f_I(i) = \left(\begin{array}{ccc} X_1(i) & \xrightarrow{f_1(i)} & Y_1(i) \\ \downarrow{\scriptstyle \eta(i)} & & \downarrow{\scriptstyle \theta(i)} \\ X_2(i) & \xrightarrow[f_2(i)]{} & Y_2(i) \end{array}\right)$$

both $f_1(i)$ and $f_2(i)$ are homotopy equivalences.

With this notation we have the following result:

THEOREM 5.1. *Let* $f: X \to Y$ *in* $\text{pro}(\text{Kan}_0)$ *be such that both* X *and* Y *are 1-connected. If* $\pi_r(f) = 0$, *with the exception of the values*

$$r = m_1, m_2, \dots, m_k, \dots, \text{ for } m_1 < m_2 < \dots < m_k < \dots,$$

some increasing sequence, then there is, for each k, *an object* Y^k *in the category* $\text{pro}(\text{Kan}_0)$ *and two maps*

$$f^k: X \to Y^k, \quad q^k: Y^k \to Y^{k-1}$$

(where $Y^0 = Y$ *) satisfying:*

(i) $f = q^1 q^2 \dots q^k f^k$,

(ii) q^k *is a fibration with fibre a* $K(\pi_{m_k}(f), m_{k-1})$,

(iii) $\pi_r(f^k) = 0$, $r < m_{k+1}$, $\pi_r(f^k) \sim \pi_r(f)$, $r \geqslant m_{k+1}$.

The set $\{ f^k, q^k \mid k = 1, 2, \dots \}$ *is called a homotopy decomposition of* f *and the classes*

$$\{ k_{m_s} \in H^{m_s}(Y^{s-1}; \pi_{m_s}(f)) \}$$

which correspond at each stage to the factorization $f^{s-1} = q^s f^s$ *are called the k-classes or k-invariants of the decomposition.*

PROOF OF 5.1. Suppose $\pi_r(f) = 0$, $r < n$ and $\pi_n(f) \neq 0$. The morphism

$$H^n(f; \pi_n(f)) \xrightarrow{h} Hom(H_n(f), \pi_n(f))$$

of the universal coefficient Theorem is an isomorphism, since the group $N_n(C(f); \pi_n(f))$ is a subgroup of $Ext^1(H_{n-1}(f), \pi_n(f))$ and by the Hurewicz Theorem $H_{n-1}(f) = 0$, so $N_n(C(f); \pi_n(f)) = 0$. So there is, in $H^n(f; \pi_n(f))$, a class $\{v, u\}$ such that $h\{v, u\} = \phi_n$, where ϕ_n is, as before, the Hurewicz homomorphism. The definition of $H^n(f; \pi_n(f))$ means that one can represent the pair $\{v, u\}$ up to homotopy by a diagram in the category pro(Kan_0) :

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "f"'] & \Gamma K(G,n) \arrow[d, "p_{G,n}"] \\ Y \arrow[r, "u"] & K(G,n) \end{tikzcd} , \quad G = \pi_n(f).$$

We can form a new diagram

$$\begin{tikzcd} X \arrow[r, "f^1"] \arrow[d, "f"'] & Y^1 \arrow[r, "w"] \arrow[d, "q^1"] & \Gamma K(G,n) \arrow[d, "p_{G,n}"] \\ Y \arrow[r, "1_Y"'] & Y \arrow[r, "u"'] & K(G,n) \end{tikzcd}$$

by forming the pullback of the pair $(p_{G,n}, u)$, the limit being formed in

pro(Kan_o) by reindexing and taking indexwise limits of the resulting diagrams. By construction the fibres of q^1 and $p_{G,n}$ are isomorphic and are $K(G,n-1)$'s, being isomorphic to $\Omega K(G,n)$.

$$(v,u) = (w,u).(f^1, 1_Y),$$

and $f = q^1 f^1$. Since the fibres are equivalent,

$$(w,u)_* : \pi_*(q^1) \sim \pi_*(p_{G,n}),$$

and so

$$\pi_r(q^1) = 0 \quad \text{if } r \neq n \quad \text{and} \quad (f^1, 1)_* : \pi_n(f) \sim \pi_n(q^1),$$

since $(v,u)_* : \pi_n(f) \sim \pi_n(p_{G,n})$, by choice. Using the long exact sequence for the factorization $f = q^1 f^1$, we get $\pi_n(f^1) = 0$, and also

$$\pi_r(f^1) \sim \pi_r(f) \quad \text{for } r \neq n.$$

By induction, one obtains at each step a factorization of $f^{s-1} = q^s f^s$ and then one factorizes f^s, etc... The morphisms

$$u_s : Y^{s-1} \to K(\pi_{m_s}(f), m_s)$$

give the cohomology classes, denoted by

$$k_{m_s} \in H^{m_s}(Y^{s-1}; \pi_{m_s}(f)),$$

which are the k-classes mentioned above.

REMARKS. (1) Given the k-classes one can theoretically build, by a sequence of pullbacks, the original map f, up to homotopy equivalence in the category Hopro($\mathrm{Kan}_{o,\mathrm{pairs}}$).

(2) For the case

$$Y = e, \quad f = \text{the unique map } X \to e,$$

the above theorem gives a series of k-invariants in the groups

$$H^q(Y^{s-1}; \pi_{m_s}(X)), \quad \text{where } \pi_r(X) = 0$$

except for the values $r = m_1 < m_2 < \ldots$, etc... This is basically the same decomposition as the one obtained by using the various coskeleton functors.

(3) If $X = e$, the above process gives a pro-simplicial version of the

Cartan-Serre-Whitehead approach to killing homotopy classes.

(4) As far as I know, there is no analogous homology decomposition theorem possible for maps or objects in pro(Kan_0). This is caused by the failure of the Moore space construction to be functorial.

(5) Given f in Hopro(Kan_0), we can factorise it as

$$X \xleftarrow{s} X' \xrightarrow{f'} Y$$

where s is in Σ and f' is the image of a map in pro(Kan_0); decomposing f' by the above method we get a decomposition of f. Thus we can extend 5.1 to maps in Hopro(Kan_0).

6. THE WHITEHEAD THEOREM.

In [2] Artin and Mazur prove what amounts to a form of the Whitehead Theorem valid in pro(K_0), namely the following:

Let $f: X \to Y$ be a map in pro(K_0); then the following are equivalent:

(i) $\mathrm{cosk}_n f: \mathrm{cosk}_n X \to \mathrm{cosk}_n Y$ is an isomorphism for each n.

