CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 19, nº 3 (1978), p. 269-293

<http://www.numdam.org/item?id=CTGDC_1978__19_3_269_0>

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Vol. XIX-3 (1978)

SYMMETRIC MONOIDAL CLOSED CATEGORIES GENERATED BY COMMUTATIVE ADJOINT MONADS

by William F. KEIGHER

Kock (1971) has shown that if V is a symmetric monoidal closed category with equalizers and T is a commutative V-monad on V, then V^T, the category of T-algebras in V, is a closed category. The primary purpose of the present paper is to extend Kock's result by showing that if V is a symmetric monoidal closed category with coequalizers and T is a commutative V-monad on V, then V^T is a symmetric monoidal category. We also show that the pair of adjoint functors connecting V^T and V are symmetric monoidal functors, and that the adjunction natural transformations are monoidal natural transformations. Moreover, if we assume that V_0 has equalizers, then we show that V^T is symmetric monoidal closed. We also examine the relationship between the category of commutative monoids in V^T and a category of algebras in the category of commutative monoids in V. We conclude with several examples which illustrate the results.

1. PRELIMINARIES.

Throughout the paper we assume that

$$V = ([V_0, \otimes, I, a, r, l, c], p, [V_0, V, hom V, I, i, j, L])$$

is a symmetric monoidal closed category in the sense of Eilenberg and Kelly (1966), page 535. We also adopt the notation and terminology contained therein unless otherwise noted.

We recall that a monoid in V is a triple (A, e, m) where $A \in V_0$, and $e: I \to A$ and $m: A \otimes A \to A$ are morphisms in V_0 such that

$$m . A\otimes e . r_A^{-1} = A = m . e\otimes A . l_A^{-1} \quad \text{and} \quad m . m\otimes A = m . A\otimes m . a_{AAA} .$$

The monoid (A, e, m) is commutative if $m = m . c_{AA}$. A monoid morphism

$f:(A,e,m)\to(A',e',m')$ in V is a morphism $f: A\to A'$ in V_0 such that

$$f.m = m'.f\otimes f \quad \text{and} \quad f.e = e'.$$

We denote the category of monoids in V by M(V) and the full subcategory of commutative monoids in V by CM(V). Further, V^V denotes the category of V-functors from V into V, and Adj(V) denotes the full subcategory of V^V consisting of all V-functors $T: V\to V$ having a right V-adjoint. We denote by MAdj(V) the category of adjoint V-monads on V (i.e., V-monads (T,η,μ) on V where T has a right V-adjoint) and by CMAdj(V) the full subcategory of MAdj(V) consisting of those adjoint V-monads on V which are commutative in the sense of Kock (1971) or Kock (1970). We note that there are obvious forgetful functors

$$U: M(V)\to V_0,\quad U': CM(V)\to V_0,\quad U_1: MAdj(V)\to Adj(V)$$
$$\text{and}\quad U_1': CMAdj(V)\to Adj(V).$$

LEMMA 1.1. *The functor* $\Phi: V_0\to Adj(V)$ *defined by the rules*

$$\Phi A = A\otimes(-) \quad \textit{and} \quad \Phi f = f\otimes(-)$$

is a monoidal equivalence of categories. There are also equivalences

$$\Phi_1: M(V)\to MAdj(V) \quad \textit{and} \quad \Phi_1': CM(V)\to CMAdj(V)$$

such that $\Phi U = U_1\Phi_1$ *and* $\Phi U' = U_1'\Phi_1'$.

PROOF. The first statement follows from Bunge (1969), Theorem 3.8, page 89, while the second follows from Bunge (1969), Corollary 3.9, page 90, and Wolff (1973), Proposition 2.7, page 119.

In the interest of brevity many strings of equations appear in the proofs of the results herein, and many equality signs in these strings have been marked according to the following scheme (as in Kock (1971)) to indicate the reason for the equality. An equality sign with a letter above it, as in $\overset{\eta}{=}$, indicates equality as a result of naturality of that named natural transformation. One marked with a numeral within parentheses, as in $\overset{(3.4)}{=\!=}$, follows from that numbered theorem, equation, diagram, or whatever in this paper. One marked with a numeral without parentheses, as in $\overset{2}{=}$, indicates

that the reason for the equality will be explained below. Lastly, an unmarked equality sign denotes an obvious or trivial equation.

2. COMMUTATIVE ADJOINT MONADS AND CATEGORIES OF ALGEBRAS.

We first obtain the main result which when considered in conjunction with Kock's Theorem will imply that the category of T-algebras for a commutative adjoint V-monad T on V is a symmetric monoidal closed category, provided V_0 has both equalizers and coequalizers.

THEOREM 2.1. *Let* V_0 *have coequalizers and let* T *be a commutative adjoint* V*-monad on* V. *Then* V^T, *the category of* T*-algebras in* V, *is a symmetric monoidal category.*

PROOF. By Lemma 1.1, it is equivalent to assume that T is induced by a commutative monoid (A, e, m) in V, and so we denote the monad T by $\underline{A}$ $(A\otimes(-), \eta^A, \mu^A)$, where

$$\eta^A = e\otimes(-).\, l^{-1}_{(-)} \quad \text{and} \quad \mu^A = m\otimes(-).\, a^{-1}_{AA(-)}.$$

(i) We construct the tensor product $(M\otimes_A N, \rho_{M\otimes_A N})$ of two $\underline{A}$-algebras (M, ρ_M) and (N, ρ_N) as follows. Consider the pair of morphisms $\tau_M, \tau_N : A\otimes(M\otimes N) \to M\otimes N$, where

$$\tau_M = \rho_M \otimes N.\, a^{-1}_{AMN} \quad \text{and} \quad \tau_N = M\otimes\rho_N .\, a_{MAN}.\, c_{AM}\otimes N.\, a^{-1}_{AMN},$$

and let $q_{MN}: M\otimes N \to M\otimes_A N$ be the coequalizer of τ_M and τ_N. Note that since $\underline{A}$ is an adjoint V-monad, $A\otimes q_{MN}$ is the coequalizer of $A\otimes\tau_M$ and $A\otimes\tau_N$. Now consider the following diagram in V_0 :

$$(2.1.1)\qquad \begin{array}{ccccc}
A\otimes(A\otimes(M\otimes N)) & \underset{A\otimes\tau_N}{\overset{A\otimes\tau_M}{\rightrightarrows}} & A\otimes(M\otimes N) & \xrightarrow{A\otimes q_{MN}} & A\otimes(M\otimes_A N) \\
\downarrow{\scriptstyle \mu^A_{M\otimes N}} & & {\scriptstyle\tau_M}\downarrow\downarrow{\scriptstyle\tau_N} & & \downarrow{\scriptstyle \rho_{M\otimes_A N}} \\
A\otimes(M\otimes N) & \underset{\tau_N}{\overset{\tau_M}{\rightrightarrows}} & M\otimes N & \xrightarrow{q_{MN}} & M\otimes_A N
\end{array}$$

