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Symmetric monoidal closed categories generated by commutative adjoint monads

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Kock (1971) has shown that if $V$ is a symmetric monoidal closed category with equalizers and $T$ is a commutative $V$-monad on $V$, then $V^T$, the category of $T$-algebras in $V$, is a closed category. The primary purpose of the present paper is to extend Kock’s result by showing that if $V$ is a symmetric monoidal closed category with coequalizers and $T$ is a commutative $V$-monad on $V$, then $V^T$ is a symmetric monoidal category. We also show that the pair of adjoint functors connecting $V^T$ and $V$ are symmetric monoidal functors, and that the adjunction natural transformations are monoidal natural transformations. Moreover, if we assume that $V_o$ has equalizers, then we show that $V^T$ is symmetric monoidal closed. We also examine the relationship between the category of commutative monoids in $V^T$ and a category of algebras in the category of commutative monoids in $V$. We conclude with several examples which illustrate the results.

1. PRELIMINARIES.

Throughout the paper we assume that

$$V = ([V_o, \Theta, l, a, r, l, c], p, [V_o, V, \hom V, l, i, j, L])$$

is a symmetric monoidal closed category in the sense of Eilenberg and Kelly (1966), page 535. We also adopt the notation and terminology contained therein unless otherwise noted.

We recall that a monoid in $V$ is a triple $(A, e, m)$ where $A \in V_o$, and $e : l \to A$ and $m : A \otimes A \to A$ are morphisms in $V_o$ such that

$$m \cdot A \otimes e \cdot r^l_A = A = m \cdot e \otimes A \cdot r^l_A \quad \text{and} \quad m \cdot m \otimes A = m \cdot A \otimes m \cdot a_{AAA}.$$

The monoid $(A, e, m)$ is commutative if $m = m \cdot c_{AAA}$. A monoid morphism
f : (A, e, m) → (A', e', m') in V is a morphism f : A → A' in V₀ such that

\[ f \cdot m = m' \cdot f \circ f \] and \[ f \cdot e = e'. \]

We denote the category of monoids in V by M(V) and the full subcategory of commutative monoids in V by CM(V). Further, V^V denotes the category of V-functors from V into V, and Adj(V) denotes the full subcategory of V^V consisting of all V-functors T : V → V having a right V-adjoint. We denote by MAdj(V) the category of adjoint V-monads on V (i.e., V-monads (T, η, μ) on V where T has a right V-adjoint) and by CMAdj(V) the full subcategory of MAdj(V) consisting of those adjoint V-monads on V which are commutative in the sense of Kock (1971) or Kock (1970). We note that there are obvious forgetful functors

\[ U : M(V) → V₀, \quad U' : CM(V) → V₀, \quad U_I : MAdj(V) → Adj(V) \]
and \[ U_I' : CMAdj(V) → Adj(V). \]

LEMMA 1.1. The functor \( \Phi : V₀ → Adj(V) \) defined by the rules

\[ \Phi A = A \otimes (-) \] and \[ \Phi f = f \otimes (-) \]

is a monoidal equivalence of categories. There are also equivalences

\[ \Phi_I : M(V) → MAdj(V) \] and \[ \Phi'I : CM(V) → CMAdj(V) \]
such that \( \Phi U = U_I \Phi_I \) and \( \Phi' U' = U_I' \Phi'I \).

PROOF. The first statement follows from Bunge (1969), Theorem 3.8, page 89, while the second follows from Bunge (1969), Corollary 3.9, page 90, and Wolff (1973), Proposition 2.7, page 119.

In the interest of brevity many strings of equations appear in the proofs of the results herein, and any equality signs in these strings have been marked according to the following scheme (as in Kock (1971)) to indicate the reason for the equality. An equality sign with a letter above it, as in \( \eta = \), indicates equality as a result of naturality of that named natural transformation. One marked with a numeral within parentheses, as in \( (3.4) = \), follows from that numbered theorem, equation, diagram, or whatever in this paper. One marked with a numeral without parentheses, as in \( 2 = \), indicates
that the reason for the equality will be explained below. Lastly, an unmark-
ed equality sign denotes an obvious or trivial equation.

2. COMMUTATIVE ADJOINT MONADS AND CATEGORIES OF ALGEBRAS.

We first obtain the main result which when considered in conjunction
with Kock's Theorem will imply that the category of T-algebras for a commu-
tative adjoint V-monad T on V is a symmetric monoidal closed category,
provided \( V_0 \) has both equalizers and coequalizers.

**THEOREM 2.1.** Let \( V_0 \) have coequalizers and let T be a commutative ad-
joint V-monad on V. Then \( V^T \), the category of T-algebras in V, is a sym-
metric monoidal category.

**PROOF.** By Lemma 1.1, it is equivalent to assume that T is induced by a
commutative monoid \( (A, e, m) \) in V, and so we denote the monad T by
\( T(A(-), \eta_A, \mu_A) \), where
\[
\eta_A = e \otimes (-), \mu_A = m \otimes (-) \cdot a(-)_{A(-)}^{-1}.
\]

(i) We construct the tensor product \( (M \otimes_A N, \rho_{M \otimes_A N}) \) of two \( A \)-alge-
bras \( (M, \rho_M) \) and \( (N, \rho_N) \): as follows. Consider the pair of morphisms
\( r_M, r_N : A \otimes (M \otimes N) \to M \otimes N \), where
\[
r_M = \rho_M \cdot a_{A \otimes N}(M) \quad \text{and} \quad r_N = M \otimes \rho_N \cdot a_{M \otimes A \otimes N}(M) \cdot a_{A \otimes N}(M),
\]
and let \( q_{MN} : M \otimes N \to M \otimes_A N \) be the coequalizer of \( r_M \) and \( r_N \). Note that
since \( A \) is an adjoint V-monad, \( A \otimes q_{MN} \) is the coequalizer of \( A \otimes r_M \) and
\( A \otimes r_N \). Now consider the following diagram in \( V_0 \):

\[
\begin{array}{ccc}
A \otimes (A \otimes (M \otimes N)) & \xrightarrow{A \otimes r_M} & A \otimes (M \otimes N) & \xrightarrow{A \otimes q_{MN}} & A \otimes (M \otimes_A N) \\
A \otimes r_N & & & & \rho_{M \otimes_A N} \\
A \otimes (M \otimes N) & \xrightarrow{r_M} & M \otimes N & \xrightarrow{q_{MN}} & M \otimes_A N \\
\end{array}
\]
It is easy to check that both squares on the left hand side of (2.1.1) (one involving the \( r_M \)'s, the other the \( r_N \)'s) commute. It follows that \( q_{MN} \cdot r_M = q_{MN} \cdot r_N \) coequalizes \( A \otimes r_M \) and \( A \otimes r_N \), and hence there exists a unique morphism \( \rho_{M \otimes A N} : A \otimes (M \otimes AN) \rightarrow M \otimes A N \) such that

\[
(2.1.2) \quad \rho_{M \otimes A N} \cdot A \otimes q_{MN} = q_{MN} \cdot r_M = q_{MN} \cdot r_N.
\]

Since \( q_{MN} \) and \( A \otimes (A \otimes q_{MN}) \) are epimorphisms of \( V_o \), one sees that

\[
(M \otimes A N, \rho_{M \otimes A N}) = (M, \rho_M) \otimes_A (N, \rho_N)
\]

is an \( A \)-algebra. If

\[
f : (M, \rho_M) \rightarrow (M', \rho_{M'}) \quad \text{and} \quad g : (N, \rho_N) \rightarrow (N', \rho_{N'})
\]

are morphisms of \( A \)-algebras, there is a unique morphism

\[
f \otimes_A g : M \otimes A N \rightarrow M' \otimes A N'
\]

such that

\[
(2.1.3) \quad f \otimes_A g \cdot q_{MN} = q_{M'N'} \cdot f \otimes g,
\]

and one checks that \( f \otimes_A g \) is a morphism of \( A \)-algebras. Hence we have a functor \( \otimes_A : V_0^d \times V_0^d \rightarrow V_0^d \).

(ii) We take the unit object in \( V_0^d \) to be the \( A \)-algebra \( (A, m) \).