(ii) $\pi_n(f): \pi_n(X) \to \pi_n(Y)$ is an isomorphism in pro(Gp) for each n.

($\mathrm{cosk}_n X$ is the n^{th}-coskeleton of X, see Artin and Mazur [2], 2.4 p. 21).

It is natural to expect that, if $f: X \to Y$ in Hopro(Kan_0), then a corresponding result would say the statements:

(i) $\mathrm{cosk}_n f: \mathrm{cosk}_n X \to \mathrm{cosk}_n Y$ is an isomorphism in Hopro(Kan_0) for each n,

and

(ii) $\pi_n(f): \pi_n(X) \to \pi_n(Y)$ is an isomorphism in pro(Gp) for each n

are equivalent.

Unfortunately, in general, the inverse of f given by the Artin and Mazur's result will not be coherent; it will however in one very special case.

THEOREM 6.1. *If* $\pi_*(X) \simeq 0$ *for* X *in* pro(Kan_0), *then* $\mathrm{cosk}_n X$ *is coherently contractible for each* n.

PROOF. Artin and Mazur's Whitehead Theorem gives an inverse in pro(K_0) to the canonical map $e \to \mathrm{cosk}_n X$. This inverse may be represented by the

unique map $\mathrm{cosk}_n X \to e$ in $\mathrm{pro}(\mathrm{Kan}_0)$. We have that

$$\mathrm{cosk}_n X \to e \to \mathrm{cosk}_n X$$

is the identity map of $\mathrm{cosk}_n X$ (in $\mathrm{pro}(K_0)$) ; therefore, supposing that X is given by a functor $X: I \to \mathrm{Kan}_0$, we have : for each index i, there is an index $a(i)$ «finer» than i (i.e. there is a map $a(i) \to i$) such that

$$\mathrm{cosk}_n p_i^{a(i)}: \mathrm{cosk}_n X(a(i)) \to \mathrm{cosk}_n X(i)$$

is nullhomotopic. By a result of Vogt [30], page 28 Proposition 2.2, if we replace $\mathrm{cosk}_n p_i^{a(i)}$ by a trivial map e, we obtain a map in $\mathrm{Hopro}(\mathrm{Kan}_0)$ isomorphic to that given by

$$\{\mathrm{cosk}_n p_i^{a(i)}: \mathrm{cosk}_n X(a(i)) \to \mathrm{cosk}_n(X(i))\}_{i \in I}.$$

Since this is the identity map on $\mathrm{cosk}_n X$, we have that, in the category $\mathrm{Hopro}(\mathrm{Kan}_0)$, the trivial map and the identity map on $\mathrm{cosk}_n X$ are isomorphic, thus $\mathrm{cosk}_n X \simeq e$ in $\mathrm{Hopro}(\mathrm{Kan}_0)$.

REMARK. The difficulty over using

$$e \to \mathrm{cosk}_n X \quad \text{and} \quad \mathrm{cosk}_n X \to e$$

to give an isomorphism $e \simeq X$ is that the definition of isomorphism in the category $\mathrm{Hopro}(\mathrm{Kan}_0)$ is so different from that in $\mathrm{pro}(K_0)$ that one cannot be certain that, if f, g are coherent (e.g. pro-simplicial) and the induced maps are inverse in $\mathrm{pro}(K_0)$, then f and g are inverse in the category $\mathrm{Hopro}(\mathrm{Kan}_0)$.

The obvious next step would be to prove a relative form of 6.1, i.e. that, if $\pi_*(X,A) \simeq 0$ for (X,A) in $\mathrm{pro}(\mathrm{Kan}_{0,\mathrm{pairs}})$, then

$$\mathrm{cosk}_n i : \mathrm{cosk}_n A \to \mathrm{cosk}_n X$$

is in $\mathrm{Hopro}(\mathrm{Kan}_0)$ for each n. The obvious «proof» does not work, however, and I would like to thank David Edwards for drawing my attention to this fact. I would also like to thank Peter Hilton for a comment which provided the key to the proof given below (which does work !).

We shall, in fact, derive this result from the Whitehead Theorem. The usual relation between these two has been the inverse one ; one proved

results about a cofibration $A \to X$ and then converted a general $f: X \to Y$ into the cofibration $X \to M_f$, M_f being the mapping cylinder of f. We will prove the Whitehead Theorem for Hopro(Kan_0) directly making use only of a technical Lemma and a result on «principal» fibrations in Hopro(Kan_0); the general form of the Whitehead Theorem is then reduced to the result for principal fibrations using a Postnikov-Moore decomposition of the map, as described in Section 5.

LEMMA 6.2. *Let* $p: E \to B$ *be a fibration in* Kan_0 *with fibre an Eilenberg-MacLane space of type* (π, r) *for* $r > 1$. *Let* (X, A) *be a pair in* Kan_0 *which satisfies* $\pi_k(X, A) = 0$ *for* $k < r$ *and suppose given a map of maps*

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ A \arrow[u, "f_2"] \arrow[r, "i"] & X \arrow[u, "f_1"'] \end{tikzcd},$$

which induces the zero map from $\pi_r(X, A)$ *to* $\pi_r(p) \simeq \pi$; *then there is a map* $f: X \to E$ *making the diagram commute, i. e.*

$$pf = f_1 \quad \text{and} \quad fi = f_2$$

(f *is unique up to homotopy).*

PROOF. First note that the obstruction to p being a homotopy equivalence - interpreted as the obstruction to lifting in the diagram

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ E \arrow[u, equal] \arrow[r] & M_p \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

- is an element of

$$H^r(M_p, E; \pi_r(p)) = H^r(p; \pi_r(p))$$

(where, as in Section 5, homotopy, homology and cohomology groups of maps are interpreted as in Hilton [12]). It suffices to show that the induced map

$$(f_1, f_2)^* : H^r(p; \pi_r(p)) \to H^r(i; \pi_r(p))$$

sends this obstruction to zero - in fact we will show that this map is zero. Noting that $\pi_1(B)$ may act non-trivially on homology etc... and hence that we must work in the category of $\pi_1(B)$-modules and must interpret cohomology as having twisted coefficients, we can use the fact that the induced map on r^{th} homotopy groups is $\pi_r(i) \xrightarrow{0} \pi_r(p)$ and the Hurewicz Theorem for map groups (Hilton [12] page 45) to show

$$H_{r-1}(i) = H_{r-1}(p) = 0$$

and that the homology induced maps

$$(f_1, f_2)_* : H_r(i) \to H_r(p)$$

is zero. Now we use the Universal Coefficient Theorem, with coefficients $\pi_r(p)$ to deduce that

$$(f_1, f_2)^* = 0 : H^r(p; \pi_r(p)) \to H^r(i; \pi_r(p)) .$$

The uniqueness of f follows similarly.

