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Extensions of functors and their applications

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This paper is, in fact, devoted to the search of functors $F$ making
the following diagram commute

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
K \\
A \\
U
\end{array}
\end{array}
\xrightarrow{F}
\begin{array}{c}
\begin{array}{c}
M \\
T \\
\end{array}
\end{array}
\xrightarrow{L}
\begin{array}{c}
\begin{array}{c}
B \\
V
\end{array}
\end{array}
\end{array}
\end{array}
\]  

This problem was investigated by M. Hušek (see [11]) who introduced a
construction producing a solution $F$ which is the greatest one. His con-
struction works for functors $V$ of the topological type (in this case the pro-
blem (1) was dealt with in [1] and [2], too). In general this construction
gives only a functor $L : A \to B$ and a natural transformation $\lambda : U \to VL$.

We propose another construction which tests the solvability of (1)
in the following sense: whenever a solution $F$ exists, then our construction
yields a solution (which is the smallest one). This construction consists
in a transfinite modification of the functor $L : A \to B$ (for algebraic $V$ one
step suffices) and it was introduced in [15] in a special case.

The search for a functor $F$ is the same thing as the study of exten-
sions in the 2-category $E_X$ consisting of categories over $X$. It is advanta-
geous to work in a more general 2-category $D_X$. Then, roughly speaking,
the Hušek's construction corresponds to the situation that a left extension
of $T$ along $K$ in $D_X$ sits in $E_X$, while our construction computes a left
extension of $T$ along $K$ in $D_X$ and modifies it into $E_X$.

Left extensions in $D_X$ were investigated by R. Guitart [5], where
it was given one sufficient condition for their existence. This condition was
We present another sufficient condition which describes «pointwise» left extensions in $D_X$. These extensions are not pointwise in the sense of Street [20] but they seem to be the right ones. For instance, they induce a good notion of density. As in any 2-category left extensions in $D_X$ lead to the concept of a density comonad. In addition, in $D_X$ the construction of a density comonad can be parametrized by comonads in $X$. Many of these results will be stated in their dual version.

If we put in (1) $A = B$ and $U = P \circ V$, where $P : X \to X$ is a given functor, then functors $F$ are lifting of $P$ along $V$ and our results bring a general point of view to questions investigated by M. Sekanina in [18, 19]. We touch also liftings of monads using a variation on parametrized codensity monads. Extensions of functors presented here cover techniques of extensions of full and faithful functors developed in [15, 16, 17].

The Appendix of this paper is devoted to the study of 2-categories $E^C_X$ and $D^C_X$ arising from a 2-category $C$ in the same way as $E_X$ and $D_X$ from the 2-category $CAT$ of categories. There are touched their properties (comma objects, 2-completeness) depending on those of $C$. Especially, it is investigated what they have from the structure of a cosmos, when $C$ is a cosmos in the sense of Street [21]. For $C = CAT$ these questions are related to the construction of the initial completion of a faithful functor (cf. [8, 10, 23]).

I am indebted to M. Sekanina who stimulated the origin of this investigation, to R. Guitart who hinted me at the possibility to work in $D_X$, and to both of them for many valuable discussions. A part of this paper has rised during my stay in Paris and I would like to express my gratitude to Prof. Charles and Andrée Ehresmann for their encouragement and interest in my work.


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1. EXTENSIONS IN $D_X$.

1.1. DEFINITION. Let $X$ be a category. Denote by $D_X$ the 2-category whose objects are couples $(A, U)$ where $U: A \to X$ is a functor, arrows are couples $(F, \phi): (A, U) \to (B, V)$ where $F: A \to B$ is a functor and $\phi: U \to VF$ a natural transformation, and 2-cells $\alpha: (F, \phi) \to (F', \phi')$ are natural transformations $\alpha: F \to F'$ such that $\phi' = Va \cdot \phi$.

Arrows of $D_X$ are composed as follows:

$$(F', \phi').(F, \phi) = (F'F, \phi'F \cdot \phi).$$

Further denote by $D_X^e$ the 2-category which has the same objects as $D_X$, arrows $(F, \phi): (A, U) \to (B, V)$ where $\phi: VF \to U$, and 2-cells $\alpha: (F, \phi) \to (F', \phi')$ where $\alpha: F \to F'$ is a natural transformation such that $\phi' = Va = \phi$.

Such 2-categories $D_X^C$ and $D_X^C$ can be defined for every 2-category $C$ and for every object $X$ of it. Then

$$D_X = D_X^{CAT} \text{ and } D_X^e = D_X^{eCAT}.$$ Clearly $D_1 = CAT$ where 1 is the one-morphism category. Denote by $C^{op}$ the 2-category which arises from $C$ by the reversing of 2-cells.

1.2. LEMMA. $D_X^e = (D_X^{op})^{op}$.

PROOF. The isomorphism $D_X^e \to (D_X^{op})^{op}$ is given by

Categories $D_X$ are investigated in [5] and [6]. We will be interested in left extensions in $D_X$ and therefore, with respect to 1.2, in right extensions in $D_X^e$. We recall that having arrows $K: M \to A$ and $T: M \to B$ in a
2-category $C$, then a left extension of $T$ along $K$ in $C$ is a couple $L$, $a$ consisting of an arrow $L : A \to B$ and a 2-cell $a : T \to L K$ such that, for any extension of $T$ along $K$, i.e. for any couple $S : A \to B$, $\beta : T \to SK$, there is a unique 2-cell

$$\gamma : L \to S$$

such that $\gamma K \cdot a = \beta$.

All basic concepts concerning 2-categories can be found in [14].

In the sequel we will suppose that we have the following situation in $CAT$:

1.3. LEMMA. Let $L : A \to B$ be a functor and $\lambda : U \to VL$, $\bar{\lambda} : T \to L K$ natural transformations such that $\lambda K \cdot \kappa = V \bar{\lambda} \cdot \tau$. Then $(L, \lambda), \bar{\lambda}$ is a left extension of $(T, \tau) : (M, W) \to (B, V)$ along $(K, \kappa) : (M, W) \to (A, U)$ in $D_X$ iff, for every functor $S : A \to B$ and for every natural transformations

$$\sigma : U \to VS, \quad \bar{\sigma} : T \to SK$$

such that $\sigma K \cdot \kappa = V \bar{\sigma} \cdot \tau$,

there is a unique natural transformation $\alpha : L \to S$ such that

$$\sigma = V \alpha \cdot \lambda \quad \text{and} \quad \bar{\sigma} = \alpha K \cdot \bar{\lambda}.$$

PROOF is evident.

1.4. EXAMPLE. Consider the following special case of (2)

\[ \begin{array}{c}
\begin{array}{c}
A \\
\downarrow \kappa \\
U \\
\downarrow \tau \\
X
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
M \\
\downarrow K \\
T \\
\downarrow W \\
B
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
A \\
\downarrow F \\
G \\
\downarrow \beta \\
X
\end{array}
\end{array}
\]

\[ (2) \]
Then \((L, \lambda), \bar{\lambda}\) is a left extension of \((F, \beta)\) along \((1_A, \alpha)\) in \(D_X\) iff

\[
\begin{array}{ccc}
G & \xrightarrow{a} & H \\
\beta \downarrow & & \downarrow \lambda \\
F & \xrightarrow{\bar{\lambda}} & L
\end{array}
\]

is a pullback.

1.5. DEFINITION (see [9], 26.3). Let \(V : B \to X\) be a functor. A morphism \(f : x \to V b\) is said to be \(V\)-generating provided that

\[V(r). f = V(s). f\]

implies \(r = s\).

Dually, we define a \(V\)-cogenerating morphism.

1.6. REMARK. The characterization 1.3 of left extensions in \(D_X\) is more simple when \(r\) is \(V\)-generating. Namely, the second equality demanded for \(a\) is a consequence of the first. Indeed,

\[V\bar{\sigma}. r = \sigma K. \kappa = V a K. \lambda K. \kappa = V(a K. \bar{\lambda}). r\]

and thus \(\bar{\sigma} = a K. \bar{\lambda}\).

1.7. DEFINITION (see [22]). A functor \(V : B \to X\) is called a \(q\)-functor if every diagram in \(X\) of the form

\[
\begin{array}{ccc}
x & \xrightarrow{v} & y \\
\downarrow u & & \\
V b &
\end{array}
\]

has a universal solution. This means that there are

\[\bar{b} \in B, \quad \bar{v} : b \to \bar{b}\]

and

\[\bar{u} : y \to V \bar{b}\]

such that \(\bar{u}. v = V(\bar{v}). u\),

and for every

\[b' \in B, \quad v' : b \to b'\]

and

\[u' : y \to V b'\]

such that \(u'. v = V(v'). u\),

there is a unique \(t : \bar{b} \to b'\) such that \(t . \bar{v} = v'\) and \(V(t). \bar{u} = u'\).

1.8. DEFINITION (see [3]). Let \(v : V b \to y\) be a morphism in \(X\). A mor-
phism $\bar{v} : b \to \overline{b}$ in the universal solution of a diagram (3) in which we put $u = 1_{Vb}$ is called a $V$-quasiquotient of $v$. We say that $V$ is a functor with quasiquotients if a $V$-quasiquotient exists for every $v : Vb \to y$.

1.9. REMARK. Any q-functor is a functor with quasiquotients. If $B$ has pushouts, then the converse implication holds (see [22]). If $B$ has pushouts and $V$ has a left adjoint, then $V$ has quasiquotients (see [4]). If $B$ has finite limits, intersections and coequalizers and $V$ preserves finite limits and intersections, then each regular epimorphism has a $V$-quasiquotient.

In the sequel we will need to know how properties of a $V$-quasiquotient of $v$ depend on properties of $v$. Clearly a $V$-quasiquotient of an epimorphism is an epimorphism. If $X$ is an (extremally epi-mono)-category and $V$ preserves monomorphisms, then a $V$-quasiquotient of an extremal epimorphism is extremally epi. If $B$ has and $V$ preserves kernel pairs, then a $V$-quasiquotient of a regular epimorphism is a regular epimorphism.

1.10. THEOREM (see [5, 6, 22]). Let there exist left Kan extensions of $T$ and $W$ along $K$ and $V$ be a q-functor. Then a left extension of $(T, \tau)$ along $(K, \kappa)$ in $D_X$ exists.