(iii) We construct a natural isomorphism

\[
\tilde{r}_{(M, \rho_M)} : (M \otimes_A A, \rho_{M \otimes_A A}) \rightarrow (M, \rho_M) \quad \text{in} \quad V_0^d
\]

as follows. Since \((A, e, m)\) is commutative, we see that for any \( A \)-algebra \((N, \rho_N)\) there is a unique morphism \( \tilde{r}_{(M, \rho_M)} : M \otimes_A A \rightarrow M \) such that

\[
(2.1.4) \quad \tilde{r}_{(M, \rho_M)} : q_{MA} \cdot M \otimes e = \rho_M \cdot c_{MA},
\]

and it is clear that \( \tilde{r}_{(M, \rho_M)} \) is a morphism of \( A \)-algebras. We claim that \( \tilde{r}_{(M, \rho_M)} \) is a natural isomorphism, having as its inverse \( q_{MA} \cdot M \otimes e \cdot r_M^{-1} \).

Consider then the following equations. We have

\[
\tilde{r} \cdot q \cdot 1 \otimes e \cdot r_M^{-1} \quad (2.1.4) \quad r \cdot c \cdot 1 \otimes e \cdot r^{-1} \quad c = \rho \cdot e \otimes l \cdot r \cdot l^{-1} \quad \frac{1}{2} = M.
\]

where 1 follows from Eilenberg and Kelly (1966), Proposition 1.1, page 512, and 2 since \( \rho_M \) is an \( A \)-structure map. On the other hand, since there
is a unique morphism $f : M \otimes_A A \to M \otimes_A A$ such that $f . q_{MA} = q_{MA} \cdot M \otimes A$, it suffices to show that

$$q \cdot l \otimes e \cdot r^{-1} \cdot \tilde{r} \cdot q = M \otimes_A A \cdot q.$$

Hence we have

$$q \cdot l \otimes e \cdot r^{-1} \cdot \tilde{r} \cdot q = q \cdot l \otimes e \cdot r^{-1} \cdot \rho \cdot c = q \cdot l \otimes e \cdot r^{-1} \cdot c = q \cdot \rho \otimes 1 \cdot l \otimes e \cdot r^{-1} \cdot c = q \cdot l \otimes (l \otimes e) \cdot l \otimes r^{-1} = q \cdot M \otimes A \cdot q,$$

where 1 follows since $q$ is a coequalizer, 2 from coherence and naturality of $(\cdot) \otimes e$, and 3 since $(A, e, m)$ is a monoid.

(iv) Similarly we construct a natural isomorphism

$$\tilde{l}_{(M, \rho_M)} : (A \otimes_A M, \rho_A \otimes_A M) \to (M, \rho_M)$$

in $V^d_0$ by noting that for any $A$-algebra $(M, \rho_M)$ there is a unique morphism

$$\tilde{l}_{(M, \rho_M)} : A \otimes_A M \to M$$

such that

$$(2.1.5) \quad \tilde{l}_{(M, \rho_M)} \cdot q_{AM} = \rho_M.$$

One checks as above that $\tilde{l}_{(M, \rho_M)}$ is a natural isomorphism of $A$-algebras with inverse $q_{AM} \cdot e \otimes M \cdot l^{-1}_M$.

(v) We construct a natural isomorphism $\tilde{a}_{(M, \rho_M)(N, \rho_N)(P, \rho_P)}$:

$$((M \otimes_A N) \otimes_A P, \rho_{(M \otimes_A N) \otimes_A P}) \to (M \otimes_A (N \otimes_A P), \rho_{M \otimes_A (N \otimes_A P)})$$

in $V^d_0$, where $(M, \rho_M)$, $(N, \rho_N)$ and $(P, \rho_P)$ are any $A$-algebras in $V$, by considering the diagram (2.1.6) (next page) in $V_0$. In (2.1.6),

$$\sigma_M = r_M \otimes P \cdot a^{-1}_M, M \otimes N, P, \quad \sigma_N = \tau_N \otimes P \cdot a^{-1}_M, M \otimes N, P,$$

$$\sigma_N = M \otimes r_N \cdot a_M, A, N \otimes P, a^{-1}_M, A, M \otimes P$$

and

$$\sigma_P = M \otimes r_P \cdot a_M, A, N \otimes P, c_{AM} \otimes (N \otimes P), a^{-1}_M, A, M \otimes P.$$

An analysis of the diagram (2.1.6) shows that the composite
coequalizes the pair of maps $\sigma_M$ and $\sigma_N$, and since $q_{MN} \otimes P$ is the coequalizer of this pair, there is a unique morphism
\[
f : (M \otimes_A N) \otimes P \to M \otimes_A (N \otimes_A P)
\]
such that
\[
f \cdot q_{MN} \otimes P = q_{MN} \otimes P \cdot M \otimes q_{NP} \cdot a_{MNP}.
\]
In turn $f$ coequalizes the pair $r_{M \otimes_A N}$ and $r_P$ and hence, induces the unique map
\[
\tilde{a} = \tilde{a}_{(M, \rho_M \otimes N, \rho_N \otimes P, \rho_P)} : (M \otimes_A N) \otimes P \to M \otimes_A (N \otimes_A P)
\]
such that $f = \tilde{a} \cdot q_{MN} \otimes P$. It follows then that
\[
(2.1.7) \quad \tilde{a}_{(M, \rho_M \otimes N, \rho_N \otimes P, \rho_P)} : q_{MN} \otimes P = q_{MN} \otimes P \cdot M \otimes q_{NP} \cdot a_{MNP},
\]
and a lengthy but straightforward verification shows that $\tilde{a}$ is a natural isomorphism of $A$-algebras.

(vi) We construct a natural isomorphism
\[
\tilde{c}_{(M, \rho_M \otimes N, \rho_N)} : (M \otimes_A N, \rho_M \otimes_A N) \to (N \otimes_A M, \rho_N \otimes_A M)
\]
in $\mathcal{V}_0^A$ by noting that for any $A$-algebras $(M, \rho_M)$ and $(N, \rho_N)$ there is a natural isomorphism of $A$-algebras $\tilde{c}_{(M, \rho_M \otimes N, \rho_N)} : M \otimes_A N \to N \otimes_A M$ such that
\[
(2.1.8) \quad \tilde{c}_{(M, \rho_M \otimes N, \rho_N)} \cdot q_{MN} = q_{MN} \cdot c_M.
\]
where $q'_{NM}: N \otimes M \rightarrow N \otimes_A M$ is the coequalizer of the corresponding morphisms $r'_N, r'_M: A \otimes (N \otimes M) \rightarrow N \otimes M$.

We now claim that

$$\mathcal{V}^A = (\mathcal{V}_0^A, \Theta_A, (A, m), \tilde{r}, \tilde{l}, \tilde{a}, \tilde{c})$$

is a symmetric monoidal category. The commutativity of the diagrams needed to show this, i.e., MC1-MC7 in Eilenberg and Kelly (1966), page 472 and page 512, follows from the commutativity of the corresponding diagrams for $\mathcal{V}$, from the defining relations for $\tilde{r}, \tilde{l}, \tilde{a}$ and $\tilde{c}$, from the fact that the $q_{MN}$'s are epimorphisms and from the fact that $M \otimes (-)$ preserves epimorphisms for any $M \in \mathcal{V}_0$.

Associated with the category $\mathcal{V}^A$ of $A$-algebras in $\mathcal{V}$ is a pair of adjoint functors $F^A: \mathcal{V}_0 \rightarrow \mathcal{V}_0^A$ and $U^A: \mathcal{V}_0^A \rightarrow \mathcal{V}_0$ defined on objects by

$$F^A(M) = (A \otimes M, \mu_M) \text{ and } U^A(M, \rho_M) = M,$$

with $F^A$ left adjoint to $U^A$. We now show that, relative to the symmetric monoidal structure on $\mathcal{V}^A$ as given in Theorem 2.1, these functors can be given the structure of symmetric monoidal functors and the adjunction transformations can be given the structure of monoidal natural transformations as well.

**Theorem 2.2.** Let $\mathcal{V}_0$ have coequalizers and let $T$ be a commutative adjoint $\mathcal{V}$-monad on $\mathcal{V}$. Then the adjoint functors

$$F^T: \mathcal{V}_0 \rightarrow \mathcal{V}_0^T \text{ and } U^T: \mathcal{V}_0^T \rightarrow \mathcal{V}_0$$

are symmetric monoidal functors and the adjunction transformations

$$\epsilon^T: F^T U^T \rightarrow I_{\mathcal{V}_0^T} \text{ and } \eta^T: I_{\mathcal{V}_0} \rightarrow U^T F^T$$

are monoidal natural transformations.