Combining 6.1 and 6.2 we get a limited form of the Whitehead Theorem as follows:

THEOREM 6.3. *Let* $p: E \to B$ *be a basic principal fibration in* pro(Kan$_0$) *such that the fibre* F *has* $\pi_s(F) = 0$ *if* $s \neq r$ *and* $\pi_r(F)$ *isomorphic to zero in* pro(Ab) *for some* $r > 1$ *; then* p *is an isomorphism in the category* Hopro(Kan$_0$).

PROOF. Our method of proof will be to consider the diagram

$$(*) \qquad \begin{tikzcd} E \arrow[r, "p"] \arrow[d, equal, "="'] & B \\ E \arrow[r, "k"] & M_p \arrow[u, "j"'] \end{tikzcd}$$

in pro(Kan$_0$). j induces an isomorphism in Hopro(Kan$_0$), so it suffices to prove the existence of a «lifting» $M_p \to E$ making the diagram commute.

Assume, as always, that $E \to B$ is represented by a diagram of fibrations:

$$\{ E(i) \xrightarrow{p(i)} B(i) \}_{i \in I}$$

so given any i in I we may use 5.1 on the fibre F to find $\alpha(i)$ finer than i such that the induced map $F(\alpha(i)) \to F(i)$ is nullhomotopic. We can represent (*) by the diagram of squares (we assume j finer than i implies $\alpha(j)$ finer than $\alpha(i)$)

$$\left(\begin{array}{ccc} E(i) & \xrightarrow{\;p(i)\;} & B(i) \\ {\scriptstyle p_i^{\alpha(i)}} \Big\uparrow & & \Big\uparrow {\scriptstyle p_j^{\alpha(i)} \circ j(\alpha(i))} \\ E(\alpha(i)) & \xrightarrow{\;k(\alpha(i))\;} & M_p(\alpha(i)) \end{array} \right)_{i \in I}$$

By using 6.2, we can find a map $f(i)\colon M_p(\alpha(i)) \to E(i)$ making the diagram commute for each i. It would seem impossible to check that the $f(i)$ form a pro-simplicial map $M_p \to E$, however we shall show that $\{f(i)\}$ is a coherent map in the sense of Vogt [32].

Since the $f(i)$ are unique up to homotopy, we can, for each j finer than i, find (and specify) a homotopy between $p_i^j f(i)$ and $f(i) p_{\alpha(i)}^{\alpha(j)}$; the obstruction to do this is zero since the group $H^{r-1}(p(\alpha(j)); \pi_r(p(i)))$ is zero. If, in fact, we assume that we have proved the coherence up to the n^{th} stage and

$$i_n \to i_{n-1} \to \dots \to i_1 \to i_0$$

is an n-simplex in I, then the obstruction to coherence at level n and for this simplex is in the group $H^{r-n}(p(\alpha(i_n)); \pi_r(p(i_0)))$, which is zero for all $n \geqslant 0$. Now, of course, we use the identification of

$$\text{Copro}(\text{Kan}_0) \quad \text{and} \quad \text{Hopro}(\text{Kan}_0)$$

to produce a map $f\colon M_p \to E$ so that

$$fk = Id_E \quad \text{and} \quad pf = j \quad \text{in Hopro}(\text{Kan}_0).$$

Clearly fj^{-1} is the required inverse for p.

We next need a definition. We will say that X in pro(Kan₀) is of homotopical dimension $< n$ if the canonical map $X \to \text{cosk}_{n+1} X$ is an isomorphism in Hopro(Kan₀). This means that X can be replaced in all cases by a pro-simplicial set satisfying $\pi_r(X) = 0$ for all $r > n$. Thus given any

map $f: X \to Y$ of 1-connected pro-simplicial sets with the homotopical dimension of both X and Y less than or equal to n, we can use the methods of Section 5 to obtain an n-stage Postnikov-Moore homotopy decomposition of f in Hopro(Kan$_0$) as

$$f = q^1 q^2 \dots q^n f^n,$$

where q^k is a basic principal fibration with fibre a $K(\pi_k(f), k\text{-}1)$ and f^n is an isomorphism in Hopro(Kan$_0$). We can then apply 6.3 to each q in turn for the case when $\pi_k(f) = 0$ for $k = 2, 3, \dots, n$ and is equal to zero after that. To summarize we get:

THEOREM 6.4. *If* X, Y *are 1-connected and of finite homotopical dimension in* Hopro(Kan$_0$) *and* $f: X \to Y$ *is a map in* Hopro(Kan$_0$) *which induces an isomorphism* $f_*: \pi_r(X) \to \pi_r(Y)$ *for each* r, *then* f *is an isomorphism in* Hopro(Kan$_0$).

COROLLARY 6.5. *If* (X, A) *is in* pro(Kan$_{0,\text{pairs}}$) *and both* X *and* A *are 1-connected, then, if* X *and* A *are of finite homotopical dimension,*

$$\pi_r(X, A) \simeq 0 \quad \textit{for all} \quad r$$

implies the inclusion $A \to X$ *is an isomorphism in* Hopro(Kan$_0$).

If we introduce the notion of weak isomorphism (cf. ♮-isomorphism in Artin and Mazur [2]) to mean:

$f: X \to Y$ is a weak isomorphism if the induced map

$$\operatorname{cosk}_n f: \operatorname{cosk}_n X \to \operatorname{cosk}_n Y$$

is an isomorphism for each n,

then we obtain the full analogue of the Artin-Mazur form of the Whitehead Theorem for Hopro(Kan$_0$):

COROLLARY 6.6. *If* X *and* Y *are 1-connected pro-simplicial sets and* $f: X \to Y$ *induces an isomorphism of the progroups* $\pi_n(X)$ *and* $\pi_n(Y)$ *for each* n, *then* f *is a weak isomorphism in* Hopro(Kan$_0$).

COROLLARY 6.7 *(Essential uniqueness of Eilenberg-MacLane pro-objects). Suppose* X *in* pro(Kan$_0$) *is such that* X *is 1-connected,*

$$\pi_q(X) \simeq 0 \text{ if } n \neq q \ (q > 1) \text{ and } \pi_q(X) \simeq G \text{ in } \mathrm{pro}(\mathrm{Ab});$$

then, if X *has finite homotopical dimension,* X *is isomorphic to* $K(G, q)$ *in* Hopro(Kan$_0$), *otherwise* X *is weakly isomorphic to* $K(G, q)$.