SKETCH OF THE PROOF. Let

$$L_1, \epsilon_1 : T \to L_1 K \quad \text{or} \quad L_2, \epsilon_2 : W \to L_2 K$$

be left Kan extensions of $T$ or $W$ resp. along $K$. We get natural transformations $a : L_2 \to V L_1$, $\beta : L_2 \to U$ such that

$$aK.\epsilon_2 = V\epsilon_1.\tau \quad \text{and} \quad \beta K.\epsilon_2 = \kappa.$$

Following [22] $A^V$ is a q-functor. Thus the diagram

$$\begin{array}{ccc}
L_2 & \xrightarrow{a} & VL_1 \\
\downarrow{\beta} & & \\
U & & \\
\end{array}$$

has a universal solution

$$L, \bar{a} : U \to VL, \quad \bar{\beta} : L_1 \to L.$$
Now, \((L, a), \beta K \cdot \epsilon_I\) is the desired left extension.

The same theorem holds in every 2-category \(C\) with a 2-terminal object. Now, we are going to state another condition for the existence of left extensions in \(D_X\).

1.11. CONSTRUCTION. Suppose that for every \(a \in A\) there exist \(L a \in B\), \(\lambda_a: U a \rightarrow V L a\) and a natural transformation \(\beta^a: A(K-, a) \rightarrow B(T-, La)\) such that, for any \(f: K m \rightarrow a\), it holds

\[
\lambda_a \cdot U f \cdot \kappa_m = V \beta^a_m(f) \cdot \tau_m,
\]

with the following universal property: for every

\[
b \in B, \quad u: U a \rightarrow V b \quad \text{and} \quad a: A(K-, a) \rightarrow B(T-, b)
\]

such that \(u \cdot U f \cdot Tm = V a_m(f) \cdot \tau_m\) there is a unique morphism \(t: L a \rightarrow b\) such that

\[
V t \cdot \lambda_a = u \quad \text{and} \quad B(T, t) \cdot \beta^a = a.
\]

Clearly \(L: A \rightarrow B\) is a functor and \(\lambda: U \rightarrow V L\) a natural transformation. Namely, \(L g\) is defined, if \(g: a \rightarrow a'\), by the universal property as:

Furthermore, the equality \(\overline{\lambda}_m = \beta^m_K(1_{Km})\) defines a natural transformation \(\overline{\lambda}: T \rightarrow L K\).

1.12. THEOREM. \((L, \lambda), \overline{\lambda}\) is a left extension of \((T, \tau)\) along \((K, \kappa)\), in the 2-category \(D_X\).

PROOF. \((L, \lambda), \overline{\lambda}\) is an extension because

\[
V \overline{\lambda}_m \cdot \tau_m = V \beta^m_K(1_{Km}) \cdot \tau_m = \lambda_{Km} \cdot U(1_{Km}) \cdot \kappa_m = \lambda_{Km} \cdot \kappa_m.
\]
Let \((S, \sigma), \tilde{\sigma}\) be another extension. The universal property defining \(L a\) produces a natural transformation \(a : L \to S\) as follows:

\[
\begin{array}{cccc}
W_m & \xrightarrow{Uf. \kappa_m} & Ua \\
\downarrow r_m & & \downarrow \lambda_a \\
VT_m & \xrightarrow{V\beta_m^a(f)} & VL_a - V\alpha_a \\
\downarrow V(Sf. \tilde{\sigma}_m) & & \downarrow VSa \\
\end{array}
\]

From 1.3 we obtain that \((L, \lambda), \tilde{\lambda}\) is the desired left extension.

1.13. DEFINITION. The left extension constructed in 1.11 will be called pointwise.

1.14. REMARK. If \(X\) has pullbacks, then \(D_X\) is a representable 2-category (in the sense of [20]). It means that \(D_X\) has comma objects and 2-pullbacks. A comma object for an op-span

\[
\begin{array}{ccc}
A & \xrightarrow{F} & C \\
U & \downarrow \alpha & \downarrow \beta \\
X & \xrightarrow{V} & B
\end{array}
\]

in \(D_X\) is the span \(((D_0, \phi), (F/G, Z), (D_1, \psi))\) together with the 2-cell \(\lambda : (F, \alpha). (D_0, \phi) \to (G, \beta). (D_1, \psi)\),

where the span \((D_0, F/G, D_1)\) together with the 2-cell \(\lambda : FD_0 \to GD_1\) is a comma object for the op-span \((F, G, G)\) in \(CAT\) and the following diagram is a pullback in \(CAT(F/G, X)\):

\[
\begin{array}{ccc}
Z & \xrightarrow{\psi} & VD_1 \\
\downarrow \phi & & \downarrow \beta D_1 \\
UD_0 & \xrightarrow{W\lambda. \alpha D_0} & WD_1
\end{array}
\]
Pointwise left extensions in $D_X$ in our sense do not agree with the pointwise left extensions in the sense of [20]; the last one are too strong. Our pointwise left extensions are precisely those having the preservation property defining pointwise left extensions in [20] for arrows

$$(G, \gamma): (C, Z) \rightarrow (A, U)$$

such that $\gamma$ is iso.

Hence both concepts agree in the case $X = 1$.

1.15. LEMMA. Let $\tau$ be pointwise $V$-generating (it means that each component of $\tau$ is $V$-generating). Then $(L, \lambda)$, $\tilde{\lambda}$ is a pointwise left extension of $(T, \tau)$ along $(K, \kappa)$ in $D_X$ iff the morphisms

$$\lambda_a: U a \rightarrow VL a, \quad L f \cdot \tilde{\lambda}_m: T m \rightarrow L a \text{ for } f: K m \rightarrow a$$

have the following universal property: For any $b \in B$, $u: U a \rightarrow V b$ and $\tilde{f}: T m \rightarrow b$

such that $u \circ U f \cdot \kappa_m = V \tilde{f} \cdot \tau_m$ for every $f: K m \rightarrow a$, there is a unique morphism $t: L a \rightarrow b$ such that $V t \cdot \lambda_a = u$.

PROOF. The assignment $a_m(f) = \tilde{f}$ defines a natural transformation

$$a: A(K-, a) \rightarrow B(T-, b)$$

because $\tau_m$ is $V$-generating. Namely, for every $g: m \rightarrow n$, it holds

$$V(a_n(f).T g).\tau_m = V \tilde{f}.\tau_n \cdot W g = u \cdot U f \cdot \kappa_n \cdot W g = u \cdot U(f. K g).\kappa_m = V(f. K g).\tau_m = V a_m(f. K g).\tau_m.$$

Then the assertion follows from 1.6 and 1.12.

1.16. PROPOSITION. Let $\tau$ be pointwise $V$-generating. Then each of the following two conditions ensures the existence of a pointwise left extension of $(T, \tau)$ along $(K, \kappa)$ in $D_X$:

1° $B$ is complete, well-powered, has a cogenerating set of objects and $V$ preserves limits.

2° $M$ is small, $B$ and $X$ have sums and $V$ is a $q$-functor.

PROOF. It is sufficient to produce a universal solution from 1.15. Supposing 1, one can show that the category of all possible solutions is complete,
well-powered and has a cogenerating set of objects. Thus it has an initial object which is the desired universal solution. If $M$ is small and $B$, $X$ have sums, then instead of searching a universal solution from 1.15, one can search a universal solution of the diagram

$$
\begin{array}{ccc}
\sum_{f:Km \to a} Wm & \xrightarrow{v} & Va \\
\downarrow u & & \\
\sum_{f:Km \to a} Tm
\end{array}
$$

But now the property of being a $q$-functor makes the job.

1.17. PROPOSITION. Let $\kappa$ be an isomorphism, $K$ full and $(L,\lambda), \tilde{\lambda}$ a pointwise left extension of $(T,\tau)$ along $(K,\kappa)$ in $D_X$. Then each of the following two conditions ensures that $\tilde{\lambda}$ is an isomorphism:

1. $K$ is faithful;
2. $\tau$ is pointwise $V$-generating.

PROOF. Put $\beta_{Kn}^m(f) = T f'$ for every $f:Km \to Kn$ such that $f = K f'$. It holds

$$
\tau^* \cdot \kappa^{-1} \cdot U f \cdot \kappa_m = \tau^* \cdot \kappa^{-1} \cdot U K f' \cdot \kappa_m = \tau^* \cdot W f' =
$$

$$
= V T f' \cdot \tau_m = V \beta_{Kn}^m(f) \cdot \tau_m.
$$

In the case of 2, it implies that $\beta_{Kn}^m: A(K^*,Kn) \to B(T^*,Tn)$ is a natural transformation (compare with the proof of 1.15). Let $K$ be faithful and $g: m' \to m$. Then

$$
\beta_{Kn}^m(f) \cdot T g = T(f',g) = T((f,Kg)' ) = \beta_{Kn}^m(f,Kg).
$$

So $\beta_{Kn}^m$ is again a natural transformation. It suffices to show that $Tn, \tau_n \cdot \kappa^{-1}$ and $\beta_{Kn}^m$ have the universal property from 1.11. Let us have $b, u, a$ from 1.11. Then $a_n(1_{Kn}) : Tn \to b$ is the desired morphism $t$.

1.18. LEMMA. Let $\tau$ be (pointwise) $V$-generating and $(L,\lambda), \tilde{\lambda}$ a (pointwise) left extension of $(T,\tau)$ along $(K,\kappa)$ in $D_X$. Then $\lambda$ is (pointwise) $V$-generating.
PROOF is evident.

1.19. LEMMA. Let \((L, \lambda), \overline{\lambda}\) be a left extension of \((T, r)\) along \((K, \kappa)\), in \(D_X\). Then each of the following two conditions ensures that \(\lambda\) is pointwise mono:

1. \(\kappa\) and \(\overline{\lambda}\) are iso, \(r\) is pointwise mono and
\[
\{ U h \mid h : a \to Km, m \in M \}
\]
is mono for every \(a \in A\).

2. There is an extension \((S, \sigma), \overline{\sigma}\) of \((T, r)\) along \((K, \kappa)\) in \(D_X\) such that \(\sigma\) is pointwise mono.

PROOF is evident.