**Proof.** We first observe that by Lemma 1.1, it is equivalent to take

$$T = A = (A \otimes (-), \eta^A, \mu^A)$$

as in the proof of Theorem 2.1. We show first that $U^A$ is a symmetric monoidal functor. To do so, we need to construct a natural transformation
\[ \hat{U}^d_{(M, \rho_M)(N, \rho_N)} : U^d(M, \rho) \otimes U^d(N, \rho) \to U^d(M \otimes_A N, \rho_M \otimes_A N) \]

and a morphism \((U^d)_p : I \to U^d(A, m)\) in \(V_0\). We take

\[ \hat{U}^d_{(M, \rho_M)(N, \rho_N)} = q_{MN}, \]

where \(q_{MN}\) is the coequalizer of \(\tau_M\) and \(\tau_N\) as defined in the proof of Theorem 2.1, and we take \((U^d)_p = e\). We must verify MF 1-MF 4 in Eilenberg and Kelly (1966), pages 473 and 513, for \((U^d, \hat{U}^d, (U^d)_p)\). In this case, MF 1 is simply \(l_{(M, \rho_M)} : q_{AM} \circ \rho_M = l_M\), which holds as in the proof of Theorem 2.1. Similarly we have \(\hat{r}_{(M, \rho_M)} : q_{MA} \cdot M \otimes e = \tau_M\) for MF 2, again from the proof of Theorem 2.1. Also, the equations needed to verify MF 3 and MF 4 are simply the defining relations (2.1.7) and (2.1.8) for the natural isomorphisms \(\tilde{\alpha}\) and \(\tilde{\gamma}\) in \(V_0^d\).

To see that \(F^d\) is a symmetric monoidal functor, we need to construct a natural transformation

\[ \hat{F}^d_{MN} : F^d(M) \otimes_A F^d(N) \to F^d(M \otimes_A N) \]

and a morphism \((F^d)_p : (A, m) \to F^d(I)\) in \(V_0^d\) which satisfy MF 1-MF 4. For \(\hat{F}^d_{MN}\), consider the following diagram (2.2.1):

\[
\begin{array}{ccc}
A \otimes (A \otimes M) & \xrightarrow{r_{A \otimes M}} & A \otimes (A \otimes N) & \xrightarrow{q_{A \otimes M, A \otimes N}} & (A \otimes M) \otimes_A (A \otimes N) \\
A \otimes \kappa_{MN} & & & A \otimes (A \otimes (M \otimes N)) & \xrightarrow{\mu^d_{A \otimes (M \otimes N)}} & A \otimes (A \otimes (M \otimes N)) \\
A \otimes (A \otimes (A \otimes (M \otimes N))) & \xrightarrow{\mu^d_{A \otimes (M \otimes N)}} & A \otimes (A \otimes (M \otimes N)) & \xrightarrow{\mu^d_{A \otimes (M \otimes N)}} & A \otimes (M \otimes N) \\
\end{array}
\]

In (2.2.1),

\[ \kappa_{MN} = A \otimes a_{MN}, A \otimes (c_{MA} \otimes N), A \otimes a_{MN}^{-1}, A \otimes a_{MN}, A \otimes a_{MN} \]

is a coherent natural isomorphism, \(r_{A \otimes M}\) and \(r_{A \otimes N}\) are constructed as in the proof of Theorem 2.1, \(q_{A \otimes M, A \otimes N}\) is the coequalizer of \(r_{A \otimes M}\) and \(r_{A \otimes N}\), and \(\hat{F}^d_{MN}\) is the unique morphism such that

\[ (2.2.2) \quad \hat{F}^d_{MN} \cdot q_{A \otimes M, A \otimes N} = \mu^d_{A \otimes (M \otimes N)} \cdot \kappa_{MN}. \]
It is clear that $\hat{F}^d$ is natural, and one can check in a straightforward fashion that $\hat{F}^d$ is a morphism of $A$-algebras. For $r^1_{A^d}$, we take

$$ (F^d)^{\circ} = r^1_{A^d} : A \rightarrow \Theta I; $$

clearly $r^1_{A^d} : (A, m) \rightarrow (A \otimes I, \mu_I)$ is a morphism of $A$-algebras. It remains to verify MF 1 - MF 4. For MF 1, we have to show commutativity of the following diagram in $\mathcal{V}_0$:

$$
\begin{array}{c}
(A \otimes I)_A (A \otimes M) \xrightarrow{r^1_{A \otimes I}} A \otimes (I \otimes M) \\
\downarrow r^1_{A \otimes A} \\
A \otimes A (A \otimes M) \xrightarrow{L_A \otimes I} A \otimes M
\end{array}
$$

(2.2.3)

To do this, we use the fact that $\tilde{r}_{A \otimes M}$ is the unique morphism

$$ f : A \otimes (A \otimes M) \rightarrow A \otimes M \quad \text{satisfies} \quad f \cdot q_{A, A \otimes M} = u^d_M $$

But we have

$$ 1 \otimes 1 \cdot \hat{F}^d \cdot r^1_{A \otimes I} \cdot q = 1 \otimes 1 \cdot q \cdot r^1_{A \otimes I} \quad \text{by (2.2.2)} $$

where 1 follows from the coherence of $r$, $\mu$, and $r$. Hence the commutativity of (2.2.3) follows. For MF 2, we have to show that

$$ \hat{r}_{M \otimes I} : \hat{F}^d_{M \otimes I} (A \otimes M) \rightarrow A \otimes M, $$

and to do so one uses the uniqueness of $\tilde{r}_{A \otimes M}$, satisfying

$$ \tilde{r}_{A \otimes M} : q_{A \otimes M, A} = u^d : T_{A \otimes M} $$

and the commutativity of $(A, e, m)$ in the category. For MF 3, we have to show (2.2.4):

$$ A \otimes a_{MNP} : \hat{F}^d_{M \otimes N, P} (A \otimes P) = \hat{F}^d_{M, N \otimes P} (A \otimes P) $$

and since both $q$ and $q \otimes I$ are epimorphisms in $\mathcal{V}_0$, it suffices to check that (2.2.4) holds when both sides are composed on the right with

$$ q_{(A \otimes M) \otimes (A \otimes N), A \otimes P} : q_{A \otimes M, A \otimes N} (A \otimes P). $$

277
Hence we have
\[ \hat{F}^d \cdot 1_{A} \hat{F}^d \cdot \alpha \cdot q \cdot q \Theta l \xrightarrow{(2.1.7)} \hat{F}^d \cdot 1_{A} \hat{F}^d \cdot q \cdot 1_{q} \cdot a \xrightarrow{q} \]
\[ \hat{F}^d \cdot q \cdot 1 \hat{F}^d \cdot 1_{q} \cdot a \xrightarrow{(2.2.2)} \mu^d \cdot \kappa \cdot 1_{\mu^d} \cdot 1_{\kappa} \cdot a \]
\[ \mu^d \cdot 1_{\mu^d} \cdot \kappa \cdot 1_{\kappa} \cdot a \xrightarrow{\kappa} \mu^d \cdot \mu^d \cdot \kappa \cdot 1_{\mu^d} \cdot 1_{\kappa} \cdot a \]
\[ 1_{\Theta_{\alpha}} \cdot \mu^d \cdot \kappa \cdot 1_{\Theta_{\alpha}} \xrightarrow{\kappa} 1_{\Theta_{\alpha}} \cdot \mu^d \cdot \mu^d \cdot 1_{\Theta_{\alpha}} \cdot 1_{\Theta_{\alpha}} \]
\[ 1_{\Theta_{\alpha}} \cdot \hat{F}^d \cdot q \cdot \hat{F}^d \Theta_{\alpha} \cdot q \Theta_{\alpha} \xrightarrow{q} 1_{\Theta_{\alpha}} \cdot \hat{F}^d \cdot \hat{F}^d \Theta_{\alpha} \cdot q \Theta_{\alpha} , \]

where 1 follows from the monad equation \( \mu^d \cdot \mu^d = \mu^d \cdot 1_{\Theta_{\mu^d}} \) and 2 from coherence and naturality of \( \alpha \). Finally, for \( MF_4 \), we must show that
\[ A \Theta_{c_{MN}} \cdot \hat{F}^d_{MN} = \hat{F}^d_{N,M} \cdot \hat{c}_{A} \Theta_{M,A} \Theta_{N} \]
and as above it suffices to check that \( 1_{\Theta_{c}} \cdot \hat{F}^d \cdot q = \hat{F}^d \cdot \hat{c} \cdot q \). Now we have
\[ 1_{\Theta_{c}} \cdot \hat{F}^d \cdot q \xrightarrow{(2.2.2)} 1_{\Theta_{c}} \cdot \mu^d \cdot \kappa \xrightarrow{1} \mu^d \cdot 1_{\Theta(1_{\Theta_{c}})} \cdot \kappa \]
\[ \mu^d \cdot a \cdot c \Theta_{l} \cdot a \cdot 1_{\Theta(1_{\Theta_{c}})} \cdot \kappa \xrightarrow{2} \mu^d \cdot \kappa \cdot c \xrightarrow{(2.2.2)} \]
\[ \hat{F}^d \cdot q \cdot c \xrightarrow{(2.1.8)} \hat{F}^d \cdot \hat{c} \cdot q , \]

where 1 follows from the commutativity of \( \hat{A} \) and 2 from coherence. Hence \( F^d \) is a symmetric monoidal functor.