PROOF. First we must comment that given any G in pro(Ab) one can construct an object $K(G,q)$ in pro(Kan$_0$) indexed by the same category as G with subgroups as described in the statement above; in fact the construction outlined in Section 3 will do this.

So, given X as above, the natural map $X \to \mathrm{cosk}_n X$ is a weak isomorphism (and an isomorphism if X is of finite homotopical dimension and n is large enough) and $\mathrm{cosk}_n X$ is contractible by 6.1. There is a naturally defined fibration sequence

$$\mathrm{cosk}_q X \to \mathrm{cosk}_{q+1} X \to Y,$$

where

$$\pi_q(Y) \simeq G \quad \text{and} \quad \pi_n(Y) \simeq 0 \ \text{ for } n \neq q.$$

By reindexing, if necessary, and substitution of some of the $Y(i)$ by homotopically equivalent Kan complexes, we can construct an isomorphism

$$Y \to K(G, q) \quad \text{in} \quad \mathrm{Hopro}(\mathrm{Kan}_0).$$

(Basically one is using here the classical uniqueness of $K(G(i), q)$'s for each index i and Vogt's result [32] page 28, 4.2). Since $\mathrm{cosk}_q X$ is contractible, $\mathrm{cosk}_{q+1} X \to Y$ is an isomorphism in Hopro(Kan$_0$) by 6.4. The two cases now follow easily.

7. EXTENSIONS. SOME SIMPLE CASES.

The purpose of this section is to give a solution in some simple cases to the extension problem in pro(Kan$_0$), in other words, to find algebraic invariants which answer the following question:

Suppose (X, A) is a pair of objects in pro(Kan$_0$) and suppose further that there is given a map $f: A \to Y$ in pro(Kan$_0$). Is there a map $g: X \to Y$ extending f in the sense that the following diagram commutes, and, moreover, if g and g' are two possible extensions of f are they homotopic?

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

We have, by our approach, avoided much of the work needed in Spanier [30] to answer this question, however we still need the analogue of Spanier [30], page 410 (7,7.11). We assume throughout that $n > 1$.

If Y is a $K(G,n)$ of finite homotopy dimension, then there is by 6.7 an isomorphism

$$\phi : Y \to K(G,n) \quad \text{in} \quad \mathrm{Hopro}(\mathrm{Kan}_0)\,;$$

we will say that an element $u \in H^n(Y;G)$ *is universal* if the image of u under the natural isomorphism

$$H^n(Y;G) \to [Y, K(G,n)]$$

is the isomorphism ϕ.

THEOREM 7.1. *Given any* Y *in* $\mathrm{pro}(\mathrm{Kan}_0)$ *and* $u \in H^n(Y;G)$, *there is a* $\bar{Y}$ *in* $\mathrm{pro}(\mathrm{Kan}_0)$ *and a universal element* $\bar{u} \in H^n(\bar{Y};G)$ *such that:*

(i) $\bar{Y}$ *is a* $K(G,n)$ *of finite homotopy dimension;*

(ii) there is a map $f: Y \to \bar{Y}$ *in* $\mathrm{Hopro}(\mathrm{Kan}_0)$ *with* $f^*(\bar{u}) = u$.

PROOF. Following Spanier [30], 7.7.11, we want to kill off the homotopy groups of Y in dimensions other than n and to adjust the resulting pro-objects in dimension n, however we must do this globally so as to end up with a pro-object as required.

Let $\mathrm{sk}_{n-1}Y$ denote the $n\text{-}1^{\text{th}}$ skeleton of Y ; then we can form the homotopy pushout

$$\begin{tikzcd} \mathrm{sk}_{n-1}Y \arrow[r] \arrow[d] & Y \arrow[d, "j^{(1)}"] \\ e \arrow[r] & Y^{(1)} \end{tikzcd}$$

and $Y^{(1)}$ has no non-trivial homotopy in dimensions lesser than n. Using the coskeleton functor cosk_{n+1} we get

$$j^{(2)}: Y \to Y^{(1)} \to cosk_{n+1} Y^{(1)} = Y^{(2)}$$

and $Y^{(2)}$ is a $K(H,n)$ for some progroup H.

Using the long exact sequence in cohomology for the cofibration sequence

$$sk_{n-1} Y \longrightarrow Y \xrightarrow{j^{(1)}} Y^{(1)}$$

we find that

$$j^{(1)*}: H^n(Y^{(1)}; G) \to H^n(Y; G)$$

is an epimorphism and hence we can find

$$u^{(1)} \in H^n(Y^{(1)}; G) \quad \text{such that} \quad j^{(1)*}(u^{(1)}) = u.$$

Since the cohomology group $H^n(Y^{(1)}; G)$ is isomorphic to $[Y^{(1)}, K(G,n)]$ and $K(G,n)$ has zero homotopy in dimensions $> n$, the canonical map

$$Y^{(1)} \to cosk_{n+1} Y^{(1)} = Y^{(2)}$$

induces an isomorphism

$$[Y^{(2)}; K(G,n)] \to [Y^{(1)}; K(G,n)]$$

by the characterization of $cosk_n$ given by Artin and Mazur in [2] page 21. Thus there is a $u^{(2)} \in H^n(Y^{(2)}; G)$ such that the composite

$$j^{(2)}: Y \to Y^{(1)} \to Y^{(2)}$$

sends $u^{(2)}$ to u. $u^{(2)}$ can be represented by a map $u^{(2)}: Y^{(2)} \to K(G,n)$, and if we look at the map

$$f = u^{(2)} j^{(2)}: Y \to K(G,n)$$

we find the image of the identity map Id under f^* is

$$f^*(Id) = j^{(2)*} u^{(2)*}(Id) = j^{(2)*}(u^{(2)}) = u.$$

Taking $\bar{Y} = K(G,n)$ completes the proof of the theorem.

REMARK. The above proof gives $f: Y \to K(G,n)$ in Hopro(Kan_0), but, since f can be represented as $\gamma(\bar{f})\gamma(\bar{s})^{-1}$, where $\bar{f}$ is a map in pro(Kan_0) and $\bar{s}$ a weak equivalence in pro(Kan_0), we can, in fact, replace f by $\bar{f}: Y \to \bar{Y}$ for some $K(G,n)$, $\bar{Y}$, weakly equivalent to the standard one used here.

Now we can proceed, as in [25], to obtain obstructions for simple extension problems.