2. EXTENSIONS IN \(C_X\).

Let \(C_X\) be the sub-2-category of \(D_X\) which has the same objects as \(D_X\), arrows \((F, \phi)\) such that \(\phi\) is an isomorphism and any 2-cell in \(D_X\) between arrows of \(C_X\) belongs to \(C_X\).

We will be interested in left and right extensions in \(C_X\). In this Part we will work in the situation (2) considered in \(C_X\) (i.e. \(\kappa\) and \(r\) will be isomorphisms). Evidently, if \((L, \lambda), \overline{\lambda}\) is a left extension of \((T, r)\), along \((K, \kappa)\) in \(D_X\) and \(\lambda\) is an isomorphism, then \((L, \lambda), \overline{\lambda}\) is a left extension of \((T, r)\) along \((K, \kappa)\) in \(C_X\). Similarly, one can treat right extensions.

2.1. PROPOSITION. If \((R, \rho), \overline{\rho}\) is a right extension of \((T, r^{-1})\) along \((K, \kappa^{-1})\) in \(D_X^*\) and \(\rho\) is an isomorphism, then \((R, \rho^{-1}), \overline{\rho}\) is a right extension of \((T, r)\) along \((K, \kappa)\) in \(C_X^*\).

PROOF follows from the fact that the assignment
\[
a : (F, \phi) \to (F', \phi') \mapsto a : (F, \phi^{-1}) \to (F', \phi'^{-1})
\]
gives an isomorphism \(C_X \to C_X^*\).

If the extension from 2.1 is pointwise, then we obtain the construction from [11]. A typical situation in which right extensions in \(C_X\) are des-
cribed in 2.1 is given in the following theorem. Firstly we recall some definitions from [7].

A source in $X$ is a class of morphisms $\{ f_i : x \to x_i \}_{i \in I}$ in $X$. It is called **mono** if

$$f_i \cdot r = f_i \cdot s \quad \text{for each } i \in I \quad \text{implies} \quad r = s.$$  

$X$ is called a $(E,M)$-category if $E$ is a class of epimorphisms in $X$ closed under compositions with isomorphisms, $M$ is a class of sources in $X$ closed under compositions with isomorphisms, and the following conditions hold:

(a) $X$ is $(E,M)$-factorizable, i.e. for every source $\{ f_i \}_{i \in I}$ in $X$ there exist $e \in E$ and $\{ g_i \}_{i \in I} \in M$ such that $f_i = g_i \cdot e$ for each $i \in I$.

(b) $X$ has the $(E,M)$-diagonalization property, i.e. whenever $f$ and $e$ are morphisms and $\{ g_i \}_{i \in I}$ and $\{ f_i \}_{i \in I}$ are sources in $X$ such that

$$e \in E, \quad \{ g_i \}_{i \in I} \in M \quad \text{and} \quad f_i \cdot e = g_i \cdot f \quad \text{for each } i \in I,$$

then there exists a morphism $g$ such that

$$g \cdot e = f \quad \text{and} \quad g_i \cdot g = f_i \quad \text{for each } i \in I.$$

Let $X$ be an $(E,M)$-category. We say that a functor $V : B \to X$ is $(E,M)$-topological if for every source $\{ f_i \}_{i \in I}$ in $X$ there is a source $\{ f_i \}_{i \in I}$ in $B$ and an isomorphism $f : Vb \to x$ such that

$$f_i \cdot f = Vf_i \quad \text{for each } i \in I,$$

with the following universal property: For every source $\{ g_i \}_{i \in I}$ in $B$ and every morphism $g : Vb \to x$ in $X$ such that $f_i \cdot g = Vg_i$ for each $i \in I$ there is a unique morphism $k : b' \to b$ such that

$$f \cdot Vk = g \quad \text{and} \quad f_i \cdot k = g_i \quad \text{for each } i \in I.$$  

Further, $V$ is called absolutely topological if it is $(E,M)$-topological for any $(E,M)$-structure on $X$.

2.2. THEOREM. Let $V$ be an $(E,M)$-topological functor and

$$\{ Uf \mid f : a \to Km, \ m \in M \} \in M \quad \text{for every} \quad a \in A.$$  

Then there exists a right extension of $(T, r^{-1})$ along $(K, \kappa^{-1})$ in $C_X$.  

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PROOF. Since $V$ is $(E,M)$-topological and

$$\{ \tau_m \cdot \kappa_m^{-1} \cdot Uf \mid f : a \to K m, \ m \in M \} \in M,$$

the following diagram has a universal solution

$$Ra, \ \rho_a : VRa \to Ua, \ \bar{f} : Ra \to Tm$$

such that $\rho_a$ is an isomorphism.

$$\begin{array}{ccc}
Ua & \xrightarrow{\kappa_m^{-1} Uf} & Wm \\
\downarrow \tau_m^{-1} \ & \ & \downarrow \tau_m^{-1} \\
V Tm
\end{array}$$

Since any $(E,M)$-topological functor is faithful (see [7]), $\tau$ is pointwise $V$-generating and the assertion follows from the dual of 1.15 and from 2.1.

2.3. COROLLARY. If $V$ is absolutely topological, then there exists a left extension of $(T,\tau)$ along $(K,\kappa)$ and a right extension of $(T,\tau^{-1})$ along $(K,\kappa^{-1})$ in $CX$.

PROOF follows from 2.2 and from the fact that any absolutely topological functor is absolutely co-topological (see [7]).

In [2] it is shown that this property is characteristic for absolutely topological functors.

2.4. PROPOSITION. Let any morphism of $E$ be extremally epi and $V$ be $(E,M)$-topological. If there exists an extension of $(T,\tau)$ along $(K,\kappa)$ in $CX$, then there exists a left extension, too.

PROOF. Following Lemma 6.1 from [7] there is a left extension $(L,\lambda), \bar{\lambda}$ of $(T,\tau)$ along $(K,\kappa)$ in $DX$ such that $\lambda$ is pointwise extremally epi. Hence the assertion follows from 1.19, 2°.

If $V$ is not $(E,M)$-topological, then the Hušek's construction usually does not yield a left extension in $CX$. We shall state a more general construction of left extensions in $CX$ which was introduced in a special case in [15] (compare with 2.13).
2.5. CONSTRUCTION. Let $X$ be an (extremally epi-monosource)-category and $L: A \to B$ be a functor. Take $a \in A$ and consider an (extremally epi-monosource) factorization of \( \{ V L h \mid h: a \to K m, m \in M \} \).

\[
\begin{array}{ccc}
VL a & \xrightarrow{V L h} & VL K m \\
\downarrow v_a & & \downarrow \hat{h} \\
\downarrow x_a & & \\
\end{array}
\]

Let $\gamma_a^L$ be a $V$-quasiquotient of $v_a$

\[
\begin{array}{ccc}
VL a & \xrightarrow{v_a} & x_a \\
\downarrow V \gamma_a^L & & \downarrow k_a \\
\downarrow VL a & & \\
\end{array}
\]

Let $g: a' \to a$ be a morphism in $A$. Then

\[
\hat{h} \cdot \hat{g} \cdot v_{a'} = VL (h \cdot g) = \hat{h} \cdot v_a \cdot VL g.
\]

Since $v_a$ is extremally epi and

\[
\{ \hat{h} \mid h: a \to K m, m \in M \}
\]

is a monosource, the (extremally epi-monosource)-diagonalization property of $X$ provides a unique morphism $g': x_{a'} \to x_a$ such that

\[
g' \cdot v_{a'} = v_a \cdot VL g \quad \text{and} \quad \hat{h} \cdot g' = \hat{h} \cdot \hat{g}.
\]

Hence there is a unique morphism $\bar{L} g: \bar{L} a' \to \bar{L} a$ such that

\[
\bar{L} g \cdot \gamma_{a'}^L = \gamma_a^L \cdot L g \quad \text{and} \quad VL g \cdot k_{a'} = k_a \cdot g'.
\]

Thus $\bar{L}: A \to B$ is a functor and $\gamma^L: L \to \bar{L}$ a natural transformation. Further, $\gamma^L K$ is iso because

\[
\{ VL h \mid h: Kn \to K m, m \in M \}, \text{for any } n \in M,
\]

is mono. Namely, $1_{VL Kn}$ belongs to this source.

2.6. LEMMA. Let $X$ be an (extremally epi-monosource)-category and $V \gamma^L$ be pointwise extremally epi. Then $\{ VL h \mid h: a \to K m, m \in M \}$ is a monosource for every $a \in A$. 

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PROOF. Since
\[ V \gamma_{km}^L \cdot h, v_a = V \gamma_{km}^L \cdot V h = V(L(h) \cdot \gamma_a^L), \]
\{ V \gamma_{km}^L \cdot h \}_{h} \text{ is mono and } V \gamma_a^L \text{ extremally epi, there is a unique morphism}
t : V L a \to x_a \text{ such that}
\[ t, V \gamma_a^L = v_a \text{ and } V \gamma_{km}^L \cdot h, t = V \bar L h \text{ for any } h : a \to km. \]
Clearly \( t \) is iso with the inverse \( k_a \). Thus \{ V \bar h \}_{h} \text{ is mono.}

2.7. CONSTRUCTION. Let \( L : A \to B \) be a functor. Put \( L_0 = L \). Let \( a \) be
an ordinal number. Suppose that for any \( \beta < a \) we have a functor \( L_\beta : A \to B \)
and for any \( \beta' \leq \beta < a \) a natural transformation \( \gamma_{\beta', \beta} : L_{\beta'} \to L_{\beta} \) such that
\[ \gamma_{\beta', \beta} = 1_{L_{\beta'}} \text{ and } \gamma_{\beta', \beta''}, \gamma_{\beta''} = \gamma_{\beta', \beta''}. \]
If \( a \) is isolated, we put
\[ L_a = L_{a-1} \text{ and } \gamma_{\beta, a} = \gamma_{L_{\beta}, \beta} = \gamma_{L_{\beta}, a-1} \text{ for every } \beta < a. \]
For a limit, \( \gamma_{\beta, a} : L_\beta \to L_a \) are components of a colimit cone of a diagram
having \( L_\beta, \beta < a \) as objects and
\[ \gamma_{\beta', \beta''}, \beta' \leq \beta < a \]
as morphisms.