We now show that \( \epsilon^d : F^d U^d \rightarrow \mathcal{V}^d_c \) is a monoidal natural transformation, i.e., satisfies MN1 and MN2 in Eilenberg and Kelly (1966), page 474. Observe that, for any \( A \)-algebra \((M, \rho_M)\),
\[ \epsilon^d_{(M, \rho_M)} : F^d U^d (M, \rho_M) \rightarrow (M, \rho_M) \]
is the morphism \( \rho_M : (A \Theta_{M}, \mu^d_{M}) \rightarrow (M, \rho_M) \). Now in this case, MN1 becomes \( m \cdot A \Theta e. r^d_A = A \), which is one of the unit laws for \((A, e, m)\). Also MN2 becomes
\[ \rho_M \Theta_{A^N} \cdot A \Theta_{q_{MN}} \cdot \hat{F}^d_{MN} = \rho_M \Theta_{A} \rho_N , \]
and as above it suffices to check
\[ \rho_M \Theta_{A^N} \cdot A \Theta_{q_{MN}} \cdot \hat{F}^d_{MN} \cdot q_{A \Theta_{M},A \Theta_{N}} = \rho_M \Theta_{A} \rho_N \cdot q_{A \Theta_{M},A \Theta_{N}} . \]
Now:

\[
\rho_M \otimes_A \rho_N \cdot q_A \otimes_M, A \otimes N \quad (2.1.3) \quad q_{MN} \cdot \rho_M \otimes \rho_N \quad 1 = q_{MN} \cdot \tau_M \cdot A \otimes r_N \cdot \kappa_{MN} \quad 2 = \]

\[
q_{MN} \cdot \tau_M \cdot A \otimes r_N \cdot \kappa_{MN} \quad 3 = q_{MN} \cdot \tau_N \cdot \mu^{A \otimes N} \cdot \kappa_{MN} \quad (2.2.2) \]

\[
q_{MN} \cdot \tau_N \cdot \hat{F}^{A}_{MN} \cdot q_A \otimes_M, A \otimes N \quad (2.1.2) \quad \rho_M \otimes_A, A \otimes q_{MN} \cdot \hat{F}^{A}_{MN} \cdot q_A \otimes_M, A \otimes N, \]

where 1 follows from coherence and naturality, 2 since \(q_{MN}\) is the coequalizer of \(r_M\) and \(r_N\) and 3 since \(\rho_N\) (and hence \(r_N\)) is an \(A\)-structure morphism.

To complete the proof of Theorem 2.2, we show that \(\eta^A : V_\circ \to U^A F^A\) is a monoidal natural transformation, where \(\eta^A_M : M \to U^A F^A(M)\) is the morphism \(e_{\otimes M} \cdot l_M^1 : M \to A \otimes M\) in \(V_\circ\). In this case, \(MN 1\) is \(e_{\otimes I} \cdot l_I^1 = r_A^1 \cdot e\), but

\[
e_{\otimes I} \cdot l_I^1 = e_{\otimes I} \cdot r_I^1 = r_A^1 \cdot e,
\]

where 1 follows from MC 5 in Eilenberg and Kelly (1966), page 472. Also, \(MN 2\) becomes

\[
\hat{F}^{A}_{MN} \cdot q_A \otimes_M, A \otimes N \cdot (e_{\otimes M} \otimes (e_{\otimes N} \cdot l_M^1 \otimes l_N^1) = e_{\otimes (M \otimes N)} \cdot l_M^1 \otimes l_N^1.
\]

But we have

\[
\hat{F}^{A}_{MN} \cdot q_{e_{\otimes I} \otimes e_{\otimes I}} \cdot l_I^1 \otimes l_I^1 \quad (2.2.2) \quad \mu^A \cdot \kappa \cdot (e_{\otimes I} \otimes (e_{\otimes I} \cdot l_I^1 \otimes l_I^1 = \mu^A \cdot e_{\otimes I} \cdot l_I^1 \cdot e_{\otimes I} \cdot l_I^1 = e_{\otimes I} \cdot l_I^1,
\]

where 1 follows from coherence and naturality and 2 from the monad law \(\mu^A \cdot e_{\otimes I} \cdot l_I^1 = A \otimes M\). This completes the proof of Theorem 2.2.

3. SYMMETRIC MONOIDAL CLOSED CATEGORIES OF ALGEBRAS.

We have seen in Theorem 2.1 that if \(V\) is a symmetric monoidal closed category with coequalizers and \(T\) is a commutative adjoint \(V\)-monad on \(V\), then \(V^T\) is a symmetric monoidal category. If \(V_\circ\) has equalizers, Kock (1971) has shown that \(V^T\) is closed as well. The following theorem shows that these two structures on \(V^T\) are compatible, i.e., that \(V^T\) is a symmetric monoidal closed category.

Recall first from Kock (1971) the construction of the fundamental
natural transformation $\lambda_{AB} : T(AB) \to (A, TB)$, where $T$ is any $V$-endofunctor on $V$. In the case that $T$ is the functor of a commutative adjoint $V$-monad on $V$, it is equivalent by Lemma 1.1 to assume that $T = A \otimes (-)$. In this case, the natural transformation $\lambda_{MN} : A \otimes (MN) \to (M, A \otimes N)$ as constructed by Kock is defined as the following (lengthy) composite:

$$
\begin{align*}
A \otimes (MN) & \xrightarrow{u} (M, (A \otimes (MN)) \otimes M) \xrightarrow{(1, c)} \big( M, (M \otimes (A \otimes (MN))) \big) \xrightarrow{(1, u \otimes 1)} \\
(M, (M \otimes (MN)) \otimes (A \otimes (MN))) & \xrightarrow{(1, (1, c) \otimes 1)} (M, ((MN) \otimes M) \otimes (A \otimes (MN))) \xrightarrow{(1, H^A \otimes 1)} \\
(1, (1, t) \otimes 1) & \xrightarrow{(1, t) \otimes 1} (M, ((MN) \otimes N) \otimes (A \otimes (MN))) \xrightarrow{(1, t)} (M, A \otimes N).
\end{align*}
$$

In the above composite,

$$
\begin{align*}
u & = u_{MN} : M \to (N, M \otimes N) \quad \text{and} \\
t & = t_{MN} : (MN) \otimes M \to N
\end{align*}
$$

are the adjunction transformations as in Eilenberg and Kelly (1966), page 477, for the pair of adjoint functors

$$
M \otimes (-) : V_o \to V_o \quad \text{and} \quad (M, -) : V_o \to V_o,
$$

and

$$
H^A = H^A_{MN} : (MN) \to (A \otimes M, A \otimes N)
$$

is the natural transformation defined in Eilenberg and Kelly (1966), page 527, making $A \otimes (-)$ into a $V$-functor. We claim that in this case $T = A \otimes (-)$, $\lambda$ has a simpler equivalent formulation.