It is easy to check that both squares on the left hand side of (2.1.1) (one involving the τ_M 's, the other the τ_N 's) commute. It follows that $q_{MN} \cdot \tau_M = = q_{MN} \cdot \tau_N$ coequalizes $A \otimes \tau_M$ and $A \otimes \tau_N$, and hence there exists a unique morphism $\rho_{M \otimes_A N} : A \otimes (M \otimes_A N) \to M \otimes_A N$ such that

(2.1.2) $$\rho_{M \otimes_A N} \cdot A \otimes q_{MN} = q_{MN} \cdot \tau_M = q_{MN} \cdot \tau_N \, .$$

Since q_{MN} and $A \otimes (A \otimes q_{MN})$ are epimorphisms of V_o, one sees that

$$(M \otimes_A N, \rho_{M \otimes_A N}) = (M, \rho_M) \otimes_A (N, \rho_N)$$

is an $\underline{A}$-algebra. If

$$f: (M, \rho_M) \to (M', \rho_{M'}) \quad \text{and} \quad g: (N, \rho_N) \to (N', \rho_{N'})$$

are morphisms of $\underline{A}$-algebras, there is a unique morphism

$$f \otimes_A g : M \otimes_A N \to M' \otimes_A N'$$

such that

(2.1.3) $$f \otimes_A g \cdot q_{MN} = q_{M'N'} \cdot f \otimes g ,$$

and one checks that $f \otimes_A g$ is a morphism of $\underline{A}$-algebras. Hence we have a functor $\otimes_A : V_o^{\underline{A}} \times V_o^{\underline{A}} \to V_o^{\underline{A}}$.

(ii) We take the unit object in $V^{\underline{A}}$ to be the $\underline{A}$-algebra (A, m).

(iii) We construct a natural isomorphism

$$\bar{r}_{(M, \rho_M)} : (M \otimes_A A, \rho_{M \otimes_A A}) \to (M, \rho_M) \quad \text{in} \quad V_o^{\underline{A}}$$

as follows. Since (A, e, m) is commutative, we see that for any $\underline{A}$-algebra (M, ρ_M) there is a unique morphism $\bar{r}_{(M, \rho_M)} : M \otimes_A A \to M$ such that

(2.1.4) $$\bar{r}_{(M, \rho_M)} \cdot q_{MA} = \rho_M \cdot c_{MA} ,$$

and it is clear that $\bar{r}_{(M, \rho_M)}$ is a morphism of $\underline{A}$-algebras. We claim that $\bar{r}_{(M, \rho_M)}$ is a natural isomorphism, having as its inverse $q_{MA} \cdot M \otimes e \, . \, r_M^{-1}$. Consider then the following equations. We have

$$\bar{r} \cdot q \, . \, 1 \otimes e \, . \, r^{-1} \overset{(2.1.4)}{=\!=\!=} \rho \, . \, c \, . \, 1 \otimes e \, . \, r^{-1} \overset{c}{=} \rho \, . \, e \otimes 1 \, . \, c \, . \, r^{-1} \overset{1}{=} \rho \, . \, e \otimes 1 \, . \, l^{-1} \overset{2}{=} M .$$

where 1 follows from Eilenberg and Kelly (1966), Proposition 1.1, page 512, and 2 since ρ_M is an $\underline{A}$-structure map. On the other hand, since there

is a unique morphism $f: M\otimes_A A \to M\otimes_A A$ such that $f.\, q_{MA} = q_{MA}.\, M\otimes A$, it suffices to show that

$$q.\, 1\otimes e.\, r^{-1}.\, \bar{r}.\, q = M\otimes_A A.\, q.$$

Hence we have

$$q.\, 1\otimes e.\, r^{-1}.\, \bar{r}.\, q \overset{(2.1.4)}{=\!=\!=} q.\, 1\otimes e.\, r^{-1}.\, \rho.\, c \overset{r}{=} q.\, 1\otimes e.\, \rho\otimes 1.\, r^{-1}.\, c =$$
$$q.\, \rho\otimes 1.\, 1\otimes e.\, r^{-1}.\, c \overset{1}{=} q.\, 1\otimes m.\, a.\, c\otimes 1.\, 1\otimes e.\, r^{-1}.\, c \overset{2}{=}$$
$$q.\, 1\otimes m.\, 1\otimes(1\otimes e).\, 1\otimes r^{-1} \overset{3}{=} q.\, M\otimes A \overset{(2.1.3)}{=\!=\!=} M\otimes_A A.\, q,$$

where 1 follows since q is a coequalizer, 2 from coherence and naturality of $(-)\otimes e$, and 3 since (A, e, m) is a monoid.

(iv) Similarly we construct a natural isomorphism

$$\bar{l}_{(M,\rho_M)}: (A\otimes_A M,\, \rho_{A\otimes_A M}) \to (M,\, \rho_M) \quad \text{in } \mathcal{V}_o^{\underline{A}}$$

by noting that for any $\underline{A}$-algebra (M, ρ_M) there is a unique morphism

$$\bar{l}_{(M,\rho_M)}: A\otimes_A M \to M$$

such that

(2.1.5) $$\bar{l}_{(M,\rho_M)}.\, q_{AM} = \rho_M.$$

One checks as above that $\bar{l}_{(M,\rho_M)}$ is a natural isomorphism of $\underline{A}$-algebras with inverse $q_{AM}.\, e\otimes M.\, l_M^{-1}$.