THEOREM 7.2. *Let* Y *be a* $K(G,n)$ *of finite homotopy dimension, with* $n > 1$ *where* G *is, as usual, in* pro(Ab). *If* v *is a universal element in* $H^n(Y,G)$ *and* (X,A) *is a pair in* $\text{pro}(\text{Kan}_0)$, *then a map* $f: A \to Y$ *in* $\text{Hopro}(\text{Kan}_0)$ *can be extended to a map* $g: X \to Y$ *iff*

$$\delta f^*(v) = 0 \quad \text{in} \quad H^{n+1}(X,A;G).$$

PROOF. If such a g exists, then

$$f = gi \quad \text{and} \quad \delta f^*(v) = \delta i^* g^*(v) = 0,$$

since $\delta i^* = 0$. So assume that $\delta f^*(v) = 0$. Taking the homotopy pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "k"] \\ X \arrow[r, "f'"] & Y^1 \end{tikzcd}$$

the morphism of pairs $j : (X,A) \to (Y^1, Y)$ induces an isomorphism

$$j^*: H^*(Y^1, Y; G) \to H^*(X,A;G)$$

such that the diagram

$$\begin{tikzcd} H^n(Y;G) \arrow[r, "\delta"] \arrow[d, "f^*"'] & H^{n+1}(Y^1,Y;G) \arrow[d, "j^*"] \\ H^n(A;G) \arrow[r, "\delta"] & H^{n+1}(X,A;G) \end{tikzcd}$$

commutes. Since $\delta f^*(v) = 0$, it follows that $\delta(v) = 0$, but by the exactness of the sequence

$$\dots \to H^n(Y^1;G) \xrightarrow{k^*} H^n(Y;G) \longrightarrow H^{n+1}(Y^1,Y;G) \to \dots$$

there thus exists a

$$v' \in H^n(Y^1;G) \quad \text{with} \quad k^*(v') = v.$$

Using 7.1 we obtain a map $a: Y^1 \to K(G,n)$ in $\text{Hopro}(\text{Kan}_0)$ such that: $a^*(Id) = v'$. If, as before, we allow ourselves to use essentially the same symbol for $v \in H^n(Y;G)$ and for the isomorphism $v: Y \to K(G,n)$, etc...,

we have that $ak : Y \to K(G,n)$ induces $(ak)_*$ such that $(ak)_*(Id) = v$ and interpreting as maps we get

$$\begin{tikzcd} Y \arrow[rr, "ak"] \arrow[dr, "v"'] & & K(G,n) \arrow[dl, "Id"] \\ & K(G,n) & \end{tikzcd}$$

so ak is an isomorphism in Hopro(Kan_0).

We define $g : X \to Y$ in Hopro(Kan_0) by the composite

$$g = (ak)^{-1} a f' : X \to Y^1 \to K(G,n) \to Y .$$

Restricting to A we get

$$gi = (ak)^{-1} a f' i = (ak)^{-1} a k f = f ,$$

so g is an extension as required.

REMARK. If f is, to start with, a morphism in pro(Kan_0), we can apply the above result to the corresponding homotopy class in Hopro(Kan_0) but we cannot expect to obtain an extension within pro(Kan_0). The reason for this is basically that, whilst in the classical theory if a map h was an isomorphism in, for instance, $\mathrm{HoKan}_0 = K_0$, there was a map g, going in the opposite direction, such that gh and hg were homotopic to the respective identities, in Hopro(Kan_0), a map f in pro(Kan_0), which gives an isomorphism on taking homotopy is only formally invertible and there is not necessarily any map in pro(Kan_0) acting as an «inverse to f up to homotopy».

8. LIFTINGS AND PRINCIPAL FIBRATIONS.

As has been used several times before in this paper, the canonical map $p(B) : \Gamma B \to B$ in pro(Kan_0) is a basic fibration. The fibre of this fibration is the loop-object ΩB on B. If $\theta : B_1 \to B$ is a morphism in the category pro(Kan_0), the homotopy pullback

$$\begin{tikzcd} E_\theta \arrow[r] \arrow[d, "p_\theta"'] & \Gamma B \arrow[d, "p(B)"] \\ B_1 \arrow[r, "\theta"] & B \end{tikzcd}$$

gives a new fibration p_θ called *the principal basic fibration induced from* $p(B)$ via the map θ. Again p_θ has fibre ΩB up to isomorphism in the category $\mathrm{Hopro}(\mathrm{Kan}_o)$.

Now suppose, as usual, (X, A) is a pair in $\mathrm{pro}(\mathrm{Kan}_o)$ with inclusion $i: A \to X$ and consider a map $f: i \to p_\theta$ in $\mathrm{pro}(\mathrm{Kan}_{o,\mathrm{pairs}})$ given by the square

$$\begin{tikzcd} A \arrow[r, "f''"] \arrow[d, "i"'] & E_\theta \arrow[d, "p_\theta"] \\ X \arrow[r, "f'"'] & B_1 \end{tikzcd}$$

If there exists a map $\bar{f}: X \to E_\theta$ in $\mathrm{Hopro}(\mathrm{Kan}_o)$ such that $\bar{f}$ extends f'' and such that $f' = p_\theta \bar{f}$ in $\mathrm{Hopro}(\mathrm{Kan}_o)$, then $\bar{f}$ will be called a lifting of $f = (f', f'')$.

We want to find obstructions which will indicate if liftings exist. Since the diagram

$$\begin{tikzcd} E_\theta \arrow[r, "\bar{\theta}"] \arrow[d, "p_\theta"'] & \Gamma B \arrow[d, "p(B)"] \\ B_1 \arrow[r, "\theta"'] & B \end{tikzcd}$$

is a pullback, a map $f: X \to E_\theta$ in $\mathrm{Hopro}(\mathrm{Kan}_o)$, being factorised as $\gamma(\phi)\gamma(s)^{-1}$, gives maps

$$X \xleftarrow{\ s\ } X' \xrightarrow{\ \phi\ } E_\theta$$

and hence is completely determined by

$$f_1 = p_\theta \phi : X' \to B_1 \quad \text{and} \quad f_2 = \theta\phi : X' \to \Gamma B \subset B^I .$$

Using the natural transformation $ev: (?)^I \times I \to (?)$ in Kan_o, we get

$$F: X' \times I \to B \quad \text{defined by} \quad X' \times I \xrightarrow{f_2 \times I} B^I \times I \xrightarrow{ev(B)} B .$$

$\theta f_1 = p(B) f_2$, so F is effectively a «global homotopy» from the constant map to θf_1.