**Lemma 3.1.** Let $A \in V_o$, $T = A \otimes (-)$, and let $\lambda$ be defined as above. Then $\lambda_{MN}$ is equal to the following composite:

$$
\begin{align*}
A \otimes (MN) & \xrightarrow{u} (M, (A \otimes (MN)) \otimes M) \xrightarrow{(1, c)} \big( M, (A \otimes (MN)) \otimes M) \big) \xrightarrow{(1, H^A \otimes 1)} \\
(1, 1 \otimes t) & \xrightarrow{(1, t) \otimes 1} (M, A \otimes N).
\end{align*}
$$

**Proof.** Clearly it is sufficient to show that

$$
1 \otimes t \cdot a = t \cdot H^A \otimes 1 \cdot (1, t) \otimes 1 \cdot (1, c) \otimes 1 \cdot u \otimes 1 \cdot c.
$$

Now we have

$$
t \cdot H^A \otimes 1 \cdot (1, t) \otimes 1 \cdot (1, c) \otimes 1 \cdot u \otimes 1 \cdot c =
$$
Here we see that each numbered equality sign follows from Eilenberg and Kelly (1966), in particular 1 follows from page 537, (6.7), 2 from page 499, (7.1), 3 from page 480, (3.19) with \( x = a^{-1} \), 4 from coherence, 5 from page 477, (3.4) with \( x = 1 \otimes c \), 6 from page 478, (3.7) and 7 from MC 6, page 512, where \( K \) is defined on page 499 (7.1), and \( p \) is part of the given data for \( V \) as in Section 1 above.

**Lemma 3.2.** Let \( A \in V_0 \), \( T = A \otimes (-) \) and let \( \lambda \) be defined as in Lemma 3.1. Then the following diagram commutes.

\[
\begin{array}{ccc}
A \otimes (M \otimes N, P) & \xrightarrow{A \otimes p_{MN}} & A \otimes (M, (NP)) \\
\lambda_{M \otimes N, P} & & \lambda_{M, (NP)} \\
(M \otimes N, A \otimes P) & \xrightarrow{p_{M, N, A \otimes P}} & (M, (N, A \otimes P))
\end{array}
\]

**Proof.** We have

\[
p \cdot (1, 1 \otimes t). (1, a) \cdot (1, a_1) \cdot p^{-1}. (1, u) \cdot u = 1
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, a). p^{-1}. (1, u) \cdot u = 2
\]

\[
p \cdot (1, 1 \otimes t). (1, 1 \otimes a). (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 3
\]

\[
p \cdot (1, 1 \otimes t). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 4
\]

\[
p \cdot (1, 1 \otimes t). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 5
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 6
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 7
\]

Here we see that each numbered equality sign follows from Eilenberg and Kelly (1966), in particular 1 follows from page 537, (6.7), 2 from page 499, (7.1), 3 from page 480, (3.19) with \( x = a^{-1} \), 4 from coherence, 5 from page 477, (3.4) with \( x = 1 \otimes c \), 6 from page 478, (3.7) and 7 from MC 6, page 512, where \( K \) is defined on page 499 (7.1), and \( p \) is part of the given data for \( V \) as in Section 1 above.

**Lemma 3.2.** Let \( A \in V_0 \), \( T = A \otimes (-) \) and let \( \lambda \) be defined as in Lemma 3.1. Then the following diagram commutes.

\[
\begin{array}{ccc}
A \otimes (M \otimes N, P) & \xrightarrow{A \otimes p_{MN}} & A \otimes (M, (NP)) \\
\lambda_{M \otimes N, P} & & \lambda_{M, (NP)} \\
(M \otimes N, A \otimes P) & \xrightarrow{p_{M, N, A \otimes P}} & (M, (N, A \otimes P))
\end{array}
\]

**Proof.** We have

\[
p \cdot \lambda \overset{(3.1)}{=} p \cdot (1, 1 \otimes t). (1, a). (1, a). p^{-1}. (1, u) \cdot u = 1
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, a). p^{-1}. (1, u) \cdot u = 2
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 3
\]

\[
p \cdot (1, 1 \otimes t). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 4
\]

\[
p \cdot (1, 1 \otimes t). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a). (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 5
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 6
\]

\[
p \cdot (1, 1 \otimes t). (1, a). (1, 1 \otimes (t \otimes 1)). (1, 1 \otimes (p \otimes 1)) \otimes 1 \cdot (1, a \otimes 1) \cdot p^{-1}. (1, u) \cdot u = 7
\]
(1,(1,1)t)). (1,(1,a)). p. (1,(1,t)\otimes 1). (1,(1\otimes (\rho \otimes 1))\otimes 1). (1,a\otimes 1). p^{-1}. (1,u). u \\
\overset{4}{=} (1,(1,1)t)). (1,(1,a)). p. (1,(1,t)\otimes 1). (1,(1\otimes (\rho \otimes 1))\otimes 1). (1,a\otimes 1). K^N. u \\
\overset{K^N}{=} (1,(1,1)t)). (1,(1,a)). p. K^N. (1,1t). (1,1\otimes (\rho \otimes 1)). (1,a). u \\
\overset{a}{=} (1,(1,1)t)). (1,(1,a)). (1,u). (1,1t). (1,1\otimes (\rho \otimes 1)). (1,a). u \\
\overset{u}{=} (1,(1,1)t)). (1,(1,a)). (1,u). (1,1t). (1,a). (1,1\otimes (\rho \otimes 1)). u \\
\overset{\lambda 1\otimes (\rho \otimes 1)}{=} (1,\lambda). \lambda 1\otimes (\rho \otimes 1).

Of the equalities of this string of equations which follow from Eilenberg and Kelly (1966), 1 follows from page 480, (3.19), with \( x = a^{-1} \),

\[ A = A \otimes (M \otimes N, P), \quad B = M, \quad C = N, \quad D = (A \otimes (M \otimes N, P) \otimes M) \otimes N \]

and from page 477, (3.1) and (3.3), 2 from MC 3, page 472, 3 from page 480, (3.19) with \( \pi^{-1} \) in place of \( \pi \) and \( x = 1 \), and 4 from page 499, (7.1).

**THEOREM 3.3.** Let \( \mathcal{V}_o \) have equalizers and coequalizers and let \( T \) be a commutative adjoint \( V \)-monad on \( V \). Then \( V^T \) is a symmetric monoidal closed category.

**PROOF.** We have shown in Theorem 2.1 that \( V^T \) is a symmetric monoidal category, and Kock (1971) has shown that \( V^T \) is a closed category. We must show that the two structures on \( V^T \) are compatible, and for this we use Theorem 5.3, page 490, of Eilenberg and Kelly (1966). Again by Lemma 1.1 we may assume that \( T = A = (A \otimes (-), \eta^A, \mu^A) \) for a commutative monoid \( (A, e, m) \) in \( V \).

At this point we recall the construction of the internal hom functor in \( \mathcal{V}^A \) from Kock (1971). Let \( (M, \rho_M) \) and \( (N, \rho_N) \) be two \( A \)-algebras. The internal hom object \( (\text{Hom}_A(M, N), <\rho_M, \rho_N>) \) is defined as follows. The following diagram (3.3.1) is an equalizer in \( \mathcal{V}_o \).

\[
\begin{array}{ccc}
\text{Hom}_A(M, N) & \xrightarrow{e_{MN}} & (MN) \\
& \xrightarrow{H^A_{MN}} & (A \otimes M, N) \\
& \xrightarrow{(1, \rho_N)} & (A \otimes M, A \otimes N)
\end{array}
\]
We note that in Kock (1971) the internal hom object in $\mathcal{V}_o^A$ is denoted by $\text{MILN}$ and in $\mathcal{V}_o$ it is denoted by $\text{MAHN}$, rather than $\text{Hom}_A(M, N)$ and $(MN)$, respectively, which we use. The $A$-structure $\langle \rho_M, \rho_N \rangle$ on $\text{Hom}_A(M, N)$ is given by commutativity of the following diagram

\[ \begin{array}{ccc}
A \otimes \text{Hom}_A(M, N) & \xrightarrow{A \otimes e_{MN}} & A \otimes (MN) \\
\downarrow{\langle \rho_M, \rho_N \rangle} & & \downarrow{\lambda_{MN}} \\
\text{Hom}_A(M, N) & \xrightarrow{e_{MN}} & (MN)
\end{array} \]  

(3.3.2)

Moreover, if $f: (M', \rho_{M'}) \to (M, \rho_M)$ and $g: (N, \rho_N) \to (N', \rho_{N'})$ are two morphisms of $A$-algebras, there is a unique morphism of $A$-algebras $\text{Hom}_A(f, g): \text{Hom}_A(M, N) \to \text{Hom}_A(M', N')$ which is such that

\[ e_{M', N'} \cdot \text{Hom}_A(f, g) = (f, g) \cdot e_{MN}. \]  