(v) We construct a natural isomorphism $\bar{a}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}$:

$$((M\otimes_A N)\otimes_A P,\, \rho_{(M\otimes_A N)\otimes_A P}) \to (M\otimes_A(N\otimes_A P),\, \rho_{M\otimes_A(N\otimes_A P)})$$

in $\mathcal{V}_o^{\underline{A}}$, where (M, ρ_M), (N, ρ_N) and (P, ρ_P) are any $\underline{A}$-algebras in $\mathcal{V}$, by considering the diagram (2.1.6) (next page) in $\mathcal{V}_o$. In (2.1.6),

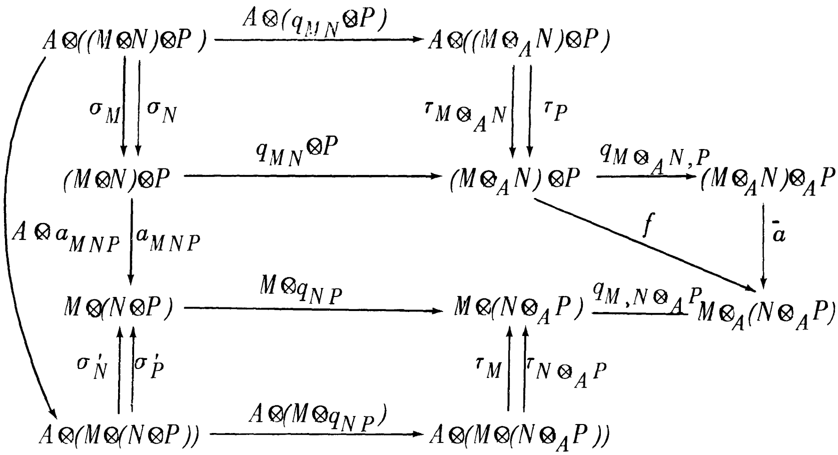
$$\sigma_M = \tau_M\otimes P.\, a^{-1}_{A,\, M\otimes N,\, P}, \quad \sigma_N = \tau_N\otimes P.\, a^{-1}_{A,\, M\otimes N,\, P},$$

$$\sigma'_N = M\otimes\tau_N.\, a_{M,\, A,\, N\otimes P}.\, c_{AM}\otimes(N\otimes P).\, a^{-1}_{A,\, M,\, N\otimes P}$$

and

$$\sigma'_P = M\otimes\tau_P.\, a_{M,\, A,\, N\otimes P}.\, c_{AM}\otimes(N\otimes P).\, a^{-1}_{A,\, M,\, N\otimes P}.$$

An analysis of the diagram (2.1.6) shows that the composite

(2.1.6) 

$$q_{M,N\otimes_A P}\cdot M\otimes q_{NP}\cdot a_{MNP}$$

coequalizes the pair of maps σ_M and σ_N, and since $q_{MN}\otimes P$ is the coequalizer of this pair, there is a unique morphism

$$f: (M\otimes_A N)\otimes P \to M\otimes_A (N\otimes_A P)$$

such that

$$f\cdot q_{MN}\otimes P = q_{M,N\otimes_A P}\cdot M\otimes q_{NP}\cdot a_{MNP}\,.$$

In turn f coequalizes the pair $\tau_{M\otimes_A N}$ and τ_P and hence, induces the unique map

$$\bar{a} = \bar{a}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}: (M\otimes_A N)\otimes_A P \to M\otimes_A(N\otimes_A P)$$

such that $f = \bar{a}\,.\, q_{M\otimes_A N,P}$. It follows then that

(2.1.7) $\bar{a}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}\cdot q_{M\otimes_A N,P}\cdot q_{MN}\otimes P = q_{M,N\otimes_A P}\cdot M\otimes q_{NP}\cdot a_{MNP}$,

and a lengthy but straightforward verification shows that $\bar{a}$ is a natural isomorphism of $\underline{A}$-algebras.

(vi) We construct a natural isomorphism

$$\bar{c}_{(M,\rho_M)(N,\rho_N)}: (M\otimes_A N,\ \rho_{M\otimes_A N}) \to (N\otimes_A M,\ \rho_{N\otimes_A M}) \text{ in } \mathcal{V}_0^{\underline{A}}$$

by noting that for any $\underline{A}$-algebras $(M,\ \rho_M)$ and $(N,\ \rho_N)$ there is a natural isomorphism of $\underline{A}$-algebras $\bar{c}_{(M,\rho_M)(N,\rho_N)}: M\otimes_A N \to N\otimes_A M$ such that

(2.1.8) $$\bar{c}_{(M,\rho_M)(N,\rho_N)}\cdot q_{MN} = q'_{NM}\cdot c_{MN}\,,$$

where $q'_{NM}: N \otimes M \to N \otimes_A M$ is the coequalizer of the corresponding morphisms $r'_N, r'_M : A \otimes (N \otimes M) \to N \otimes M$.

We now claim that

$$V^{\underline{A}} = (V_0^{\underline{A}}, \otimes_A, (A, m), \bar{r}, \bar{l}, \bar{a}, \bar{c})$$

is a symmetric monoidal category. The commutativity of the diagrams needed to show this, i.e., MC 1-MC 7 in Eilenberg and Kelly (1966), page 472 and page 512, follows from the commutativity of the corresponding diagrams for V , from the defining relations for $\bar{r}$, $\bar{l}$, $\bar{a}$ and $\bar{c}$, from the fact that the q_{MN}'s are epimorphisms and from the fact that $M \otimes (-)$ preserves epimorphisms for any $M \in V_0$.

Associated with the category $V^{\underline{A}}$ of $\underline{A}$-algebras in V is a pair of adjoint functors $F^{\underline{A}} : V_0 \to V_0^{\underline{A}}$ and $U^{\underline{A}} : V_0^{\underline{A}} \to V_0$ defined on objects by

$$F^{\underline{A}}(M) = (A \otimes M, \mu_M) \quad \text{and} \quad U^{\underline{A}}(M, \rho_M) = M,$$

with $F^{\underline{A}}$ left adjoint to $U^{\underline{A}}$. We now show that, relative to the symmetric monoidal structure on $V^{\underline{A}}$ as given in Theorem 2.1, these functors can be given the structure of symmetric monoidal functors and the adjunction transformations can be given the structure of monoidal natural transformations as well.

THEOREM 2.2. *Let* V_0 *have coequalizers and let* T *be a commutative adjoint* V*-monad on* V*. Then the adjoint functors*

$$F^T : V_0 \to V_0^T \quad \textit{and} \quad U^T : V_0^T \to V_0$$

are symmetric monoidal functors and the adjunction transformations

$$\epsilon^T : F^T U^T \to V_0^T \quad \textit{and} \quad \eta^T : V_0 \to U^T F^T$$

are monoidal natural transformations.