Suppose that for any \( a \in A \) there exists \( a_a \) such that \( a_a \beta \) is iso
for every \( \beta \geq a_a \). Put \( L^{\ast} a = L_{a_a} a \) and
\[ L^{\ast} g = L_a g \text{ for } g : a \to a', \text{ where } a = \max \{ a_a, a_a' \}. \]
In this way we obtain a functor \( L^{\ast} : A \to B \) and the equality \( \gamma_a^L = \gamma_{a, a}^L \) de-
fines a natural transformation \( \gamma_a^L : L_a \to L^{\ast} \). By 2.5, \( \gamma^L K \) is iso for any \( a \).

2.8. PROPOSITION. Let \( X \) be an (extremally epi-monosource)-category and
\( V \) a functor with quasiquotients. Then each of the following conditions en-
sures the existence of \( L^{\ast} \) for any functor \( L : \)

1o \( V \) preserves monomorphisms and extremal epimorphisms.
2o \( B \) has kernel pairs and \( V \) preserves kernel pairs and regular epi-
morphisms.

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3° $M$ is a small and $B$ a co-well-powered category with colimits of functors $D : S \to B$ where $S$ is a well-ordered set.

4° In comparison with 3, $B$ is only extremally co-well-powered, but in addition $B$ has and $V$ preserves kernel pairs.

In cases 1 and 2, it holds $L_{\ast} = \overline{L}$.

**Proof.** 1° By 1.9, $V_{\gamma}^L$ is pointwise extremally epi. Following 2.6, $L_{\ast} = \overline{L}$ exists.

2° Similarly, $L_{\ast} = \overline{L}$ exists by 1.9 and 2.6.

3° Clearly $L_{a}$ exists for every $a$. Further any $\gamma_{a}^{\beta_{a}}$, is an epimorphism. Since $B$ is co-well-powered, $L_{\ast}a$ exists for every $a$.

4° By 1.9 any $\gamma_{a}^{\alpha_{a}}$ is pointwise regularly epi. Thus $\{ \gamma_{a}^{\beta_{a}} \}_{\beta, \alpha}$ is a right multistrict analysis, in the sense of [12]. Following 2.3 of [12] any $\gamma_{a}^{\beta_{a}}$ is extremally epi. Since $B$ is extremally co-well-powered, $L_{\ast}a$ exists for every $a$.

2.9. **Theorem.** Let $X$ be an (extremally epi-monosource) category and

\[ \{ U h \mid h : a \to Km, \ m \in M \} , \ \text{for every} \ a \in A, \]

be mono. Let there exist the functor $L_{\ast}$ where $(L, \lambda), \overline{\lambda}$ is a left extension of $(T, \tau)$ along $(K, \kappa)$ in $D_{X}$ such that $\lambda$ is an isomorphism. Suppose that there exists an extension of $(T, \tau)$ along $(K, \kappa)$ in $C_{X}$. Then

\[ (L_{\ast}, V_{\gamma}^{0}.\lambda), \gamma^{0}K.\overline{\lambda} \]

is a left extension of $(T, \tau)$ along $(K, \kappa)$ in $C_{X}$.

**Proof.** Evidently, $(L_{\ast}, V_{\gamma}^{0}.\lambda), \gamma^{0}K.\overline{\lambda}$ is an extension of $(T, \tau)$ along $(K, \kappa)$ in $D_{X}$. Let $(S, \sigma), \overline{\sigma}$ be an extension of $(T, \tau)$ along $(K, \kappa)$ in $C_{X}$. By 1.3 there is a unique natural transformation $\delta^{0}: L_{\theta} = L \to S$ such that

\[ \sigma = V_{\delta^{0}}\lambda \ \text{and} \ \overline{\sigma} = \delta^{0}K.\overline{\lambda}. \]

Let $a$ be an ordinal number and suppose that, for any $\beta < a$ there is defined $\delta^{\beta}: L_{\beta} \to S$ such that

\[ \delta^{\beta}.\gamma_{\beta_{\beta_{1}}^{\beta_{2}}} = \delta^{\beta_{1}} \ \text{for any} \ \beta_{1} \leq \beta_{2} < a. \]

Let $a$ be isolated, $a \in A$, and consider the diagram.
where the triangles at the bottom express the definition of $L_{\alpha} = \overline{L}_{\alpha-1}$. Since $\{Uh\}_h$ is mono and $\sigma_{\alpha}$ iso, $\{VSh\}_h$ is mono. Thus there is a unique morphism $t$ such that
\[ t \cdot v_a = V\delta_{a}^{\alpha-1} \quad \text{and} \quad VSh \cdot t = V\delta_{Km}^{\alpha-1} \cdot \hat{h}. \]
So there is a unique morphism $\delta_{a}$ such that
\[ \delta_{a} \cdot y_{a}^{\alpha-1,a} = \delta_{a}^{\alpha-1} \quad \text{and} \quad V\delta_{a} \cdot k_{a} = t. \]
Since $y_{a}^{\alpha-1,a}$ is epi, $\delta_{a} : L_{\alpha} \to S$ is a natural transformation. For a limit, the existence of $\delta_{a}$ follows immediately from the construction of $L_{\alpha}$ as a colimit. Now, $\delta_{a} = \delta_{a}^{a}$ defines a natural transformation $\delta : L^* \to S$ such that $\delta \cdot y^{0} = \delta^{0}$.

$\delta$ is a morphism of the corresponding extensions and the unicity of $\delta$ follows from 2.1 and from the fact that $y^{0}$ is epi. It remains to prove that $V\gamma^{0}.\lambda$ is iso. By the supposition there is an extension $(S, \sigma), \overline{\sigma}$ of $(T, r)$ along $(K, \kappa)$ in $C_X$. We will show that $\sigma_{1} \cdot V\delta$ is the inverse of $V\gamma^{0}.\lambda$.

Clearly it is a left inverse. Further it holds
\[ V(\delta_{Km} \cdot L_{*}h) \cdot (V\gamma_{a}^{0} \cdot \lambda_{a}) \cdot (\sigma_{a}^{1} \cdot V\delta_{a}) = V(\delta_{Km} \cdot L_{*}h). \]

Since $L_{*}a = \overline{L}_{*}a$, 
\[ \{ VL_{*}h \mid h : a \to K_{m}, m \in M \} \]
is mono. Since $\overline{\sigma}, \overline{\lambda}$ and $\gamma^{0}K$ are isomorphisms, $\delta K$ is iso as well. Hence $\sigma_{1} \cdot V\delta_{a}$ is a right inverse of $V\gamma_{a}^{0} \cdot \lambda_{a}$.

If $(L_{*}, V\gamma^{0} \cdot \lambda), \gamma^{0}K.\overline{\lambda}$ is a left extension of $(T, r)$ along $(K, \kappa)$ in $C_X$, then $\overline{\lambda}$ has to be iso. Further, if there exists an extension of $(T, r)$
along \((K, \kappa)\) in \(C_X\), then \(\lambda\) is extremally mono.

Theorem 2.9 says that, under certain assumptions, \(L^*\) tests the ext-endability of \((T, r)\) along \((K, \kappa)\) in \(C_X\). We are going to give conditions ensuring that \(L^*\) is really a left extension.

2.10. LEMMA. Let \(X\) be an (extremally epi-mono) category, \(L : A \to B\) a functor and \(\lambda : U \to VL\) a pointwise mono natural transformation such that \(\lambda K\) is iso. Let for every \(a \in A\) be \(n \in M\) and \(h : a \to Kn\) such that \(VLh\) is mono and the monomorphism \(k\) in an extremally epi-mono factorization \(Uh = k\) is an intersection of equalizers of pairs \(Ur, Us\) such that \(r.h = s.h\). Then \(\lambda\) is iso.

PROOF. Let \(r, s : Kn \to a'\) equalize \(h\). Then \(\lambda_a \cdot Ur, \lambda_a \cdot Us\) equalize \(\lambda^I_{Kn} VLh\) and thus \(Ur, Us\) equalize \(\lambda^I_{Kn} VLh\) because \(\lambda_a\) is mono. Hence there is a unique morphism \(t\) such that \(k \cdot t = \lambda^I_{Kn} VLh\). But this implies that \(t \cdot \lambda_a = e\) and \(t\) is mono because \(VLh\) is mono. Hence \(t\) is iso, \(\lambda_a\) extremally epi and thus iso.

2.11. LEMMA. Let \(X = Set\), \(L : A \to B\) be a functor such that 
\[\{ VLh \mid h : a \to Km, m \in M \}, \text{ for every } a \in A,\]
is mono and \(\lambda : U \to VL\) be a natural mono transformation such that \(\lambda K\) is iso. Let for every \(f : Km \to a\) such that \(m \in M\) and \(a \in A - K(M)\) there exist \(n \in M\) and \(h_f : a \to Kn\) with the following properties:

(a) For any \(y \in UKm - U(h_f(f)(UKm))\) there are \(r, s\) such that \(r \cdot h_f \cdot f = s \cdot h_f \cdot f\) and \(U r(y) \neq U s(y)\).

(b) for any \(m' \in M\) and \(h : a \to Km'\) the source \(\{ Ut \mid t\text{ has domain }Km'\text{ and there is }t'\text{ such that }t \cdot h \cdot f = t' \cdot h_f \cdot f\}\)
is mono.

Further let there exist a functor \(V' : B \to Set\) such that 
\[\{ V'L f \mid f : Km \to a, m \in M \}, \text{ for any } a \in A,\]
is epi and a natural mono transformation \(\delta : V \to V'\). Then \(\delta\) is iso.