(3.3.3)

Recall also from Kock (1970), page 8, and Eilenberg and Kelly (1966), Theorem 5.2, page 445, that for each $(M, \rho_M) \in \mathcal{V}_o^A$ there is a $\mathcal{V}_o^A$-functor $\tilde{L}^{(M, \rho_M)} = \tilde{L}^M: \mathcal{V}_o^A \to \mathcal{V}_o^A$ defined for any $(N, \rho_N) \in \mathcal{V}_o^A$ by

\[ \tilde{L}^M(N, \rho_N) = (\text{Hom}_A(M, N), \langle \rho_M, \rho_N \rangle), \]

and if $(P, \rho_P), (Q, \rho_Q) \in \mathcal{V}_o^A$, we have a natural transformation

\[ (\tilde{L}^M)(P, \rho_P)(Q, \rho_Q) = \tilde{L}^M_{PQ}: (\text{Hom}_A(P, Q), \langle \rho_P, \rho_Q \rangle) \to (\text{Hom}_A(\text{Hom}_A(M, P), \text{Hom}_A(M, Q)), \langle \langle \rho_M, \rho_P \rangle, \langle \rho_M, \rho_Q \rangle \rangle) \]

defined by the commutativity of the following diagram (3.3.4).

\[ \begin{array}{ccc}
\text{Hom}_A(P, Q) & \xrightarrow{e} & (PQ) \\
\downarrow{\tilde{L}^M_{PQ}} & & \downarrow{L^M_{PQ}} \\
\text{Hom}_A(\text{Hom}_A(M, P), \text{Hom}_A(M, Q)) & \xrightarrow{(e, 1)} & (\text{Hom}_A(M, P), (MQ)) \\
\end{array} \]  

(3.3.4)
In order to verify that Theorem 5.3 of Eilenberg and Kelly (1966) applies, we define a natural isomorphism \( \tilde{p} = \tilde{p}(M, \rho_M)(N, \rho_N)(P, \rho_P) \):

\[
(\text{Hom}_A(M \Theta_A N, P), <\rho_M \Theta_A N, \rho_P>) \to (\text{Hom}_A(M, \text{Hom}_A(N, P)), <\rho_M, <\rho_N, \rho_P>>) \]

of \( A \)-algebras as follows. Consider the following diagram

\[
\begin{array}{cccc}
\text{Hom}_A(M \Theta_A N, P) & \xrightarrow{e_{M \Theta_A N, P}} & (M \Theta_A N, P) & \xrightarrow{(q_{MN}, 1)} & (M \Theta N, P) \\
\downarrow{\tilde{p}_{MN\rho}} & & \downarrow{p_{MN\rho}} & & \downarrow{p_{MN\rho}} \\
\text{Hom}_A(M, \text{Hom}_A(N, P)) & \xrightarrow{e_{M, \text{Hom}_A(N, P)}} & (M, \text{Hom}_A(N, P)) & \xrightarrow{(I, e_{NP})} & (M, (N P))
\end{array}
\]

In (3.3.5), \( p \) is the natural isomorphism from the symmetric monoidal closed category \( V \), \( q \) is the coequalizer which defines \( M \Theta_A N \) as in the proof of Theorem 2.1, and \( e \) is the equalizer defined in (3.3.1). Since the functor \( (M, \cdot) : V \to V \) has a left adjoint, \( (I, e_{NP}) \) is the equalizer of

\[
(I, (\rho_N, 1)) \quad \text{and} \quad (I, (1, \rho_P)), (I, H^A),
\]

and we claim that \( p_{MN\rho} \cdot (q_{MN}, 1) \cdot e_{M \Theta_A N, P} \) also equalizes them. To see this, note that we have

\[
\begin{align*}
(1, (\rho_N, 1)) \cdot p \cdot (q, 1) \cdot e & = p \cdot 1 \Theta N, 1 \cdot (q, 1) \cdot e \\
p \cdot (a^{-1}, 1) \cdot (c \Theta l, 1) \cdot (a, 1) \cdot (q, 1) \cdot e & = (2, 1, 2) \\
p \cdot (a^{-1}, 1) \cdot (c \Theta l, 1) \cdot (a, 1) \cdot (q, 1) \cdot e & = (2) \\
p \cdot (a^{-1}, 1) \cdot (c \Theta l, 1) \cdot (a, 1) \cdot (q, 1) \cdot e & = (1, (1, \rho_P)) \cdot (a^{-1}, 1) \cdot (c \Theta l, 1) \cdot (a, 1) \cdot (q, 1) \cdot e^3 \]
\]
In this string of equations, 1 follows from the definition of $T_N$ as in Theorem 2.1, 2 since $e$ is an equalizer, 3 from Eilenberg and Kelly (1966), page 537, (6.7) and page 499, (7.1), 4 from coherence and 5 from MCC 3 for $V$, Eilenberg and Kelly (1966), page 475. It follows that there is a unique morphism

$$p'_{MNP} : \text{Hom}_A(M \otimes_A N, P) \to (M, \text{Hom}_A(N, P))$$

such that

$$(3.3.6) \quad p_{MNP} \cdot (q_{MN}, 1) \cdot e_{M \otimes_A N, P} = (1, e_{NP}) \cdot p'_{MNP}.$$ 

Next we claim that $p'$ equalizes $(p_M, 1)$ and $(1, \langle p_N, p_P \rangle)$. HA, and since $(1, e_{NP})$ is a monomorphism, it suffices to show that

$$(1, e). (p'_M, 1) \cdot p' = (1, e). (1, \langle p_N, p_P \rangle) \cdot H^A \cdot p'.$$

We have then

$$
\begin{align*}
(1, e). (p'_M, 1) \cdot p' &= (p'_M, 1) \cdot (1, e) \cdot p' \\
&\xrightarrow{(3.3.6)} (p'_M, 1) \cdot (1, e) \cdot p' \xrightarrow{(2.1.2)} p \cdot (a, 1) \cdot (1 \otimes_q 1) \cdot (p_M \otimes_{A^N} 1) \\
&\xrightarrow{1} p \cdot (a, 1) \cdot (1 \otimes_q 1) \cdot (1, p_P) \cdot H^A \cdot e \xrightarrow{HA} p \cdot (a, 1) \cdot (1, p_P) \cdot H^A \cdot (q, 1) \\
&\xrightarrow{P} (1, (1, p_P)) \cdot (p \cdot (a, 1)) \cdot H^A \cdot (q, 1) \cdot e \\
&\xrightarrow{2} (1, (1, p_P)) \cdot (1, p_P) \cdot H^A \cdot (q, 1) \cdot e \\
&\xrightarrow{H^A} (1, (1, p_P)) \cdot (1, 1 \otimes t) \cdot (a, 1) \cdot H^A \cdot K^N \cdot p \cdot (q, 1) \cdot e \\
&\xrightarrow{3} (1, (1, p_P)) \cdot (1, 1 \otimes t) \cdot (1, a) \cdot K^N \cdot H^A \cdot p \cdot (q, 1) \cdot e \\
&\xrightarrow{P} (1, (1, p_P)) \cdot (1, 1 \otimes t) \cdot (1, a) \cdot p \cdot K^N \cdot H^A \cdot p \cdot (q, 1) \cdot e \\
&\xrightarrow{4} (1, (1, p_P)) \cdot (1, 1 \otimes t) \cdot (1, (1, a)) \cdot p \cdot K^N \cdot H^A \cdot p \cdot (q, 1) \cdot e \\
&\xrightarrow{(3.1)} (1, (1, p_P)) \cdot (1, 1 \otimes t) \cdot (1, (1, a)) \cdot (1, u) \cdot H^A \cdot p \cdot (q, 1) \cdot e \\
&\xrightarrow{(3.3.6)} (1, (1, p_P)) \cdot (1, 1 \otimes e) \cdot H^A \cdot p' \\
&\xrightarrow{(3.3.2)} (1, e) \cdot (1, \langle p_N, p_P \rangle) \cdot H^A \cdot p'.
\end{align*}
$$
In this string of equations, 1 follows since $e$ is an equalizer, 2 from Eilenberg and Kelly (1966), page 500, (7.2), 3 from Eilenberg and Kelly (1966) page 537, (6.7) and page 499, (7.1), and 4 from Eilenberg and Kelly (1966) page 499, (7.1). Since $p'$ equalizes

$$(\rho_M, I) \text{ and } (1, \langle \rho_N, \rho_P \rangle), H^A,$$

there is a unique morphism

$$\tilde{p} : \text{Hom}_A(M \otimes_A N, P) \rightarrow \text{Hom}_A(M, \text{Hom}_A(N, P))$$

such that $e \cdot \tilde{p} = p'$, and hence we have

$$(3.3.7) \quad p_{MNP} \cdot (q_{MN}, I) \cdot e_{M \otimes_A N, P} = (1, e_{N, P}) \cdot e_{M, \text{Hom}_A(N, P)} \cdot \tilde{p}_{MNP}.$$ 