PROOF. We first observe that by Lemma 1.1, it is equivalent to take

$$T = \underline{A} = (A \otimes (-), \eta^{\underline{A}}, \mu^{\underline{A}})$$

as in the proof of Theorem 2.1. We show first that $U^{\underline{A}}$ is a symmetric monoidal functor. To do so, we need to construct a natural transformation

$$\hat{U}^A_{(M,\rho_M)(N,\rho_N)} : U^A(M,\rho)\otimes U^A(N,\rho)\to U^A(M\otimes_A N, \rho_{M\otimes_A N})$$

and a morphism $(U^A)^0 : I \to U^A(A,m)$ in V_0 . We take

$$\hat{U}^A_{(M,\rho_M)(N,\rho_N)} = q_{MN} ,$$

where q_{MN} is the coequalizer of τ_M and τ_N as defined in the proof of Theorem 2.1, and we take $(U^A)^0 = e$. We must verify MF 1 - MF 4 in Eilenberg and Kelly (1966), pages 473 and 513, for $(U^A, \hat{U}^A, (U^A)^0)$. In this case, MF 1 is simply $\bar{l}_{(M,\rho_M)} \cdot q_{AM} \cdot e\otimes M = l_M$, which holds as in the proof of Theorem 2.1. Similarly we have $\bar{r}_{(M,\rho_M)} \cdot q_{MA} \cdot M\otimes e = r_M$ for MF 2, again from the proof of Theorem 2.1. Also, the equations needed to verify MF 3 and MF 4 are simply the defining relations (2.1.7) and (2.1.8) for the natural isomorphisms $\bar{a}$ and $\bar{c}$ in V_0^A .

To see that F^A is a symmetric monoidal functor, we need to construct a natural transformation

$$\hat{F}^A_{MN} : F^A(M)\otimes_A F^A(N) \to F^A(M\otimes N)$$

and a morphism $(F^A)^0 : (A,m) \to F^A(I)$ in V_0^A which satisfy MF 1 - MF4. For $\hat{F}^A_{MN}$, consider the following diagram (2.2.1):

$$\begin{array}{ccccc}
A\otimes((A\otimes M)\otimes(A\otimes N)) & \underset{\tau_{A\otimes N}}{\overset{\tau_{A\otimes M}}{\rightrightarrows}} & (A\otimes M)\otimes(A\otimes N) & \xrightarrow{q_{A\otimes M, A\otimes N}} & (A\otimes M)\otimes_A(A\otimes N) \\
\downarrow{\scriptstyle A\otimes\kappa_{MN}} & & \downarrow{\scriptstyle \kappa_{MN}} & & \downarrow{\scriptstyle \hat{F}^A_{MN}} \\
A\otimes(A\otimes(A\otimes(M\otimes N))) & \underset{\mu^A_{A\otimes(M\otimes N)}}{\overset{A\otimes\mu^A_{M\otimes N}}{\rightrightarrows}} & (A\otimes(A\otimes(M\otimes N)) & \xrightarrow{\mu^A_{M\otimes N}} & A\otimes(M\otimes N)
\end{array}$$

In (2.2.1),

$$\kappa_{MN} = A\otimes a_{AMN} \cdot A\otimes(c_{MA}\otimes N) . A\otimes a^{-1}_{MAN} \cdot a_{A\,M\,A\otimes N}$$

is a coherent natural isomorphism, $\tau_{A\otimes M}$ and $\tau_{A\otimes N}$ are constructed as in the proof of Theorem 2.1, $q_{A\otimes M, A\otimes N}$ is the coequalizer of $\tau_{A\otimes M}$ and $\tau_{A\otimes N}$, and $\hat{F}^A_{MN}$ is the unique morphism such that

(2.2.2) $$\hat{F}^A_{MN} \cdot q_{A\otimes M, A\otimes N} = \mu^A_{M\otimes N} \cdot \kappa_{MN} .$$

It is clear that $\hat{F}^{A}$ is natural, and one can check in a straightforward fashion that $\hat{F}^{A}$ is a morphism of $\underline{A}$-algebras. For $(\hat{F}^{A})^{o}$, we take

$$(F^{A})^{o} = r_{A}^{-1} : A \to A \otimes I \;;$$

clearly $r_{A}^{-1} : (A, m) \to (A \otimes I, \mu_{I})$ is a morphism of $\underline{A}$-algebras. It remains to verify MF 1 - MF 4. For MF 1, we have to show commutativity of the following diagram in V_o :

$$\begin{array}{ccc} (A\otimes I)\otimes_{A}(A\otimes M) & \xrightarrow{\hat{F}^{A}_{IM}} & A\otimes(I\otimes M) \\ \uparrow{\scriptstyle r_{A}^{-1}\otimes_{A}(A\otimes M)} & & \downarrow{\scriptstyle A\otimes l_{M}} \\ A\otimes_{A}(A\otimes M) & \xrightarrow[\bar{l}_{A\otimes M}]{} & A\otimes M \end{array} \qquad (2.2.3)$$

To do this, we use the fact that $\bar{l}_{A\otimes M}$ is the unique morphism

$$f : A\otimes_{A}(A\otimes M) \to A\otimes M \quad \text{such that} \quad f \cdot q_{A, A\otimes M} = \mu_{M}^{A}$$

But we have

$$1\otimes l \,.\, \hat{F}^{A} \,.\, r^{-1}\otimes_{A} 1 \,.\, q \;\overset{q}{=}\; 1\otimes l \,.\, \hat{F}^{A} \,.\, q \,.\, r^{-1}\otimes 1 \;\overset{(2.2.2)}{=\!=\!=}$$

$$= 1\otimes l \,.\, \mu^{A} \,.\, \kappa \,.\, r^{-1}\otimes 1 \;\overset{\mu^{A}}{=}\; \mu^{A} \,.\, (1\otimes l) \,.\, \kappa \,.\, r^{-1}\otimes 1 \;\overset{1}{=}\; \mu^{A} ,$$

where 1 follows from the coherence of l, κ and r. Hence the commutativity of (2.2.3) follows. For MF 2, we have to show that

$$A\otimes r_{M} \,.\, \hat{F}^{A}_{MI} \,.\, (A\otimes M)\otimes_{A} r_{A}^{-1} = \bar{r}_{A\otimes M} ,$$

and to do so one uses the uniqueness of $\bar{r}_{A\otimes M}$ satisfying

$$\bar{r}_{A\otimes M} \cdot q_{A\otimes M, A} = \mu^{A}_{A\otimes M, A}$$

and the commutativity of (A, e, m) in the verification. For MF 3, we have to show (2.2.4):

$$A\otimes a_{MNP} \cdot \hat{F}^{A}_{M\otimes N, P} \cdot \hat{F}^{A}_{MN}\otimes_{A}(A\otimes P) = \hat{F}^{A}_{M, N\otimes P} \cdot (A\otimes M)\otimes_{A}\hat{F}^{A}_{NP} \cdot a_{A\otimes M\, A\otimes N\, A\otimes P}$$

and since both q and $q\otimes 1$ are epimorphisms in V_o, it suffices to check that (2.2.4) holds when both sides are composed on the right with

$$q_{(A\otimes M)\otimes_{A}(A\otimes N), A\otimes P} \cdot q_{A\otimes M, A\otimes N}\otimes(A\otimes P).$$