PROOF. Let \(a \in A - K(M)\) and \(x \in VL a\). Since \(\{ V'L f \mid f : Km \to a\}\) is epi
one can find
\[ m \in M, \quad z \in V' L K m \text{ and } f : K m \to a \]
such that \( \delta_{L a}(x) = V' L f(z) \). Suppose that
\[ y = \lambda_{K a}^{-1} V L h f(x) \downarrow U(h f)(U K m). \]
Considering \( r, s : K n \to a' \), from (a) we get that
Since \( \delta_{L a'} \) and \( \lambda_{a'} \) are mono, it holds \( U r(y) = U s(y) \), which is a contradiction. Thus \( y \in U(h f)(U K m) \) and so there is a \( w' \in U K m \) such that \( y = U(h f)(w') \). Hence
\[ V L h f(x) = V L(h f)(w), \text{ where } w = \lambda_{K m}(w'). \]
Consider \( h : a \to K m' \) and take \( t : K m' \to a' \) such that \( t . h . f = t'. h f, f \) for a suitable \( t' \). Successively it holds
\[ \delta_{L a} \cdot V L(t . h)(x) = V' L(t . h) \cdot \delta_{L a}(x) = V' L(t . h . f)(z) = V' L(t . h . f)(z) = \delta_{L a} \cdot V L(t . h . f)(w) \]
\[ V L(t . h)(x) = V L(t . h . f)(w) \text{ and } U t . \lambda_{K m}^{-1} V L(h)(x) = U t . \lambda_{K m}^{-1} V L(h . f)(w). \]
Following (b),
\[ V L(h)(x) = V L(h . f)(w). \]
Since \( \{ V L h \}_{h} \) is mono, \( x = V L f(w) \). We have proved that
\[ \{ V L f \mid f : K m \to a, m \in M \} \]
is epi. It immediately implies that \( \lambda_{a} \) is epi and thus iso.

The condition (a) in 2.11 says that the monomorphism \( k \) in an extremally epi-mono factorization \( U(h f, f) = k . e \) is an intersection of equalizers of pairs \( U r, U s \) such that \( r . h f, f = s . h f, f \). This supposition is
considerably less restrictive than the corresponding condition in 2.10.

2.12. **THEOREM.** Let $X$ be an (extremally epi-monosource)-category and 
\[ \{ U_h \mid h : a \rightarrow Km, m \in M \} \]
be mono. Let for every $a \in A$ there exist $n \in M$ and $h_0 : a \rightarrow Kn$ with the following properties:

(a) A monomorphism $k$ in an extremally epi-mono factorization $Uh_0 = k.e$ is an intersection of equalizers of pairs $Ur, Us$ such that $r.h_0 = s.h_0$.

(b) The source 
\[ \{ U_t \mid t \text{ has domain } Km \text{ and there exists } t' \text{ such that } t.h = t'.h_0 \} \]
is mono for any $m \in M$ and $h : a \rightarrow Km$.

Let there exist a functor $L^*$, where $(L, \lambda)$ is a left extension of $(T, \tau)$ along $(K, \kappa)$ in $D_X$ such that $\lambda$ is iso. Then $(L^*, V\gamma^0.\lambda), \gamma^0 K.\lambda$ is a left extension of $(T, \tau)$ along $(K, \kappa)$ in $C_X$.

**PROOF.** With respect to 2.9 it suffices to show that $V\gamma^0.\lambda$ is iso. Similarly as in 1.19, one deduces that $V\gamma^0.\lambda$ is pointwise mono. Since $\lambda$ is iso, $(V\gamma^0.\lambda)K$ is iso as well. This fact, (b) and the fact that 
\[ \{ VL^*h \mid h : a \rightarrow Km, m \in M \} \]
is mono imply that $VLh_0$ is mono. By 2.10, $V\gamma^0.\lambda$ is iso.

Analogously it is possible to formulate a sufficient condition for $L^*$ to be a left extension in $C_X$ based on 2.11.

2.13. **EXAMPLE.** Let $M$ be a small category and consider the following special case of (2):

\[ A(KM, -) \]

where $A(KM, -)$ assigns to each $a \in A$ the set of all morphisms $Km \rightarrow a$
where \( m \in M \). The effect on morphisms is defined by composition and \( M(M, \cdot) \) and \( B(TM, \cdot) \) are defined in the same way. Natural transformations \( \kappa \) and \( \tau \) are defined by the assignments

\[
\kappa_m(f) = Kf, \quad \tau_m(f) = Tf.
\]

It can be shown that \((L, \lambda), \tilde{\lambda}\) is a left extension of \((T, \tau)\) along \((K, \kappa)\) in \( D_X \) iff \( L, \tilde{\lambda} \) is a left Kan extension of \( T \) along \( K \). If \( K \) and \( T \) are full and faithful, then \( \kappa \) and \( \tau \) are isomorphisms and extensions of \((T, \tau)\) along \((K, \kappa)\) in \( C_X \) correspond to left \( M \)-full and left \( M \)-faithful functors in the sense of [15]. Now, the construction 2.7 provides just the functor \( L_* \) from [15]. Then Theorem 2.9 is a generalization of Proposition 2 from [15]. Similarly 2.10 and 2.12 go out from Theorem 3.5 of [16].

### 3. Parametrized Codensity Monads.

#### 3.1. Definition.

Let \( V: A \to X \) be a functor and \((P, \eta, \mu)\) a monad in \( X \). If \((S, \tilde{\eta}, \tilde{\mu})\) is a monad in \( A \) and \( \sigma: VS \to PV \) a natural transformation such that

\[
\sigma . V \tilde{\eta} = \eta V \quad \text{and} \quad \sigma . V \tilde{\mu} = \mu V . P \sigma . \sigma S,
\]

then we say that \((S, \sigma)\) (more precisely \(((S, \tilde{\eta}, \tilde{\mu}), \sigma))\) is a lax lifting of \((P, \eta, \mu)\) along \( V \).

Morphisms of liftings \( a: (S, \sigma) \to (S', \sigma') \) are taken as morphisms of monads

\[
a: S \to S' \quad \text{such that} \quad \sigma'. Va = \sigma.
\]

In this way we get the category \( Z(P, V) \) of lax liftings of a monad \( P \) along the functor \( V \).

Liftings of \((P, \eta, \mu)\) along \( V \) are lax liftings \((S, \sigma)\) such that:

\[
\sigma = I_P V.
\]

Any lax lifting \((S, \sigma)\) determines a functor \( \tilde{V}: A_S \to X_P \) between Kleisli categories and a natural transformation \( \tilde{\sigma}: VG_S \Rightarrow G_P \tilde{V} \) where

\[
G_S: A_S \to A \quad \text{and} \quad G_P: X_P \to X
\]

are the underlying functors.

Let us have functors
a natural transformation \( \kappa : V K \to W \) and a monad \((P, \eta, \mu)\) in \( X \). Consider the following special case of the dual of (2):

\[
\begin{array}{ccc}
A & \xrightarrow{V} & X \\
\downarrow{P_K} & & \downarrow{V} \\
M & \xleftarrow{K} & A
\end{array}
\]

3.2. THEOREM. Let \((R, \rho)\), \(\tilde{\rho}\) be a right extension of \((K, P_K, \eta V K)\) along \((K, P_K)\) in \(D_X^*\). Then there are natural transformations \(\tilde{\eta}, \tilde{\mu}\) exhibiting \((R, \tilde{\eta}, \tilde{\mu})\) as a monad in \( A \) such that \((R, \rho)\) is a lax lifting of \(P\) along \(V\) and \(\tilde{\rho}\) is an action of \(R\) on \(K\). In addition, for any monad \((S, \sigma)\) in \(A\), any lax lifting \((S, \sigma)\) of \(P\) along \(V\) and any action \(\sigma\) of \(S\) on \(K\) such that

\[
P_K, \eta V K, \sigma = P_K, \sigma K,
\]

there exists a unique morphism of liftings \(\alpha : (S, \sigma) \to (R, \rho)\) such that:

\[
\tilde{\sigma} = \tilde{\rho} \cdot \alpha K \quad \text{(i.e. \(\alpha\) is a morphism of actions, too)}.
\]

PROOF. Clearly \((I_A, \eta V), I_K\) is an extension of \((K, P_K, \eta V K)\) along \((K, P_K)\) in \(D_X^*\). Thus there is a unique natural transformation \(\tilde{\eta} : I_A \to R\) such that \(\rho \cdot V \tilde{\eta} = \eta V\) and \(\tilde{\rho} \cdot \tilde{\eta} K = I_K\). Further, \((R R, \mu V, P \rho, \rho R, \tilde{\rho} . R \tilde{\rho})\) is such an extension as well, because the following diagram commutes:
Thus there is a unique natural transformation $\tilde{\mu} : R R \rightarrow R$ such that

$$\rho \cdot V \tilde{\mu} = \mu V \cdot P \rho \cdot \rho R \quad \text{and} \quad \tilde{\rho} \cdot \tilde{\mu} K = \tilde{\rho} \cdot R \tilde{\rho}.$$ 

If we show that $(R, \tilde{\eta}, \tilde{\mu})$ is a monad, then $(R, \rho)$ will be a lax lifting and $\tilde{\rho}$ an action. The commutative diagrams

and the universality of $(R, \rho)$, $\tilde{\rho}$ imply the associativity of $(R, \tilde{\eta}, \tilde{\mu})$. Similarly the commutative diagrams

and the universality of $(R, \rho)$, $\tilde{\rho}$ imply the associativity of $(R, \tilde{\eta}, \tilde{\mu})$. Similarly the commutative diagrams
complete the proof that \((R, \eta, \mu)\) is a monad.

Let \((S, \eta, \mu)\) be a monad in \(A\), \((S, \sigma)\) be a lax lifting of \(P\) along \(V\), \(\bar{\sigma}\) be an action of \(S\) on \(K\) and suppose that

\[ p_\kappa \cdot \eta VK \cdot V \bar{\sigma} = P_\kappa \cdot \sigma K. \]

Thus \((S, \sigma), \bar{\sigma}\) is an extension of \((K, P_\kappa \cdot \eta VK)\) along \((K, P_\kappa)\), which provides a unique natural transformation \(\alpha: S \to R\) such that \(\sigma = p_\rho \cdot V \alpha\) and \(\bar{\sigma} = \bar{\rho} \cdot \alpha K\). It remains to show that \(\alpha\) is a morphism of monads. This follows from the commutative diagrams
3.3. PROPOSITION. Let $V: A \to X$ be a functor, $(P, \eta, \mu)$ a monad in $X$, $S: A \to A$ a functor and $\bar{\eta}: 1_A \to S$, $\bar{\mu}: SS \to S$ natural transformations. Let $\sigma: VS \to PV$ be a $V$-cogenerating natural transformation such that there are satisfied the conditions for a lax lifting from 3.1. Then $(S, \bar{\eta}, \bar{\mu})$ is a monad.