Now since $q$ is a coequalizer, $(1, q)$ is an equalizer, and also $\tilde{p}$ is the unique morphism such that

$$p \cdot (q, I) \cdot e = (1, e) \cdot e \cdot \tilde{p}.$$ 

Hence, since $p$ is a natural isomorphism, it follows that $\tilde{p}$ is a natural isomorphism as well. It remains to show that $\tilde{p}$ is a morphism of $A$-algebras, i.e., that we have

$$(3.3.8) \quad \tilde{p}_{MNP} \cdot \langle \rho_M \otimes_A N, \rho_P \rangle = \langle \rho_M, \langle \rho_N, \rho_P \rangle \rangle, A \otimes \tilde{p}_{MNP}.$$ 

and as before it suffices to show that

$$(1, e) \cdot \tilde{p} \cdot \langle \rho_M \otimes_A N, \rho_P \rangle = (1, e) \cdot \rho_M \cdot \langle \rho_N, \rho_P \rangle, 1 \otimes \tilde{p}.$$ 

We have

$$(1, e) \cdot \tilde{p} \cdot \langle \rho_M \otimes_A N, \rho_P \rangle = (3.3.7) \quad p \cdot (q, I) \cdot e \cdot \langle \rho_M \otimes_A N, \rho_P \rangle = (3.3.2)$$

$$p \cdot (q, I) \cdot (1, \rho_P), \lambda \cdot 1 \otimes e = p \cdot (q, I) \cdot 1 \otimes (q, 1). 1 \otimes e = p$$

$$(1, (1, \rho_P)). p \cdot \lambda \cdot 1 \otimes (q, 1). 1 \otimes e = (3.2)$$

$$(1, (1, \rho_P)). (1, \lambda). 1 \otimes (1, e). 1 \otimes e = (3.3.7)$$

$$(1, (1, \rho_P)). (1, \lambda). 1 \otimes (1, e). 1 \otimes e = (3.3.2)$$

$$(1, (1, \rho_P)). (1, \lambda). (1, 1 \otimes e). 1 \otimes e. 1 \otimes \tilde{p} = (3.3.2)$$

$$(1, \rho_P). (1, \rho_P). (1, \otimes e). 1 \otimes e. 1 \otimes \tilde{p} = (3.3.2)$$

$$(1, e). (1, \langle \rho_N, \rho_P \rangle). \lambda \cdot 1 \otimes e. 1 \otimes \tilde{p} = (3.3.2)$$

$$(1, e). e \cdot \langle \rho_M, \langle \rho_N, \rho_P \rangle \rangle. 1 \otimes \tilde{p}.$$
Hence we have that \( \tilde{p} : (\eta_M, \rho_M \otimes N, \rho_N \otimes P, \rho_P) \):

\[
(\text{Hom}_A(M \otimes_A N, P, \rho_M \otimes A N, \rho_P), \rho_M \otimes A N, \rho_P) \rightarrow (\text{Hom}_A(M, \text{Hom}_A(N, P)), \rho_M, \rho_N, \rho_P)
\]

is a natural isomorphism of \( A \)-algebras.

We now claim that for any \((M, \rho_M), (N, \rho_N) \in \mathcal{V}_0^A\), \( \tilde{p} \) induces a \( \mathcal{V}_0^A \)-natural isomorphism of \( \mathcal{V}_0^A \)-functors

\[
\tilde{p} : \text{Hom}_A(M \otimes_A N, P) \rightarrow \text{Hom}_A(M, \text{Hom}_A(N, P)).
\]

To see this we need to verify that VN in Eilenberg and Kelly (1966), page 466, holds for \( \tilde{p} \), so that we must check that the following diagram commutes where \((P, \rho_P), (Q, \rho_Q) \in \mathcal{V}_0^A:\)

\[
\begin{array}{ccc}
\text{Hom}_A(P, Q) & \xrightarrow{\text{Hom}_A(M \otimes_A N, P, \text{Hom}_A(M \otimes_A N, Q))} & \text{Hom}_A(M, \text{Hom}_A(N, P)) \\
\downarrow & \downarrow & \downarrow \\
\text{Hom}_A(\text{Hom}_A(N, P), \text{Hom}_A(N, Q)) & \xrightarrow{\text{Hom}_A(M \otimes_A N, \text{Hom}_A(N, Q))} & \text{Hom}_A(M, \text{Hom}_A(N, Q)) \\
& \downarrow & \downarrow \\
\text{Hom}_A(M, \text{Hom}_A(N, P)) & \xrightarrow{\text{Hom}_A(M, \text{Hom}_A(N, Q))} & \text{Hom}_A(M, \text{Hom}_A(N, Q))
\end{array}
\]

Now since \( e \) is a monomorphism in \( \mathcal{V}_0 \), it suffices to check that

\[
(1, (1, e)), (1, e), \cdot \text{Hom}_A(1, \tilde{p}), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot \text{Hom}_A(\tilde{p}, 1), \cdot L^M \otimes A N.
\]

We have that

\[
(1, (1, e)), (1, e), \cdot \text{Hom}_A(1, \tilde{p}), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot L^M \otimes A N, \quad (3.3.7)
\]

\[
(1, (1, e)), (1, e), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot L^M \otimes A N, \quad (3.3.4)
\]

\[
(1, (1, e)), (1, e), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot L^M \otimes A N, \quad e
\]

\[
(1, (1, e)), (1, e), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot e
\]

\[
(1, (1, e)), (1, e), \cdot L^M \otimes A N = (1, (1, e)), (1, e), \cdot L^M \otimes A N, \quad e
\]

\[
287
\]
In this string of equations, 1 follows from MCC 3' in Eilenberg and Kelly (1966), page 475.

It is now clear that the object \((MOA N, pMOA N)\) in \(VA\) and the \(V^4\)-natural 1 isomorphism \(p : LMOAN \rightarrow LM L N\) form a representation of the \(VA\)-functor \(LMLN : VA \rightarrow Vi\) in the sense of Eilenberg and Kelly (1966), Remark 10.11, page 471, so that by Theorem 5.3 of Eilenberg and Kelly, the closed category \(VA\) admits enrichment to a (symmetric) monoidal closed category \(VA\). This completes the proof of Theorem 3.3.

We observe that Day (1970) has shown that for a symmetric monoidal closed category \(V\) such that \(V_0\) has all small limits and colimits and for a commutative monoid \(M\) in \(V\), the category of \(M\)-modules, viewed as a functor category, is also symmetric monoidal closed. We note that Day's proof depends heavily on the completeness and cocompleteness of \(V_0\), while the proof given herein is motivated by the obvious examples and requires only the existence of equalizers and coequalizers in \(V_0\).

4. COMMUTATIVE MONOIDS AND ALGEBRAS FOR A MONAD.

The following theorem describes a connection between the categories \(CM(V)\) of commutative monoids in \(V\) and \(V^T\) of \(T\)-algebras for a
commutative adjoint $V$-monad $T$ on $V$. Briefly stated, it says that there is a monad $T_1$ on $CM(V)$ with the property that $T_1$-algebras in $CM(V)$ are (isomorphic to) commutative monoids in $V^T$. We denote by $U: CM(V) \to V_0$ the functor given on objects by $U(A, e, m) = A$.

**Theorem 4.1.** Let $V_0$ have coequalizers and let $T = (T, \eta, \mu)$ be a commutative adjoint $V$-monad on $V$. Then there is a monad $T_1 = (T_1, \eta_1, \mu_1)$ on $CM(V)$ such that

$$UT_1 = TU, \quad U\eta_1 = \eta U, \quad U\mu_1 = \mu U,$$

and $CM(V^T)$ is isomorphic to $CM(V)^T_1$ over $V$.