Hence we have

$$\hat{F}^{\underline{A}}.\, 1\otimes_A \hat{F}^{\underline{A}}.\, \bar{a}.\, q.\, q\otimes 1 \overset{(2.1.7)}{=\!=\!=} \hat{F}^{\underline{A}}.\, 1\otimes_A \hat{F}^{\underline{A}}.\, q.\, 1\otimes q.\, a \overset{q}{=}$$

$$\hat{F}^{\underline{A}}.\, q.\, 1\otimes \hat{F}^{\underline{A}}.\, 1\otimes q.\, a \overset{(2.2.2)}{=\!=\!=} \mu^{\underline{A}}.\, \kappa.\, 1\otimes \mu^{\underline{A}}.\, 1\otimes \kappa.\, a \overset{\kappa}{=}$$

$$\mu^{\underline{A}}.\, 1\otimes \mu^{\underline{A}}.\, \kappa.\, 1\otimes \kappa.\, a \overset{1}{=} \mu^{\underline{A}}.\, \mu^{\underline{A}}.\, \kappa.\, 1\otimes \kappa.\, a \overset{2}{=}$$

$$1\otimes a.\, \mu^{\underline{A}}.\, \mu^{\underline{A}}.\, \kappa.\, \kappa\otimes 1 \overset{\kappa}{=} 1\otimes a.\, \mu^{\underline{A}}.\, \kappa.\, \mu^{\underline{A}}\otimes 1.\, \kappa\otimes 1 \overset{(2.2.2)}{=\!=\!=}$$

$$1\otimes a.\, \hat{F}^{\underline{A}}.\, q.\, \hat{F}^{\underline{A}}\otimes 1.\, q\otimes 1 \overset{q}{=} 1\otimes a.\, \hat{F}^{\underline{A}}.\, \hat{F}^{\underline{A}}\otimes_A 1.\, q.\, q\otimes 1\ ,$$

where 1 follows from the monad equation $\mu^{\underline{A}}.\, \mu^{\underline{A}} = \mu^{\underline{A}}.\, 1\otimes \mu^{\underline{A}}$ and 2 from coherence and naturality of a. Finally, for MF 4, we must show that

$$A\otimes c_{MN}.\, \hat{F}^{\underline{A}}_{MN} = \hat{F}^{\underline{A}}_{NM}.\, \bar{c}_{A\otimes M, A\otimes N}$$

and as above it suffices to check that $1\otimes c.\, \hat{F}^{\underline{A}}.\, q = \hat{F}^{\underline{A}}.\, \bar{c}.\, q$. Now we have

$$1\otimes c.\, \hat{F}^{\underline{A}}.\, q \overset{(2.2.2)}{=\!=\!=} 1\otimes c.\, \mu^{\underline{A}}.\, \kappa \overset{\mu^{\underline{A}}}{=} \mu^{\underline{A}}.\, 1\otimes(1\otimes c).\, \kappa \overset{1}{=}$$

$$\mu^{\underline{A}}.\, a.\, c\otimes 1.\, a^{-1}.\, 1\otimes(1\otimes c).\, \kappa \overset{2}{=} \mu^{\underline{A}}.\, \kappa.\, c \overset{(2.2.2)}{=\!=\!=}$$

$$\hat{F}^{\underline{A}}.\, q.\, c \overset{(2.1.8)}{=\!=\!=} \hat{F}^{\underline{A}}.\, \bar{c}.\, q,$$

where 1 follows from the commutativity of $\underline{A}$ and 2 from coherence. Hence $F^{\underline{A}}$ is a symmetric monoidal functor.

We now show that $\epsilon^{\underline{A}}: F^{\underline{A}}U^{\underline{A}} \to V_0^{\underline{A}}$ is a monoidal natural transformation, i.e., satisfies MN1 and MN2 in Eilenberg and Kelly (1966), page 474. Observe that, for any $\underline{A}$-algebra (M, ρ_M),

$$\epsilon^{\underline{A}}_{(M,\rho_M)}: F^{\underline{A}}U^{\underline{A}}(M, \rho_M) \to (M, \rho_M)$$

is the morphism $\rho_M: (A\otimes M, \mu^{\underline{A}}_M) \to (M, \rho_M)$. Now in this case, MN1 becomes $m.\, A\otimes e.\, r_A^{-1} = A$, which is one of the unit laws for (A, e, m). Also MN2 becomes

$$\rho_{M\otimes_A N}.\, A\otimes q_{MN}.\, \hat{F}^{\underline{A}}_{MN} = \rho_M \otimes_A \rho_N\ ,$$

and as above it suffices to check

$$\rho_{M\otimes_A N}.\, A\otimes q_{MN}.\, \hat{F}^{\underline{A}}_{MN}.\, q_{A\otimes M, A\otimes N} = \rho_M \otimes_A \rho_N.\, q_{A\otimes M, A\otimes N}.$$

Now:

$$\rho_M \otimes_A \rho_N \cdot q_{A\otimes M, A\otimes N} \overset{(2.1.3)}{=\!=} q_{MN} \cdot \rho_M \otimes \rho_N \overset{1}{=} q_{MN} \cdot \tau_M \cdot A\otimes\tau_N \cdot \kappa_{MN} \overset{2}{=}$$

$$q_{MN} \cdot \tau_N \cdot A\otimes\tau_N \cdot \kappa_{MN} \overset{3}{=} q_{MN} \cdot \tau_N \cdot \mu^{A}_{M\otimes N} \cdot \kappa_{MN} \overset{(2.2.2)}{=\!=}$$

$$q_{MN} \cdot \tau_N \cdot \hat{F}^{A}_{MN} \cdot q_{A\otimes M, A\otimes N} \overset{(2.1.2)}{=\!=} \rho_{M\otimes_A N} \cdot A\otimes q_{MN} \cdot \hat{F}^{A}_{MN} \cdot q_{A\otimes M, A\otimes N},$$

where 1 follows from coherence and naturality, 2 since q_{MN} is the coequalizer of τ_M and τ_N and 3 since ρ_N (and hence τ_N) is an $\underline{A}$-structure morphism.