PROOF. The assertion follows from the diagrams (3), (4) and (5) from the proof of 3.2 (considered for $(S, \bar{\eta}, \bar{\mu})$ instead of $(R, \bar{\eta}, \bar{\mu})$). Here the universality of $\rho, \bar{\rho}$ is replaced by the fact that $\sigma$ is $V$-cogenerating.

If $(P, \eta, \mu)$ is the identity monad in $X$, then the monad $(R, \bar{\eta}, \bar{\mu})$ from 3.2 is the codensity monad induced by $(K, \kappa)$ in $D^*_X$. So Theorem 3.2 tells us that the construction of a codensity monad in $D^*_X$ admits a para-
metrization by monads in $X$. In the case $X = 1$ there is possible only the parametrization by the identity monad and we get the usual codensity monad in $\text{CAT}$.

3.4. DEFINITION. We say that an arrow $(K, \kappa):(M, W) \rightarrow (A, V)$ is co-dense in $D_X^\star$ if $(I_A, I_V)$, $I_K$ is a pointwise right extension of $(K, \kappa)$ along $(K, \kappa)$ in $D_X^\star$.

3.5. PROPOSITION. Let $V$ be faithful. Then an arrow

$$(K, I_{VK})(M, VK) \rightarrow (A, V)$$

is codense in $D_X^\star$ iff for a morphism $f: V a \rightarrow V a'$, $f = V f'$ holds for a morphism $f': a \rightarrow a'$ whenever for any $z \in A$ and any morphism $h: a' \rightarrow z$ there exists a morphism $h': a \rightarrow z$ such that $V h' = V h \cdot f$.

PROOF follows from 1.15.

The condition from the last proposition was often considered in the litterature (e.g. [11, 8 or 16]) and it means that the codensity in our sense is a right one (at least for the full subcategory of $D_X^\star$ consisting of faithful functors $V: A \rightarrow X$). Results 1.5 and 1.8 from [16] can be generalized to the following statements concerning «the reflection of a codensity along a change of base».

3.6. LEMMA. Let $G: X \rightarrow Y$ be a functor and suppose that

$$(K, G\kappa):(M, GW) \rightarrow (A, GV)$$

is codense in $D_Y^\star$. Then, each of the two following conditions ensures that $(K, \kappa)$ is codense in $D_X^\star$:

1° $G$ is faithful.

2° $\kappa$ is pointwise mono and $\{ Vf | f: Km \rightarrow a, m \in M \}$ epi for any $a \in A$.

PROOF is straightforward.

4. LIFTINGS OF FUNCTORS AND MONADS.

Let $E_X$ be the sub-2-category of $C_X$ having the same objects as $C_X$ and such that an arrow $(F, \phi):(A, U) \rightarrow (B, V)$ in $C_X$ belongs to $E_X$
iff $VF = U$ and $\phi = 1_U$, and any 2-cell in $C_X$ between arrows of $E_X$ belongs to $E_X$.

Under a mild supposition arrows (and thus also extensions) in $E_X$ are given by those in $C_X$. We say that a functor $V: B \to X$ has the property of transfer if for every object $b$ of $B$ and every isomorphism $g: x \to V b$ there is an isomorphism $f'$ of $B$ such that $V f' = f$. Now, if $f$ has the property of transfer, then for every arrow $(F, \phi): (A, U) \to (B, V)$ in $C_X$ there exists an arrow in $E_X$ isomorphic with it.

$E_X$ is a representable 2-category. A comma object for an opspan

$$(A, U) \xrightarrow{F} (C, W) \xleftarrow{G} (B, V)$$

in $E_X$ is a full subcategory of a comma object $F/G$ in $CAT$ determined by all $f: Fa \to Gb$ such that $W f = 1$. This description shows that pointwise left extensions in the sense of 1.13 (considered for arrows in $E_X$) do not agree with those in $E_X$ in the sense of [20]; the last ones are again too strong.

Let us have functors $U: A \to X$ and $V: B \to X$ and denote briefly the category of arrows $E_X((A, U), (B, V))$ by $E(U, V)$. By the definition $E(U, V)$ is the category of liftings of $U$ through $V$ (i.e. $VF = U$).

Recall that this category has as morphisms natural transformations

$$a: F \to F'$$

such that $V a = 1$.

We will be interested in the structure of this category.

$V$ has the property of unicity if every isomorphism $f$ of $B$ such that $V f = 1$ is the identity.

4.1. Lemma. If $V$ is faithful and has the property of unicity, then the category of arrows $E(U, V)$ is an ordered class for every object $U$ of $E_X$. If, in addition, $V$ reflects isomorphisms, $E(U, V)$ is a discrete category.

Proof is evident.

Any functor $K: M \to A$ induces a functor

$$E(K, V): E(U, V) \to E(UK, V).$$
If a right extension $R_K(T)$ of $T$ along $K$ in $E_X$ exists for each functor $T : M \to B$ such that $VT = UK$, then we obtain a functor

$$R_K(-) : E(UK, V) \to E(U, V)$$

which is right adjoint to $E(K, V)$. We are going to show that it enables us to partly recognize the structure of $E(U, V)$ from the structure of $E(UK, V)$, i.e. to recognize the structure of liftings of $U$ through $V$ on a suitable full subcategory $M$ of $A$.

4.2. **THEOREM.** Let $V$ be an $(E, M)$-topological functor having the property of unicity and transfer,

$$\{ U f \mid f : a \to Km, m \in M \} \in M \quad \text{for every} \quad a \in A,$$

and $K$ be a full functor. Then $E(K, V)$ is an isotone map of the ordered class $E(U, V)$ onto the ordered class $E(UK, V)$ inducing a bijection between maximal elements of these classes.

**PROOF.** Following 2.2 there is an isotone map

$$R_K(-) : E(UK, V) \to E(U, V).$$

By 1.17, $R_K(-)$ is a right inverse to $E(K, V)$. Thus $E(K, V)$ is surjective and since $R_K(-)$ is a right adjoint to $E(K, V)$, $E(K, V)$ induces a bijection between the classes of maximal elements of $E(UK, V)$ and of $E(U, V)$.

4.3. **THEOREM.** Let $X$ be an (extremally epi-monosource)-category with sums, $M$ be small and $B$ cocomplete and co-well-powered. Let $V$ be a faithful $q$-functor having the property of unicity and transfer,

$$\{ U f \mid f : a \to Km, m \in M \}, \quad \text{for any} \quad a \in A,$$

be mono and $K$ be a full functor. Then, $E(K, V)$ induces a bijection between minimal elements of $E(U, V)$ and those of $\{ FK \mid F \in E(U, V) \}$. In addition, if $V$ reflects isomorphisms, then $E(K, V)$ is injective.

**PROOF** follows similarly as above from 1.16, 2, 1.17, 2, 2.8, 3, 2.9 and 4.1.

4.4. **EXAMPLE.** Let $A$ be the category of distributive lattices, $Ord$ the cat-
egory of ordered sets, and $V: \text{Ord} \to \text{Set}$ be the forgetful functor. Let $U: A \to \text{Set}$ be the functor which assigns to each distributive lattice the set of all its sublattices and to each homomorphism $f$ the mapping $Uf$ carrying a sublattice to its image in $f$. Functors $F \in E(U, V)$ correspond to functorial orderings of the set of all sublattices. Such functors are investigated by M. Sekanina in [19]. He has shown that

$$\{ Uf \mid f: a \to 4 \}, \text{ for any} \ a \in A,$$

is mono, where $4$ is the four-element Boolean algebra. Since $V$ is (extremally epi-monosource)-topological, following 4.2 there is only finitely many maximal $F$. 4.3 makes possible to study liftings of $U$ along the forgetful functor $A \to \text{Set}$.

Let $P: X \to X$ be a functor. If we put $U = PV$, then $E(U, V)$ is the category of all liftings of $P$ along $V$.

4.5. EXAMPLE. Let $P^+: \text{Set} \to \text{Set}$ be the covariant power-set functor. It is easy to show that $\{ P^+Vf \mid f: b \to 3 \}$ is mono for any ordered set $b$, where $3$ is the three-element chain. Let $K$ be the inclusion of the full subcategory of $\text{Ord}$ generated by $3$. By 4.2, $E(K, V)$ induces a bijection between maximal elements of $E(U, V)$ and $E(UK, V)$. Hence maximal elements of $E(U, V)$ form a finite set. These maximal liftings are dealt with in [18].

Now, let $(P, \eta, \mu)$ be a monad in $X$. We will be interested in liftings of the monad $P$ along $V$, i.e. in monads $(S, \tilde{\eta}, \tilde{\mu})$ in $B$ such that:

$$VS = PV, \quad V\tilde{\eta} = \eta V \quad \text{and} \quad V\tilde{\mu} = \mu V.$$

These liftings form a full subcategory $Z_E(P, V)$ of the category $Z(P, V)$ from 3.1.

4.6. LEMMA. Let $V$ be faithful. Then $Z_E(P, V)$ is a full subcategory of $E(PV, V)$.

PROOF. Let

$$(S_i, \eta_i, \mu_i) \in Z_E(P, V) \quad \text{for} \ i = 1, 2,$$
and \( s_1 \to s_2 \) be a morphism in \( E(PV, V) \). It holds
\[
V(\alpha \cdot \eta_1) = V \eta_1 = \eta V = V \eta_2
\]
and similarly
\[
V(\alpha \cdot \mu_1) = V(\mu_2 \cdot (\alpha \circ \alpha)).
\]
Since \( V \) is faithful, \( \alpha \) is a morphism of monads.

The study of the structure of \( Z_E(P, V) \) analogous to the preceding examination of liftings of functors is based on the following result. Consider the diagram

\[
\begin{array}{ccc}
K & \xrightarrow{M} & T \\
\downarrow{\delta} & & \downarrow{\delta} \\
A & \xleftarrow{PV} & A
\end{array}
\]

and regard the following special case of the dual of (2):

\[
\begin{array}{ccc}
[K, T] & \xleftarrow{M+M} & [T, T] \\
\downarrow{[\delta, \delta]} & & \downarrow{[\delta, \delta]} \\
A & \xleftarrow{PV, [\mu, [K, P \delta]]} & A
\end{array}
\]

Here \( M+M \) is the sum of two copies of the category \( M \) and \( [K, T], \ldots, [\delta, \delta] \) are induced functors and natural transformations.