If on the other hand we are given that a map $F: X' \times I \to B$ is a «glo-

bal homotopy» between θf_1 and the constant map fixed at the base point; then using the natural transformation $(?) \to ((?) \times I)^I$, we get

$$X \longrightarrow (X \times I)^I \xrightarrow{F^I} B^I$$

in pro(Kan$_0$) which maps into $\Gamma B \subset B^I$. Thus there is a correspondence between maps in Hopro(Kan$_0$) from X to E_θ which are liftings of f and global homotopies (in pro(Kan$_0$)) between the constant map and θf_1 which are fixed at the base point. We will not claim that this correspondence is 1-1 since the interplay of the morphisms in Σ with the «global homotopies» complicates the argument. However if we are given a map $f: i \to p_\theta$, we can, as in the classical case, find a map

$$\theta(f): (A \times I \cup X \times \dot{I},\ X \times 0) \to (B, b_0)$$

such that $\theta(f) \mid X \times 0$ is the constant map with image b_0, $\theta(f) \mid X \times 1$ is the map $\theta f'$ and $\theta(f) \mid A \times I$ is the «global homotopy» given by the existence of the map $\bar{\theta} f''$ from A to B^I in Hopro(Kan$_0$). If $\theta(f)$ can be extended to a «global homotopy» $F: X \times I \to B$ in Hopro(Kan$_0$), then F can be written as

$$X \times I \xleftarrow{s} X' \xrightarrow{\phi} B$$

and applying the construction above we get

$$f: X \to (X \times I)^I \xrightarrow{s^I} \Gamma X' \xrightarrow{\Gamma\phi} \Gamma B$$

such that

$$fi = f'' \quad \text{and} \quad p_\theta f = f'$$

in Hopro(Kan$_0$). Thus the problem of finding a lifting will be solved, if we know the answer to the extension problem for $\theta(f)$, and vice versa.

If B is a $K(G,n)$ of finite homotopy dimension, then by 7.2, $\theta(f)$ extends iff $\delta\theta(f)^*(v) = 0$, where $v \in H^n(B;G)$ is universal.

If we define

$$\tau: H^n(X,A;G) \to H^{n+1}((X,A)\times(I,\dot{I});\ G)$$

by $\tau(u) = u \times \bar{I}$, for $\bar{I}$ the generator of $H^1(I, \dot{I}; Z)$, where the cross-pro-

duct is defined in the obvious way, then the *obstruction to lifting* f is the element $c(f) \in H^n(X, A; G)$ given by

$$\delta\theta(f)^*(v) = (-1)^n r\, c(f).$$

A lifting of f exists iff $c(f) = 0$.

9. OBSTRUCTIONS IN THE GENERAL CASE.

We have now all the ingredients to attack the general obstruction problem; we can find obstructions to lifting over principal basic fibrations with fibres an Eilenberg-MacLane object and we can split an arbitrary map between 1-connected objects into a composite of such basic principal fibrations.

Let $p: E \to B$ be a map in $\mathrm{Hopro}(\mathrm{Kan}_0)$. By the remark (5) after 6.1 we can decompose p for each $k > 0$ (if E and B are 1-connected) as a composite

$$E \xleftarrow{s} E' \xrightarrow{p^k} B^k \xrightarrow{q^k} B^{k-1} \to \dots \to B^2 \to B' \xrightarrow{q'} B$$

where each q^s is a principal basic fibration with fibre a $K(\pi_{m_s}(p), m_s)$, etc... If, as in Section 8, we have a square

(1)
$$\begin{tikzcd} A \arrow[r, "f''"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f'"'] & B \end{tikzcd}$$

in $\mathrm{Hopro}(\mathrm{Kan}_0)$, we can replace it by a map f:

(2)
$$\begin{tikzcd} A \arrow[r, "\bar{f}''"] \arrow[d, "i"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "\bar{f}'"'] & B' \end{tikzcd}$$

in $\mathrm{Hopro}(\mathrm{Kan}_{0,\mathrm{pairs}})$ and a lifting in one square will give us a lifting in the other; hence we will assume (1) is given as a map in the category $\mathrm{Hopro}(\mathrm{Kan}_{0,\mathrm{pairs}})$.

For each k we can decompose p to give a diagram

$$(3,k) \qquad \begin{tikzcd} A \arrow[r, "p^k f''"] \arrow[d, "i"'] & B^k \arrow[d, "q^1 q^2 \dots q^k"] \\ X \arrow[r, "f'"] & B \end{tikzcd}$$

We will write $f''_k = p^k f''$, and f_k the map corresponding to the diagram $(3,k)$. A lifting of (f', f''_k), say $\bar{f}_k : X \to B^k$, will give us a lifting $q_k \bar{f}_k$ of (f', f''_{k-1}).

If liftings exist for each k in a compatible way to give a map from X to $\{B^k\}$, then taking the homotopy limit of the sequence of fibrations and identifying it with E will give a lifting of the original map provided $\pi_r(p)$ is not-zero for only finitely many r.

Starting with $k = 1$, we obtain, by the methods of Section 8 an obstruction

$$c(f_1) \in H^m(X, A; \pi_{m_1}(p))$$

to the existence of a lifting f_1 in the diagram

$$\begin{tikzcd} A \arrow[r, "f'_1"] \arrow[d, "i"'] & B^1 \arrow[d, "q^1"] \\ X \arrow[r, "f'"] \arrow[ur, dashed, "f_1"] & B \end{tikzcd}$$

If $c(f_1) = 0$, then there exists at least one such f_1 and we can form a new diagram

$$\begin{tikzcd} A \arrow[r, "f'_2"] \arrow[d, "i"'] & B^2 \arrow[d, "q^2"] \\ X \arrow[r, "f_1"] & B^1 \end{tikzcd}$$

There is an obstruction to a lifting in this diagram (which we will call

$$g(f_1) = (f_1, f''_2) : i \to q^2),$$

the obstruction being

$$c(g(f_1)) \in H^{m_2}(X, A; \pi_{m_2}(p)).$$

We have the possibility of several liftings f_1 in the previous square and as we are only interested in finding a lifting, we define

$$\mathcal{O}_2(f) = \{ c(g(f_1)) \mid f_1 \text{ a lifting of } f_1 \}.$$

A lifting of f to the second level exists iff

$$c(f_1) = 0 \quad \text{and} \quad c(g(f_1)) = 0$$

for some lifting f_1 of f_1, i.e. iff $0 \in \mathcal{O}_2(f)$.

Recursively we define

$$\mathcal{O}_s(f) = \{ c(g(f_{s-1})) \mid f_{s-1} \text{ a lifting of } f_{s-1} \},$$

which is the s^{th} obstruction to lifting f.

$$\mathcal{O}_s(f) \subset H^{m_s}(X, A; \pi_{m_s}(p))$$

is defined only if $0 \in \mathcal{O}_{s-1}(f)$.