To complete the proof of Theorem 2.2, we show that $\eta^{\underline{A}} : V_0 \to U^{\underline{A}} F^{\underline{A}}$ is a monoidal natural transformation, where $\eta^{\underline{A}}_M : M \to U^{\underline{A}} F^{\underline{A}}(M)$ is the morphism $e\otimes M . l^{-1}_M : M \to A\otimes M$ in V_0. In this case, MN1 is $e\otimes I . l^{-1}_I = r^{-1}_A . e$, but

$$e\otimes I . l^{-1}_I \overset{1}{=} e\otimes I . r^{-1}_I \overset{r}{=} r^{-1}_A . e,$$

where 1 follows from MC5 in Eilenberg and Kelly (1966), page 472. Also, MN2 becomes

$$\hat{F}^{A}_{MN} \cdot q_{A\otimes M, A\otimes N} \cdot (e\otimes M)\otimes(e\otimes N) . l^{-1}_M \otimes l^{-1}_N = e\otimes(M\otimes N) . l^{-1}_{M\otimes N} .$$

But we have

$$\hat{F}^{\underline{A}} . q . (e\otimes I)\otimes(e\otimes I) . l^{-1}\otimes l^{-1} \overset{(2.2.2)}{=\!=} \mu^{\underline{A}} . \kappa . (e\otimes I)\otimes(e\otimes I) . l^{-1}\otimes l^{-1} \overset{1}{=}$$

$$\mu^{\underline{A}} . e\otimes I . l^{-1} . e\otimes I . l^{-1} \overset{2}{=} e\otimes I . l^{-1},$$

where 1 follows from coherence and naturality and 2 from the monad law $\mu^{\underline{A}} . e\otimes I . l^{-1} = A\otimes M$. This completes the proof of Theorem 2.2.

3. SYMMETRIC MONOIDAL CLOSED CATEGORIES OF ALGEBRAS.

We have seen in Theorem 2.1 that if V is a symmetric monoidal closed category with coequalizers and T is a commutative adjoint V-monad on V, then V^T is a symmetric monoidal category. If V_0 has equalizers, Kock (1971) has shown that V^T is closed as well. The following theorem shows that these two structures on V^T are compatible, i.e., that V^T is a symmetric monoidal closed category.

Recall first from Kock (1971) the construction of the fondamental

natural transformation $\lambda_{AB}: T(AB) \to (A, TB)$, where T is any V-endofunctor on V. In the case that T is the functor of a commutative adjoint V-monad on V, it is equivalent by Lemma 1.1 to assume that $T = A \otimes (-)$, In this case, the natural transformation $\lambda_{MN}: A \otimes (MN) \to (M, A \otimes N)$ as constructed by Kock is defined as the following (lengthy) composite:

$$A \otimes (MN) \xrightarrow{u} (M, (A \otimes (MN)) \otimes M) \xrightarrow{(1,c)} (M, M \otimes (A \otimes (MN))) \xrightarrow{(1, u \otimes 1)}$$

$$(M, ((MN), M \otimes (MN)) \otimes (A \otimes (MN))) \xrightarrow{(1,(1,c)\otimes 1)} (M, ((MN), (MN) \otimes M) \otimes (A \otimes (MN)))$$

$$\xrightarrow{(1,(1,t)\otimes 1)} (M, ((MN), N) \otimes (A \otimes (MN))) \xrightarrow{(1, H^A \otimes 1)}$$

$$(M, (A \otimes (MN), A \otimes N) \otimes (A \otimes (MN))) \xrightarrow{(1,t)} (M, A \otimes N).$$

In the above composite,

$$u = u_{MN}: M \to (N, M \otimes N) \quad \text{and} \quad t = t_{MN}: (MN) \otimes M \to N$$

are the adjunction transformations as in Eilenberg and Kelly (1966), page 477, for the pair of adjoint functors

$$M \otimes (-): V_0 \to V_0 \quad \text{and} \quad (M, -): V_0 \to V_0,$$

and

$$H^A = H^A_{MN}: (MN) \to (A \otimes M, A \otimes N)$$

is the natural transformation defined in Eilenberg and Kelly (1966), page 527, making $A \otimes (-)$ into a V-functor. We claim that in this case $T = A \otimes (-)$, λ has a simpler equivalent formulation.

LEMMA 3.1. *Let $A \in V_0$, $T = A \otimes (-)$, and let λ be defined as above. Then λ_{MN} is equal to the following composite:*

$$A \otimes (MN) \xrightarrow{u} (M, (A \otimes (MN)) \otimes M) \xrightarrow{(1,a)} (M, A \otimes ((MN) \otimes M))$$

$$\xrightarrow{(1, 1 \otimes t)} (M, A \otimes N).$$

PROOF. Clearly it is sufficient to show that

$$1 \otimes t . a = t . H^A \otimes 1 . (1,t) \otimes 1 . (1,c) \otimes 1 . u \otimes 1 . c .$$

Now we have

$$t . H^A \otimes 1 . (1,t) \otimes 1 . (1,c) \otimes 1 . u \otimes 1 . c \overset{H}{=}$$

$$t.(1,1\otimes t)\otimes 1.(1,1\otimes c)\otimes 1.H^{A}\otimes 1.u\otimes 1.c \overset{1}{=}$$
$$t.(1,(1\otimes t)\otimes 1).(1,1\otimes c)\otimes 1.(c,c)\otimes 1.K^{A}\otimes 1.u\otimes 1.c \overset{2}{=}$$
$$t.(1,1\otimes t)\otimes 1.(1,1\otimes c)\otimes 1.(c,c)\otimes 1.p^{-1}\otimes 1.(1,u)\otimes 1.u\otimes 1.c \overset{3}{=}$$
$$t.(1,1\otimes t)\otimes 1.(1,1\otimes c)\otimes 1.(c,c)\otimes 1.(1,a^{-1})\otimes 1.u\otimes 1.c \overset{4}{=}$$
$$t.(1,1\otimes t)\otimes 1.(1,a)\otimes 1.(1,c)\otimes 1.(c,1\otimes c)\otimes 1.u\otimes 1.c \overset{5}{=}$$
$$t.(1,1\otimes t)\otimes 1.(1,a)\otimes 1.(1,c)\otimes 1.u\otimes 1.c \overset{t}{=} 1\otimes t.a.c.t.u\otimes 1.c$$
$$\overset{6}{=} 1\otimes t.a.c.c \overset{7}{=} 1\otimes t.a.$$

Here we see that each numbered equality sign follows from Eilenberg and Kelly (1966), in particular 1 follows from page 537, (6.7), 2 from page 499, (7.1), 3 from page 480, (3.19) with $x = a^{-1}$, 4 from coherence, 5 from page 477, (3.4) with $x = 1\otimes c$, 6 from page 478, (3.7) and 7 from MC6, page 512, where K is defined on page 499 (7.1), and p is part of the given data for V as in Section 1 above.