4.7. PROPOSITION. Let us have a natural transformation \( \nu : K \to T \) such that \( \eta VK = \delta V \nu \) and suppose that \( \delta \) is \( V \)-cogenerating. Let \((R, \rho), [\rho_0, \rho_1]\) be a right extension of \(([T, T], [\delta, \delta])\) along
\[
([K, T], [1_{PVK}, \mu VK, P \delta])
\]
in \( D^*_X \). Then there exist natural transformations \( \tilde{\eta}, \tilde{\mu} \) such that \((R, \tilde{\eta}, \tilde{\mu})\) is a monad and \((R, \rho)\) a lax lifting of \( P \) along \( V \). Further for any monad \((S, \tilde{\eta}, \tilde{\mu})\) in \( A \), any lax lifting \((S, \sigma)\) of \( P \) along \( V \) and any natural trans-
formations $\sigma_0: SK \to T$ and $\sigma_1: ST \to T$ such that
$$\delta \cdot V\sigma_0 = \sigma K \quad \text{and} \quad \delta \cdot V\sigma_1 = \mu VK \cdot P\delta \cdot \sigma T,$$
there exists a unique morphism of liftings $a: (S, \sigma) \to (R, \rho)$.

PROOF. Denote
$$\Lambda = ([T, T], [\delta, \delta]) \quad \text{and} \quad \Lambda' = ([K, T], [1_{PVK}, \mu VK \cdot P\delta]).$$
Since $\eta VK = \delta \cdot V\nu$ and
$$\mu VK \cdot P\delta \cdot \eta VT = \mu VK \cdot \eta PVK \cdot \delta = \delta,$$
$(1_A, \eta V), (\nu, I_T)$ is an extension of $\Lambda$ along $\Lambda'$ in $D^*_X$. Thus there is a unique natural transformation
$$\tilde{\eta}: I_A \to R \quad \text{such that} \quad \rho \cdot V\tilde{\eta} = \eta V$$
(see the dual of 1.6). The diagrams

\begin{align*}
VRK & \xrightarrow{VR\rho_0} VRT & \xrightarrow{VR\rho_1} VT \\
\rho RK & \xrightarrow{PV\rho_0} PVT & \xrightarrow{PV\rho_1} \delta \\
P\rho K & \xrightarrow{P\rho K} P\rho K & \xrightarrow{PV K} P\rho K
\end{align*}

imply that
$$(RR, \mu V \cdot P\rho \cdot \rho R), [\rho_1 \cdot R\rho_0, \rho_1 \cdot R\rho_1]$$
is an extension of \( \Delta \) along \( \Delta' \) in \( D_X^* \). Thus there is a natural transformation

\[ \tilde{\mu} : RR \to R \]

such that \( \rho \cdot V\tilde{\mu} = \mu V \rho \cdot \rho R \).

Following 3.3 and the dual of 1.18, \((R, \tilde{\eta}, \tilde{\mu})\) is a monad and \((R, \rho)\) is a lax lifting because \([\delta, \delta]\) is \(V\)-cogenerating.

Consider \( S \) from the theorem. Hence \((S, \sigma), [\sigma_0, \sigma_1]\) is an extension of \( \Delta \) along \( \Delta' \) in \( D_X^* \). Thus there is a natural transformation

\[ \alpha : S \to R \]

such that \( \sigma = \rho \cdot Va \).

The diagrams (6) and (7) from the proof of 3.2 imply that \( \alpha \) is a morphism of monads.

4.8. REMARK. (a) Following the dual of 1.6, \( \rho_I \) is an action of \( R \) on \( T \) and it holds

\[ \rho_0 \cdot \tilde{\eta}K = \nu \quad \text{and} \quad \rho_0 \cdot \tilde{\mu}K = \rho_1 \cdot R\rho_0. \]

(b) For the existence of a unique natural transformation \( \alpha : S \to R \) preserving unit and multiplication it is not necessary to suppose that \((S, \tilde{\eta}, \tilde{\mu})\) satisfies the monad axioms.

(c) If we do not assume that \( \delta \) is \( V\)-cogenerating, then the assertion of the above proposition holds except the fact that \( \eta \) is a right unit.

4.9. LEMMA. Let \((S, \tilde{\eta}, \tilde{\mu})\) be a monad in \( A \), \((S, \sigma)\) a lax lifting of \( P \) along \( V \) and \( \sigma_0 : SK \to T \) a natural isomorphism such that \( \delta \cdot V\sigma_0 = \sigma K \). If we put \( \sigma_I = \sigma_0 \cdot \tilde{\mu}K \cdot S\sigma^{-1}_0 \), it holds \( \delta \cdot V\sigma_I = \mu VK \cdot P\delta \cdot \sigma T \).

4.10. THEOREM. Let the suppositions of 4.7 be satisfied and, in addition,
K be full and R pointwise. Then the following statements are equivalent:

1° $\rho_0$ is iso (and one can choose R such that $\rho_0 = 1_T$).

2° There is a monad $(S, \eta, \mu)$ in A, a lax lifting $(S, \sigma)$ of P along V and a natural isomorphism $\sigma_0 : SK \to T$ such that $\delta \cdot V \sigma_0 = \sigma K$.

3° For any $m, n \in M$ and any $g : Kn \to Tm$ there exists

$$\tilde{g} : Tn \to Tm$$

such that $\delta_m \cdot V \tilde{g} = \mu V_{Kn} \cdot P \delta_m \cdot PVg \cdot \delta_n$.

**Proof.** 1 $\implies$ 2 by 4.7. Let 2 hold and consider $g : Kn \to Tm$. Put

$$\tilde{g} = \sigma_{0,m}^{-1} \cdot \mu_{Kn}^{-1} \cdot S g \cdot \sigma_{0,n}^{-1}.$$ 

Then 3 follows from the diagram

Let 3 hold. Then

$$\delta_n : VTn \to PVKn, \quad T \mu : Tn \to Tm, \quad \tilde{g} : Tn \to Tm,$$

is a universal solution of the diagram

where $f$ ranges over morphisms $Kn \to Km$ and $g$ over $Kn \to Tm$ with $m \in M$.

Namely it is a solution by 3 and the naturality of $\delta$ and for another solution

$$a, u : V a \to PVKn, \quad \hat{f} : a \to Tm, \quad \hat{g} : a \to Tm$$

the desired morphism $a \to Tn$ is equal to $\hat{1}_{Kn}$. Hence $RKn = Tn$ by the dual of 1.15, and thus 1 holds.
Denote by $Z(P, V, \delta)$ the subcategory of $Z(P, V)$ consisting of lax liftings $(S, \sigma)$ such that $SK = T$ and $\sigma K = \delta$

and morphisms of liftings $a$ such that $a K = 1_T$.

4.11. THEOREM. Let the suppositions of 4.10 be satisfied. Then $Z(P, V, \delta)$ is non-void iff the condition 3 from 4.10 holds and in this case $(R, \rho)$ from 4.7 is a terminal object of $Z(P, V, \delta)$.

PROOF. The first assertion follows from 4.10 and the second one from 4.7, 4.9 and 4.10.

Let $Z_E(P, V, T)$ be the full subcategory of $Z_E(P, V)$ consisting of all liftings $S \in Z_E(P, V)$ such that $SK = T$.

4.12. COROLLARY. Let us have functors $K : M \to A$, $T : M \to A$, $V : A \to X$

and a monad $(P, \eta, \mu)$ in $X$ such that $VT = PV K$. Let $V$ be an $(E, M)$-topological functor having the property of unicity and transfer, $K$ be a full functor and

$$\{ PV f \mid f : a \to km, \ m \in M \} \in M \ \text{for any} \ a \in A.$$

Then $Z_E(P, V, T) \neq \emptyset$ iff there is a natural transformation $\nu : K \to T$ such that $VV \nu = \eta VK$ and for any $m, n \in M$ and any $g : Kn \to Tm$ there is

$$\hat{g} : Tn \to Tm \ \text{such that} \ V\hat{g} = \mu V K m . PV g.$$

In this case a right extension of

$$([T, T], [1_{VT}, 1_{VT}]) \ \text{along} \ ([K, T], [1_{VT}, \mu VK])$$

in $D_X$ is the greatest element of $Z_E(P, V, T)$.

PROOF. Since $X$ is an $(E, M)$-category, $\{ PV f \mid f \in E \}$ implies that

$$\{ PV f \mid f \in E \} \cup \{ \mu V K m . PV g \mid g : a \to Tm, \ m \in M \} \in M$$

for any $a \in A$. By 2.2 there exists a pointwise right extension

$$(R, \rho), [\rho_0, \rho_1] \ \text{of} \ ([T, T], [1_{VT}, 1_{VT}])$$

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along \([K, T], [I_{VT}, \mu VK]\) in \(D_X^\{1\}\) and one can choose \(p = 1_{P V}\). Now, the assertion follows from 4.7, 4.9 and 4.10 for the case \(\delta = 1_{VT}\).

The construction of diagram (8) is not functorial in \(T\). So we had to state Theorem 4.11 locally, i.e. as a result asserting the existence of a terminal lifting for each \(\delta\) and not the existence of a right adjoint for the functor from \(Z(P, V)\) to the category of \(T\)'s. Consequently Corollary 4.12 admits the case where \(S\) is maximal in \(Z_E(P, V)\) but \(S\) is not maximal among restrictions of functors from \(Z_E(P, V)\) on \(M\).

Liftings of the monad \(P^+\) on \(Ord\) (compare 4.5) were dealt with by M. Sekanina.

**APPENDIX. PRESHEAF CONSTRUCTION IN \(E_X\).**

We have indicated that 2-categories \(D^C_X\) and \(E^C_X\) can be defined for any 2-category \(C\). Formally \(E^C_X\) is the comma object \(1_{C/X}\) of the opspan

\[
\begin{array}{ccc}
C & \xrightarrow{1} & X \\
\downarrow & & \downarrow \\
C & \xrightarrow{X} & \end{array}
\]

in \(2-CAT\) (see [20]) and \(D^C_X\) is the lax comma object \(1_{C//X}\), in \(2-CAT\) (in the sense of [13]). We have also indicated that certain previous results hold in this general context (e.g. Theorems 1.10 and 3.2). Now, we will be interested in properties of \(D^C_X\) and \(E^C_X\).