REMARK. A difficulty arises if $\pi_r(p) \neq 0$ for infinitely many r since then the promaps $\{ f_k \}$ may not give a promap $f: X \to E$. To see why this is, take as an example the unique map $Y \to e$; then the factors in the decomposition are precisely the $\operatorname{cosk}_n Y$'s with the canonical projections, but the maps

$$\{ f_k : X \to \operatorname{cosk}_k Y \}$$

only define a map $f: X \to Y^\natural$ where $Y^\natural$ is as in Section 5 and $Y \sim Y^\natural$ iff Y has finite homotopy dimension.

APPENDIX

As mentioned in the Introduction the motivation for studying coherent prohomotopy theory in the abstract is the hope that it is applicable in geometric topology and in the study of the algebraic topology of more general spaces than are suitable for study via the classical theory. This Appendix is therefore included in order to sketch some of the constructions which translate the topological or geometric situation into a coherent prohomotopy context. It has been kept as brief as possible and it is hoped that anyone who is interested in applications will refer to the papers in which these constructions occur in more detail.

NOTATION. SS_0 will denote the category of connected semi-simplicial sets with base points and $(\ )_K : SS_0 \to Kan_0$ will denote the Kanification functor (see Gabriel and Zisman [11] page 65-69, or the Summary in [23]).

1. THE VIETORIS FUNCTOR.

Let $\mathfrak{X}$ be a pointed connected topological space and α an open cover of $\mathfrak{X}$; then $V(\mathfrak{X};\alpha)$ denotes the simplicial set whose typical n-simplexes are $(n+1)$-tuples $x_0, \dots, x_n$ of points in $\mathfrak{X}$ such that there is a $\mathfrak{U}$ in α with $\{x_0, \dots, x_n\} \subset \mathfrak{U}$. Letting α vary over open covers of $\mathfrak{X}$ gives a functor

$$V(\mathfrak{X};\ ): \mathrm{Cov}\,\mathfrak{X} \to SS_0$$

and one can easily show that the assignment

$$V(\ ;\ ): \mathfrak{X} \to V(\mathfrak{X};\ )$$

is functorial. Composing with Kanification gives a functor

$$V(\ ;\ )_K : \mathrm{Top}_0 \to \mathrm{pro}(\mathrm{Kan}_0).$$

(The references for this include [21, 22, 23] and also the survey article by Edwards [10].)

$V(\ ;\ )_K$ can thus be used to get to $\mathrm{pro}(\mathrm{Kan}_0)$ and thus to the category $\mathrm{Hopro}(\mathrm{Kan}_0)$. The arguments used in [23] show that, in fact, if we look at homotopy equivalences in Top_0, we find that $V(\ ;\ )_K$ sends them to maps in Σ, so $V(\ ;\ )_K$ induces a functor on the homotopy category.

REMARK. If one tries to work with the «un-Kanified» $V(\mathfrak{X};\ )$ and in the category $\mathrm{pro}(SS_0)$, then a map

$$f : V(\mathfrak{X};\ ) \to V(\mathfrak{Y};\ ) \quad (\text{in } \mathrm{pro}(SS_0))$$

can be shown to be the induced map of a continuous map under very mild restrictions on $\mathfrak{Y}$. I am grateful to Marcel Van der Vel for this observation.

2. THE ČECH FUNCTOR AND ITS VARIANTS.

Again let $\mathfrak{X}$ be a pointed connected topological space and let α be an open cover of $\mathfrak{X}$. $C(\mathfrak{X};\alpha)$ is the simplicial set with n-simplexes those $(n+1)$-tuples of elements of α:

$$< \mathfrak{U}_0, \dots, \mathfrak{U}_n > \quad \text{such that} \quad \bigcap_{i=0}^{n} \mathfrak{U}_i \neq \emptyset .$$

If β is a refinement of α, then a *canonical projection from* β *to* α is a function $\phi : \beta \to \alpha$ such that $\mathfrak{V} \subset \phi(\mathfrak{V})$ for all $\mathfrak{V} \in \beta$; defining a projection $p_\alpha^\beta(\phi) : C(\mathfrak{X};\beta) \to C(\mathfrak{X};\alpha)$ by

$$p_\alpha^\beta(\phi) < \mathfrak{V}_0, \dots, \mathfrak{V}_n > = < \phi(\mathfrak{V}_0), \dots, \phi(\mathfrak{V}_n) >,$$

one can hope to obtain a pro-simplicial set $C(\mathfrak{X};\ )$ as in the Vietoris construction; however it is not obvious that one can choose the canonical projections in such a way to do this. The situation is saved if it is noted that any two projections from β to α are contiguous (see Spanier [30] pages 130 and 152) and hence are coherent in a very simple way. Thus each choice of canonical projections gives an object, after Kanification, which is well defined in the category $\mathrm{Copro}(\mathrm{Kan}_0)$. If one compares the results obtained from two different choices of projections, it is apparent that they are isomorphic objects in $\mathrm{Copro}(\mathrm{Kan}_0)$.

If, in the usual fashion, one tries to make $C(\ ;\ )_K$ into a functor on Top_0, one encounters this same difficulty, but the same solution works. If $f : \mathfrak{X} \to \mathfrak{Y}$ is in Top_0, then it induces a map

$$C(f) : C(\mathfrak{X};\ )_K \to C(\mathfrak{Y};\ )_K \quad \text{in } \mathrm{Copro}(\mathrm{Kan}_0).$$

Identifying $\mathrm{Copro}(\mathrm{Kan}_0)$ with $\mathrm{Hopro}(\mathrm{Kan}_0)$ gives us finally a functor

$$C(\ ;\ )_K : \mathrm{Top}_0 \to \mathrm{Hopro}(\mathrm{Kan}_0).$$

Since Dowker showed in [9] that $C(\mathfrak{X};\alpha)$ and $V(\mathfrak{X};\alpha)$ were homotopically equivalent, in a natural way, it follows that $C(\ ;\ )_K$ is equilent to $V(\ ;\ )_K$ as functors from Top_0 to $\mathrm{Hopro}(\mathrm{Kan}_0)$ and thus that there is a functor induced by $C(\ ;\ )_K$ on the homotopy category of pointed connected topological spaces.

By restricting the covers under consideration, one obtains variants:

(i) Normal open covers (Morita [20]),

(ii) Numerable covers (Levan [15]).