LEMMA 3.2. *Let $A \in V_0$, $T = A\otimes(-)$ and let λ be defined as in Lemma 3.1. Then the following diagram commutes.*

$$\begin{array}{ccc}
A\otimes(M\otimes N, P) & \xrightarrow{\quad A\otimes p_{MNP}\quad} & A\otimes(M,(NP)) \\
 & & \downarrow{\scriptstyle \lambda_{M,(NP)}} \\
\downarrow{\scriptstyle \lambda_{M\otimes N, P}} & & (M, A\otimes(NP)) \\
 & & \downarrow{\scriptstyle (1,\lambda_{NP})} \\
(M\otimes N, A\otimes P) & \xrightarrow[\quad p_{M,N,A\otimes P}\quad]{} & (M,(N, A\otimes P))
\end{array}$$

PROOF. We have

$$p.\lambda \overset{(3.1)}{=\!=\!=} p.(1,1\otimes t).(1,a).u \overset{1}{=}$$
$$p.(1,1\otimes t).(1,a).(1,a).p^{-1}.(1,u).u \overset{2}{=}$$
$$p.(1,1\otimes t).(1,1\otimes a).(1,a).(1,a\otimes 1).p^{-1}.(1,u).u \overset{3}{=}$$
$$p.(1,1\otimes t).(1.1\otimes(t\otimes 1)).(1,1\otimes((p\otimes 1)\otimes 1)).(1,a).(1,a\otimes 1).p^{-1}.(1,u).u \overset{p}{=}$$
$$(1,(1,1\otimes t)).p.(1,1\otimes(t\otimes 1)).(1,1\otimes((p\otimes 1)\otimes 1)).(1,a).(1,a\otimes 1).p^{-1}.(1,u).u \overset{a}{=}$$
$$(1,(1,1\otimes t)).p.(1,a).(1,(1\otimes t)\otimes 1).(1,(1\otimes(p\otimes 1))\otimes 1).(1,a\otimes 1).p^{-1}.(1,u).u \overset{p}{=}$$

$$(1,(1,1\otimes t)).(1,(1,a)).p.(1,(1\otimes t)\otimes 1).(1,(1\otimes(p\otimes 1))\otimes 1).(1,a\otimes 1).p^{-1}.(1,u).u$$
$$\overset{4}{=} (1,(1,1\otimes t)).(1,(1,a)).p.(1,(1\otimes t)\otimes 1).(1,(1\otimes(p\otimes 1))\otimes 1).(1,a\otimes 1).K^N.u$$
$$\overset{K^N}{=} (1,(1,1\otimes t)).(1,(1,a)).p.K^N.(1,1\otimes t).(1,1\otimes(p\otimes 1)).(1,a).u \overset{4}{=}$$
$$(1,(1,1\otimes t)).(1,(1,a)).(1,u).(1,1\otimes t).(1,1\otimes(p\otimes 1)).(1,a).u \overset{a}{=}$$
$$(1,(1,1\otimes t)).(1,(1,a)).(1,u).(1,1\otimes t).(1,a),(1,(1\otimes p)\otimes 1).u \overset{u}{=}$$
$$(1,(1,1\otimes t)).(1,(1,a)).(1,u).(1,1\otimes t).(1,a).u.1\otimes p \overset{(3.1)}{=\!=\!=} (1,\lambda).\lambda.1\otimes p.$$

Of the equalities of this string of equations which follow from Eilenberg and Kelly (1966), 1 follows from page 480, (3.19), with $x = a^{-1}$,

$$A = A\otimes(M\otimes N,P),\quad B = M,\quad C = N,\quad D = (A\otimes(M\otimes N,P)\otimes M)\otimes N$$

and from page 477, (3.1) and (3.3), 2 from MC 3, page 472, 3 from page 480, (3.19) with π^{-1} in place of π and $x = 1$, and 4 from page 499, (7.1).

THEOREM 3.3. *Let* V_0 *have equalizers and coequalizers and let* T *be a commutative adjoint* V*-monad on* V. *Then* V^T *is a symmetric monoidal closed category.*

PROOF. We have shown in Theorem 2.1 that V^T is a symmetric monoidal category, and Kock (1971) has shown that V^T is a closed category. We must show that the two structures on V^T are compatible, and for this we use Theorem 5.3, page 490, of Eilenberg and Kelly (1966). Again by Lemma 1.1 we may assume that $T = \underline{A} = (A\otimes(-), \eta^A, \mu^A)$ for a commutative monoid (A,e,m) in V.

At this point we recall the construction of the internal hom functor in $V^{\underline{A}}$ from Kock (1971). Let (M,ρ_M) and (N,ρ_N) be two $\underline{A}$-algebras. The internal hom object $(Hom_A(M,N), <\rho_M, \rho_N>)$ is defined as follows. The following diagram (3.3.1) is an equalizer in V_0.

(3.3.1)
$$\begin{array}{ccccc} Hom_A(M,N) & \xrightarrow{e_{MN}} & (MN) & \xrightarrow{(\rho_M,1)} & (A\otimes M,N) \\ & & {\scriptstyle H^A_{MN}}\searrow & & \nearrow{\scriptstyle (1,\rho_N)} \\ & & & (A\otimes M, A\otimes N) & \end{array}$$

We note that in Kock (1971) the internal hom object in $V_0^{\underline{A}}$ is denoted by $\overline{M \pitchfork N}$ and in V_0 it is denoted by $M \pitchfork N$, rather than $Hom_A(M,N)$ and (MN), respectively, which we use. The $\underline{A}$-structure $<\rho_M, \rho_N>$ on $Hom_A(M,N)$ is given by commutativity of the following diagram

$$(3.3.2) \qquad \begin{array}{ccc} A \otimes Hom_A(M,N) & \xrightarrow{A \otimes e_{MN}} & A \otimes (MN) \\ \Big\downarrow {\scriptstyle <\rho_M, \rho_N>} & & \Big\downarrow {\scriptstyle \lambda_{MN}} \\ & & (M, A \otimes N) \\ & & \Big\downarrow {\scriptstyle (1, \rho_N)} \\ Hom_A(M,N) & \xrightarrow{e_{MN}} & (MN) \end{array}$$