Let \(C\) have comma objects. Then (as for \(C = CAT\)) any opspan

\[
(A_0, U) \xrightarrow{(F, a)} (A_2, W) \xleftarrow{(G, 1)} (A_1, GW)
\]

has a comma object

\[
\begin{array}{ccc}
(D_0, 1) & \xleftarrow{(D_1, W \lambda, a D_0)} & (D_1, W \lambda, a D_0) \\
\downarrow (F, a) & \lambda & \downarrow (G, 1) \\
(A_0, U) & \xrightarrow{(G, 1)} & (A_1, GW)
\end{array}
\]
where

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ & F/G \ar[dl]_{D_0} \ar[dr]^{D_1} & \\
A_0 \ar[r]_\lambda & A_1 & G \ar[l]_{A_2}
}
\end{array}
\end{array}
\]

is a comma object in \( C \).

Let \( C \) be representable and \( X \) be an object of \( C \) such that \( C(A, X) \) has pullbacks for all \( A \in C \) and \( C(F, X) \) preserves them for any arrow \( F \) of \( C \). Then \( D^C_X \) is representable and comma objects in \( D^C_X \) are described in the same way as in 1.14. The description of 2-pullbacks follows from the next consideration about 2-limits.

Let \( C \) be 2-complete and \( X \in C \) be complete in the previous sense. It means that \( C(\cdot, X) \) is a functor into the category of complete categories and limit preserving functors. Then \( D^C_X \) is 2-complete and the 2-limit of a diagram \( G : S \to D^C_X \) is \( (\lim_{\Delta}^C_X G, \lim_{\Delta}^C G) \) where \( \Delta^C_X : D^C_X \to C \) is the underlying functor and \( \tilde{G} : S \to C(\lim_{\Delta}^C_X G, X) \) is defined as follows:

\[
\tilde{G}(s) = U_s \cdot T_s \quad \text{where} \quad s \in S,
\]

if: \( G(s) = (A_s, U_s) \), \( T_s : \lim_{\Delta}^C_X G \to A_s \) is the component of a limit cone and \( \tilde{G}(g) = \tilde{g} T_s \), where \( g : s \to s' \) is a morphism in \( S \) and \( G(g) = (F g, \tilde{g}) \).

The limit cone in \( D^C_X \) has components

\[
(T_s, \tau_s) : \lim \tilde{G} \to G s, \quad \text{where} \quad \tau_s : \lim \tilde{G} \to \tilde{G} s
\]
is a component of a limit cone in \( C(\lim_{\Delta}^C_X G, X) \).

If \( C \) is representable, then \( E^C_X \) is representable for any \( X \). 2-pullbacks of \( E^C_X \) are those of \( C \) and the comma object of an opspan

\[
(A_0, F W) \xrightarrow{F} (A_2, W) \xleftarrow{G} (A_1, GW)
\]
is a 2-enriched op-localization (compare [21] page 167) of \( F/G \) at \( W/\lambda \) (in the notation (9)).

In the sequel we will work in a representable 2-category \( C \), endow-
ed with a 2-functor $P : C^{coop} \to C$ (i.e. both arrows and 2-cells are reversed). Further, there is specified a subclass of objects of $C$, which will be called legitimate objects and for any such legitimate object it is given an arrow $Y_A : A \to PA$. Finally, we demand that the assignment
\[ \phi^{A,B}(F) = Y_A/F \]
yields a full and faithful functor
\[ \phi^{A,B} : C(B, PA) \to Cov(A, B) \]
and these functors form the components of a pseudo-natural transformation (we recall that $Cov(A, B)$ is the category of all covering spans from $A$ to $B$, i.e. of all comma objects of opspans from $A$ to $B$). All these structures are present in a precosmos (see Street [21]). With size conditions aside (i.e. all objects are legitimate) such a $C$ is precisely a uniform precosmos. $C$ models the 2-category $CAT$ of categories. Namely, $PA$ is the category of all functors from $A^{op}$ to $Set$, legitimate objects are categories with small hom-sets and $Y_A$ is the Yoneda embedding.

Suppose that $X$ is a legitimate object of $C$ and consider $D_X^C$. Take $(A, U) \in D_X^C$ such that $A$ is legitimate and define
\[ P_X(A, U) = (P_X(A, U), D_1^{(A,U)}), \]
where
\[ \begin{array}{ccc}
D_0^{A,U} & \xrightarrow{\lambda_{(A,U)}} & D_1^{A,U} \\
\downarrow P_X(A, U) & & \downarrow P_X(A, U) \\
PA & \xleftarrow{\lambda_{(A,U)}} & PU.Y_X
\end{array} \]
is a comma object in $C$. Using the universal property of a comma object, one can complete $P_X$ to a 2-functor $P_X : (D_X^C)^{coop} \to E_X^C$. The 2-cell effect of $U$ on homs (see [21] page 144) $Y_A \to DU.Y_X.U$ induces an arrow of spans
\[ Y_{X(A,U)} : (Y_A : A, U) \to (D_0^{A,U}, P_X(A, U), D_1^{(A,U)}), \]
i.e. an arrow
The assignment 

\[ (Y^X_{(A,U)}, I_U): (A, U) \to (P_X(A, U), D^U_I(A, U)) \]

in \( D^C_X \). The assignment 

\[ \Phi^X_{(A,U), (B,V)}(F, \phi) = (Y^X_{(A,U)}, I_U)_{D^C_X} (F, \phi) \]

is neither pseudo-natural in \((A, U)\), nor faithful, nor full. But one has the following result.

**THEOREM.**

\[ \Phi^X_{(A,U), (B,V)}: E^C_X((B, V), P_X(A, U)) \to Cov_{D^C_X}((A, U), (B, V)) \]

are fully faithful and form the components of a pseudo-natural transformation.

The proof is rather long and the computations lean on the fact that 

\[ Y_A / D^0_0(A, U) = Y^X_{(A,U), 1}. \]

The author does not know whether \( D^C_X \) or \( E^C_X \) resp. is a precosmos. But it seems not to be so because comma objects in \( D^C_X \) and \( E^C_X \) are bad. For instance, the prescription \( P_X(A, U) = P A \times X \) does not work. Good comma objects are those which appear in the just stated theorem and these are the same which yield pointwise left extensions in the sense of 1.13. Namely, we mean comma objects in \( D^C_X \) of opspans from \( E^C_X \).

Let the 2-functor \( P: C^{coop} \to C \) have a left 2-adjoint \( P^*: C \to C^{coop} \) (i.e. \( C \) is a cosmos). Then one can construct a left 2-adjoint

\[ P^*: E^C_X \to (E^C_X)^{coop} \]

as follows:

\[ P^*(A, U) \]

Here \( Y^*_X \) is the image of \( Y^*_X \) in the adjunction isomorphism 

\[ C(B, PA) \to C(A, P^*B)^{op}. \]
However, $P^*_X$ cannot be extended on $D_X^C$.

Consider the case $C = CAT$. Then objects of $P_X(A, U)$ are triples $(F, \delta, x)$, where

$$F: A^{op} \to Set, \quad x \in X \quad \text{and} \quad \delta: F \to X(U \cdot, x)$$

is a natural transformation. Morphisms

$$(F, \delta, x) \to (F', \delta', x')$$

of $P_X(A, U)$ are couples $(a, f)$ where

$$a: F \to F' \quad \text{and} \quad f: x \to x' \quad \text{such that} \quad \delta'. a = X(U, f). \delta.$$ 

The arrow $Y^X_{(A, U)}$ is given by

$$Y^X_{(A, U)}(a) = (A(\cdot, a), \delta, U a), \quad \text{where} \quad \delta(h) = U h.$$ 

This construction is closely related to the initial completion of a faithful functor (see [8, 10, 23]). Namely consider the full sub-2-category $E_X$ of $ECAT$ having objects $(A, U)$ such that $U$ is faithful. Then the functor $D_1(A, U): P_X(A, U) \to X$ need not be faithful, but it is faithful on a full subcategory $P_X(A, U)$ of $P_X(A, U)$ consisting of the $(F, \delta, x)$ such that $\delta$ is mono. Then the arrow $Y^X_{(A, U)}: (A, U) \to (P_X(A, U), D_1(A, U))$ is precisely the initial completion $E^{-2}$ of $(A, U)$ in the sense of [8]. Similarly $P^{**}_X(A, U)$ yields $E^2$ from [8].

The same construction is given in [21], page 175. There is considered the full sub-2-category $Simp$ of $E_{Set}$ consisting of all the functors $U$ from $A$ to $Set$ such that any constant mapping underlies a morphism of $A$ and the 2-functor $\bar{P}: Simp^{coop} \to Simp$ such that $\bar{P}(A, U)$ is the full subcategory of $P_{Set}(A, U)$ consisting of all $(F, \delta, x)$ such that $\delta_a(Fa)$ contains all constant mappings $U a \to x$. But $\bar{P}$ does not make, from $Simp$, a cosmos, and similarly for $P_X$ and $E_X$ (though the first assertion is stated in [21]).

EXAMPLE. Let $A$ be the subcategory of $Set$ having one object $2 = \{0, 1\}$.
and the identity and constant mappings as morphisms. Let $U : A \to \text{Set}$ be the inclusion. Then $\bar{P}(A, U)$ has objects $(x, \rho)$, where $\rho$ is a reflexive binary relation on a set $x$, and morphisms are relation preserving mappings. Let $B$ be the subcategory of $\text{Set}$ having one object 3 and the identity and constant mappings as morphisms. Let $V : B \to \text{Set}$ be the inclusion. There are functors

$$F, G : (B, V) \to (\bar{P}(A, U), \bar{D}_{I})$$

such that there is no 2-cell $F \Rightarrow G$ in $\text{Simp}$ because there are two incomparable reflexive relations on 3. But

$$\overline{Y}_{A/Simp} F = \overline{Y}_{A/Simp} G = \emptyset,$$

and thus $\overline{Y}_{A/Simp}$ is not full.
REFERENCES.


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