In both cases on passing to $\mathrm{pro}(K_0)$ one gets the shape theory defined by Mardešič in [16]. (For other constructions in shape theory see the survey of Edwards [10]; I have not checked if the definitions of intrinsic and extrinsic shape given there are coherent, but I suspect they are.) The obvious other case to consider would be:

(iii) Finite covers.

As far as I know not much has been done on the resulting homotopy theories.

3. THE LUBKIN CONSTRUCTION AND SULLIVAN'S VARIANT OF IT.

Although Lubkin [26] only gives his construction for «punctually finite» topological spaces and for locally connected schemes, it is, in fact, a very general construction which is valid in any Grothendieck topos, i.e. in any category of sheaves on a category enriched with a Grothendieck topology. (For work on Grothendieck topologies in algebraic geometry, see S.G.A. [33] or Artin's Notes [1]; for more general ideas and properties of toposes, see Kock and Wraith [13].)

Let E be a Grothendieck topos. A *point in* E is a geometric morphism of toposes $E \xrightarrow{e} \mathrm{Sets}$.

Since E is a category of sheaves on a «site» C, there is already, amongst the representable sheaves, an idea of covering. We will assume that the topology on the site C is weaker than the canonical topology, i.e. that all representable presheaves on C are sheaves.

A covering of E is a covering of the object 1 in E and by that we mean a collection of maps $\mathfrak{U} = \{p_i : E_i \to 1\}$ such that:

(C1) 1 is the union (in E) of the images of the p_i.

(C2) If (E_1, p_1) and (E_2, p_2) are in $\mathfrak{U}$ and

$$e_1 : E/_{E_1} \to \text{Sets}, \quad e_2 : E/_{E_2} \to \text{Sets}$$

are two points, e_1 in E_1 and e_2 in E_2, such that $p_1(e_1) = p_2(e_2)$ in the sense that the diagram

$$\begin{tikzcd} & E/_{E_1} \arrow[dr, "e_1"] & \\ E \arrow[ur, "p_1^*"] \arrow[dr, "p_2^*"'] & & \text{Sets} \\ & E/_{E_2} \arrow[ur, "e_2"'] & \end{tikzcd}$$

commutes, then there is a (E_3, p_3) in $\mathfrak{U}$ and a point $e_3 : E/_{E_3} \to$ Sets such that there exist morphisms $a_i : E_3 \to E_i$, $i = 1, 2$, making

$$\begin{tikzcd} & E/_{E_1} \arrow[dr, "e_1"] & \\ E \arrow[ur, "p_1^*"] \arrow[r, "p_3^*"] \arrow[dr, "p_2^*"'] & E/_{E_3} \arrow[u, "a_1^*"] \arrow[d, "a_2^*"] \arrow[r, "e_3"] & \text{Sets} \\ & E/_{E_2} \arrow[ur, "e_2"'] & \end{tikzcd}$$

commutes (E_3 plays the role of the «intersection» of the two coverings).

The covering $\mathfrak{U}$ is *punctually finite* if:

(C3) For each point e in E there are only finitely many $p_i : E_i \to 1$ in $\mathfrak{U}$ such that, for some e_i in E_i, the diagram

$$\begin{tikzcd} E \arrow[rr, "e"] \arrow[dr, "p_i^*"'] & & \text{Sets} \\ & E/_{E_i} \arrow[ur, "e_i"'] & \end{tikzcd}$$

commutes.

Lubkin [26], page 457, proves that, if $\mathfrak{U}$ is punctually finite, for a

point $E \xrightarrow{x} \text{Sets}$ there is a unique element $\mathfrak{U}_x = (E_x, p_x)$ in $\mathfrak{U}$ and a point e_x in E_x with the property that:

(i)
$$\begin{tikzcd} E \arrow[rr, "x"] \arrow[dr, "p_x^*"'] & & \text{Sets} \\ & E/E_x \arrow[ur, "e_x"'] & \end{tikzcd}$$

commutes and

(ii) given any other $p: E_i \to 1$ in $\mathfrak{U}$ and a point e in E such that

$$\begin{tikzcd} E \arrow[rr, "p^*"] \arrow[dr, "x"'] & & E/E_i \arrow[dl, "e"] \\ & \text{Sets} & \end{tikzcd}$$

commutes, there is a factorization of p_x

$$\begin{tikzcd} E \arrow[rr, "p_x^*"] \arrow[dr, "p^*"] \arrow[ddr, "x"'] & & E/E_x \arrow[ddl, "e_x"] \\ & E/E_i \arrow[ur, "\alpha"] \arrow[d, "e"'] & \\ & \text{Sets} & \end{tikzcd}$$

where α is the unique map $E_x \to E_i$ in E with the property that the diagram commutes.

If $\mathfrak{U}$ is a punctually finite covering of E, let $C(E,\mathfrak{U})$ denote the category having for objects the set of all $\mathfrak{U}_x$ for x a point in E and for morphisms, the morphisms in E between the corresponding objects, that is to say a morphism from $\mathfrak{U}_x$ to $\mathfrak{U}_y$ is merely a morphism in E from E_x to E_y. Forming the nerve of $C(E,\mathfrak{U})$ gives a simplicial set and using the obvious idea of refinement of a punctually finite covering, Lubkin defines a pro-simplicial set indexed by the ordered category of punctually finite coverings.

This pro-simplicial set, which will be denoted by $S(E;\ )$, has nice functorial properties in as much as if $f: E \to F$ is a geometric morphism,

then there is an induced promap $S(F;\ ) \to S(E;\ )$ defined by forming the induced coverings. To use coherent theory we need only Kanify and pass to the homotopy category.

The principal application of this construction has been in the case where $\mathfrak{X}$ has been a locally connected scheme and C has been the category of schemes étale over $\mathfrak{X}$ endowed with the étale topology. The resulting pro-simplicial set is then a «combinatorial invariant» of $\mathfrak{X}$ (see Lubkin [28], Chapter 1).

Lubkin also defines the pro-simplicial set $S(\mathfrak{X};\ )$ for $\mathfrak{X}$ a topological space by using the site $C = \mathrm{Open}(\mathfrak{X})$ of open sets in $\mathfrak{X}$. He claims ([28] page 466) that the resulting limit groups for homotopy and homology are a good candidate for the name of «Alexander-Spanier» homotopy and homology groups of $\mathfrak{X}$.

Sullivan [31] uses the same construction with «étale» interpreted as «finite covering space» and suggests as a variant the dropping of the «finite».

As far as the author knows, none of these Lubkin-type constructions has yielded a theory of the form of shape theory; although such a theory is clearly possible, one cannot say, at present, if it would yield results significant for algebraic geometry or, more generally, for the geometric side of topos theory.

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