**Proof.** Again by Lemma 1.1 we may assume that $T = A = (A \Theta (-), \eta^A, \mu^A)$ for a commutative monoid $(A, e, m)$ in $V$. For any commutative monoid $(A', e', m')$ in $V$, define

$$T_1(A', e', m') = (A \Theta A', e_1, m_1)$$

where

$$e_1 = e \Theta e', I_1^I \quad \text{and} \quad m_1 = m \Theta m', \sigma_{AA'},$$

and where

$$\sigma_{AA'}: (A \Theta A') \Theta (A \Theta A') \to (A \Theta A) \Theta (A' \Theta A')$$

is the so-called «middle-four interchange» of Eilenberg and Kelly (1966), page 517, a coherently natural isomorphism. A simple calculation shows that $(A \Theta A', e_1, m_1)$ is a commutative monoid in $V$, so that

$$T_1: CM(V) \to CM(V)$$

is defined on objects. For a morphism $f$ in $CM(V)$, let $T_1(f) = A \Theta f$, a morphism in $CM(V)$ as well. Define natural transformations

$$\eta_1: CM(V) \to T_1 \quad \text{and} \quad \mu_1: T_1 T_1 \to T_1$$

by

$$\eta_1(A', e', m') = (\eta^A)_A', \quad \text{and} \quad \mu_1(A', e', m') = (\mu^A)_A'.$$

Again we see that $\eta_1(A', e', m')$ and $\mu_1(A', e', m')$ are morphisms in $CM(V)$ and that $\eta_1$ and $\mu_1$ are natural. It is immediate that
and since $U$ is faithful, it follows that $T_1 = (T_1, \eta_1, \mu_1)$ is a monad on $\text{CM}(V)$.

Now define $\Phi : \text{CM}(V^d) \to \text{CM}(V)^{T_1}$ on objects by

$$\Phi((M, \rho_M), e', m') = ((M, \rho_M), \rho_M),$$

where $q_{MM} : M \Theta M \to M \Theta A M$ is the coequalizer of the two morphisms

$$r_M', r_M^\prime : A \Theta (M \Theta M) \to M \Theta M,$$

$$r_M' = \rho_M \Theta M \cdot a_{AMM}^{r'}, \quad r_M^\prime = M \Theta \rho_M \cdot a_{MAM} \cdot c_{AM} \Theta M \cdot a_{AMM}^{r'},$$

as in Theorem 2.1. The calculations showing $(M, e', e, m', q_{MM})$ to be a commutative monoid are straightforward and depend upon knowing that

$$m' : (M \Theta_A M, \rho_M \Theta_A M) \to (M, \rho_M)$$

and $e' : (A, m) \to (M, \rho_M)$ are morphisms of $A$-algebras making $((M, \rho_M), e', m')$ a commutative monoid in $V^d$, and upon recalling the definitions of the natural isomorphisms $\tilde{a}$, $\tilde{l}$ and $\tilde{c}$ in $V^d$ from Theorem 2.1. Similarly one sees that $\rho_M : A \Theta M \to M$ is a morphism in $\text{CM}(V)$ and also a $T_1$-structure on $(M, e', e, m', q_{MM})$. Defining $\Phi$ on morphisms by $\Phi(f) = f$, we have a functor

$$\Phi : \text{CM}(V^d) \to \text{CM}(V)^{T_1}.$$

To see that $\Phi$ is an isomorphism, define $\Phi' : \text{CM}(V)^{T_1} \to \text{CM}(V^d)$ on objects by

$$\Phi'((M, e', m'), \rho_M) = ((M, \rho_M), e'', m''),$$

where

$$e'' = \rho_M \cdot A \Theta e' \cdot r_A^l : A \to M$$

and $m'' : M \Theta_A M \to M$ is the unique morphism such that $m' = m'' \cdot q_{MM}$. Note that the existence of $m''$ follows since $m'$ coequalizes $r_M'$ and $r_M^\prime$. It is immediate that $(M, \rho_M)$ is an $A$-algebra, and one can easily show that

$$e'' : (A, m) \to (M, \rho_M)$$

and $m'' : (M \Theta_A M, \rho_M \Theta_A M) \to (M, \rho_M)$ are $A$-algebra morphisms making $((M, \rho_M), e'', m'')$ a commutative monoid in $V^d$. Again defining $\Phi'$ on morphisms by $\Phi'(f) = f$, we have a functor

$$\Phi' : \text{CM}(V)^{T_1} \to \text{CM}(V^d),$$

and a direct calculation shows that $\Phi' = \Phi^{-1}$, so
that $\Phi$ is an isomorphism (over $V$) as required.

5. EXAMPLES.

1. The following example provided part of the motivation (and the notation) for this paper. The category $Ab$ of abelian groups and group homomorphisms is known to be a symmetric monoidal closed category. Moreover, a commutative monoid in $Ab$ is a commutative ring $R$ with identity, and if $R$ denotes the corresponding commutative adjoint ($Ab$-)monad on $Ab$, it is clear that $Ab^R \simeq R$-$Mod$, the category of $R$-modules and $R$-linear homomorphisms. Since $Ab$ has equalizers and coequalizers, it follows from Theorem 3.3 that $R$-$Mod$ is also a symmetric monoidal closed category, which is indeed a well known result. Furthermore, one sees that $CM(R$-$Mod) \simeq R$-$Alg$, the category of commutative $R$-algebras and $R$-homomorphisms, and since $CM(Ab) \simeq \text{Comm}$, the category of commutative rings with identity, we have $R$-$Alg \simeq \text{Comm}^T_1$. Hence any $R$-algebra $S$ can be viewed as either an $R$-module $S$ with a multiplicative structure, or as a commutative ring $S$ with an action of $R$ on $S$ via a ring homomorphism $R \otimes S \to S$ (i.e., a ring homomorphism $R \to S$, since $R \otimes S \simeq R_1 S$), again a well known fact; e.g., Mac Lane (1967), page 173.

2. The category $Sets$ of sets and functions is a cartesian closed category, and a commutative monoid in $Sets$ is just an abelian monoid $M$. Hence, $Sets^M \simeq M$-$sets$, the category of $M$-sets and $M$-functions, is a symmetric monoidal closed category by Theorem 3.3.

3. The category $E(Ab)$, whose objects are pairs $(A, f)$ with $A$ an abelian group and $f$ an endomorphism on $A$, is a symmetric monoidal closed category, where $(A, f) \otimes (B, g) = (A \otimes B, f \circ g)$, with

$$(f \circ g)(a \otimes b) = f(a) \otimes b + a \otimes g(b),$$

for any $a \in A$, $b \in B$, and where

$$((A, f), (B, g)) = ((AB), \{ f, g \})$$,

with $\{ f, g \}(h) = gh - fh$ for any $h: A \to B$ as in Keigher (a) and (b). Also, a commutative monoid
in $E(Ab)$ is a differential ring $(A, d)$, and
\[ E(Ab)^{(A, d)} \cong (A, d)\text{-Mod}, \]
the category of differential modules over $(A, d)$ by Keigher (b). It follows
from Theorem 3.3 that $(A, d)\text{-Mod}$ is a symmetric monoidal closed category.
Clearly we have
\[ \text{CM}((A, d)\text{-Mod}) \cong (A, d)\text{-Alg}, \]
the category of differential algebras over $(A, d)$, and since
\[ \text{CM}(E(Ab)) \cong \text{Diff}, \]
the category of differential rings, Theorem 4.1 tells us that
\[ (A, d)\text{-Alg} \cong \text{Diff}^T, \]
as well.

4. The category $\text{Ban}$ of real Banach spaces and continuous linear transforma-
tions of norm not exceeding one is a symmetric monoidal closed cat-
egory as in Wick-Negrepontis (1973), and a commutative monoid in $\text{Ban}$ is
a Banach algebra $A$. Moreover, $\text{Ban}$ has equalizers and coequalizers, so
that the category $\text{Ban}^A$ is also symmetric monoidal closed by Theorem 3.3.
The category $\text{Ban}^A$ is called the category of Banach $A$-modules for the
Banach algebra $A$, and is of some interest to functional analysts.
REFERENCES.


W. KEIGHER (a), Closed categories of coalgebras, *Communications in Algebra* 6 (10) (1978), 995-1015.

W. KEIGHER (b), Symmetric monoidal comonads and differential algebra, *Communications in Algebra* (to appear).


H. WOLFF (1973), Monads and monoids on symmetric monoidal closed categories, *Arch. Math.* 24, 113-120.

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