Moreover, if

$$f : (M', \rho_{M'}) \to (M, \rho_M) \quad \text{and} \quad g : (N, \rho_N) \to (N', \rho_{N'})$$

are two morphisms of $\underline{A}$-algebras, there is a unique morphism of $\underline{A}$-algebras $Hom_A(f,g) : Hom_A(M,N) \to Hom_A(M',N')$ which is such that

$$(3.3.3) \qquad e_{M'N'} . Hom_A(f,g) = (f,g) . e_{MN} .$$

Recall also from Kock (1970), page 8, and Eilenberg and Kelly (1966), Theorem 5.2, page 445, that for each $(M, \rho_M) \in V_0^{\underline{A}}$ there is a $V^{\underline{A}}$-functor $\bar{L}^{(M,\rho_M)} = \bar{L}^M : V^{\underline{A}} \to V^{\underline{A}}$ defined for any $(N, \rho_N) \in V_0^{\underline{A}}$ by

$$\bar{L}^M(N, \rho_N) = (Hom_A(M,N), <\rho_M, \rho_N>),$$

and if $(P, \rho_P), (Q, \rho_Q) \in V_0^{\underline{A}}$, we have a natural transformation

$$(\bar{L}^M)_{(P,\rho_P)(Q,\rho_Q)} = \bar{L}^M_{PQ} : (Hom_A(P,Q), <\rho_P, \rho_Q>)$$
$$\to (Hom_A(Hom_A(M,P), Hom_A(M,Q)), <<\rho_M, \rho_P>, <\rho_M, \rho_Q>>)$$

defined by the commutativity of the following diagram (3.3.4).

$$(3.3.4) \qquad \begin{array}{ccccc} Hom_A(P,Q) & \xrightarrow{e} & (PQ) & \xrightarrow{L^M_{PQ}} & ((MP),(MQ)) \\ \Big\downarrow {\scriptstyle \bar{L}^M_{PQ}} & & & & \Big\downarrow {\scriptstyle (e,1)} \\ Hom_A(Hom_A(M,P), Hom_A(M,Q)) & & & & (Hom_A(M,P),(MQ)) \\ & \searrow^{e} & & \nearrow_{(1,e)} & \\ & & (Hom_A(M,P), Hom_A(M,Q)) & & \end{array}$$

In order to verify that Theorem 5.3 of Eilenberg and Kelly (1966) applies, we define a natural isomorphism $\bar{p} = \bar{p}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}$:

$$(Hom_A(M\otimes_A N, P), <\rho_{M\otimes_A N}, \rho_P>) \to (Hom_A(M, Hom_A(N,P)), <\rho_M, <\rho_N, \rho_P>>)$$

of $\underline{A}$-algebras as follows. Consider the following diagram

$$\begin{array}{ccccc}
Hom_A(M\otimes_A N, P) & \xrightarrow{e_{M\otimes_A N,P}} & (M\otimes_A N, P) & \xrightarrow{(q_{MN},1)} & (M\otimes N, P) \\
\downarrow \bar{p}_{MNP} & \searrow p'_{MNP} & & & \downarrow p_{MNP} \\
Hom_A(M, Hom_A(N,P)) & \xrightarrow[e_{M,Hom_A(N,P)}]{} & (M, Hom_A(N,P)) & \xrightarrow[(1,e_{NP})]{} & (M,(NP))
\end{array} \tag{3.3.5}$$

In (3.3.5), p is the natural isomorphism from the symmetric monoidal closed category V, q is the coequalizer which defines $M\otimes_A N$ as in the proof of Theorem 2.1, and e is the equalizer defined in (3.3.1). Since the functor $(M,-): V_0 \to V_0$ has a left adjoint, $(1, e_{NP})$ is the equalizer of

$$(1,(\rho_N,1)) \quad \text{and} \quad (1,(1,\rho_P)).(1,H^A),$$

and we claim that $p_{MNP}.(q_{MN},1).e_{M\otimes_A N,P}$ also equalizes them. To see this, note that we have

$$\begin{aligned}
&(1,(\rho_N,1)).p.(q,1).e \overset{p}{=} p.(1\otimes\rho_N,1).(q,1).e \overset{1}{=} \\
&p.(a^{-1},1).(c\otimes 1,1).(a,1).(\tau_N,1).(q,1).e \overset{(2,1.2)}{=\!=} \\
&p.(a^{-1},1).(c\otimes 1,1).(a,1).(1\otimes q,1).(\rho_{M\otimes_A N},1).e \overset{2}{=} \\
&p.(a^{-1},1).(c\otimes 1,1).(a,1),(1\otimes q,1).(1,\rho_P).H^A.e = \\
&p.(1,\rho_P).(a^{-1},1).(c\otimes 1,1).(a,1).(1\otimes q,1).H^A.e \overset{p}{=} \\
&(1,(1,\rho_P)).p.(a^{-1},1).(c\otimes 1,1).(a,1).(1\otimes q,1).H^A.e \overset{H^A}{=} \\
&(1,(1,\rho_P)).p.(a^{-1},1).(c\otimes 1,1),(a,1).H^A.(q,1).e \overset{3}{=} \\
&(1,(1,\rho_P)).p.(a^{-1},1).(c\otimes 1,1).(a,1).(c,c).p^{-1}.(1,u).(q,1).e \overset{4}{=} \\
&(1,(1,\rho_P)).p.((1\otimes c),c).(a^{-1},1).p^{-1}.(1,u).(q,1).e \overset{p}{=} \\
&(1,(1,\rho_P)).(1,(c,c)).p.(a^{-1},1).p^{-1}.(1,u).(q,1).e \overset{5}{=} \\
&(1,(1,\rho_P)).(1,(c,c)).(1,p^{-1}).p.(1,u).(q,1).e \overset{p}{=}
\end{aligned}$$

$$(1,(1,\rho_P)).(1,(c,c)).(1,p^{-1}).(1,(1,u)).p.(q,1).e$$
$$\overset{3}{=} (1,(1,\rho_P)),(1,H^A).p.(q,1).e.$$

In this string of equations, 1 follows from the definition of τ_N as in Theorem 2.1, 2 since e is an equalizer, 3 from Eilenberg and Kelly (1966), page 537, (6.7) and page 499, (7.1), 4 from coherence and 5 from MCC 3 for V, Eilenberg and Kelly (1966), page 475. It follows that there is a unique morphism

$$p'_{MNP}: Hom_A(M\otimes_A N, P) \to (M, Hom_A(N,P))$$

such that

$$(3.3.6) \qquad p_{MNP}.(q_{MN}, 1).e_{M\otimes_A N,P} = (1, e_{NP}).p'_{MNP}.$$

Next we claim that p' equalizes $(\rho_M, 1)$ and $(1,<\rho_N,\rho_P>).H^A$, and since $(1, e_{NP})$ is a monomorphism, it suffices to show that

$$(1,e).(\rho_M,1).p' = (1,e).(1,<\rho_N,\rho_P>).H^A.p'.$$

We have then

$$(1,e).(\rho_M,1).p' = (\rho_M,1).(1,e).p' \overset{(3.3.6)}{=\!=} (\rho_M,1).p.(q,1).e \overset{p}{=}$$
$$p.(a,1).(\tau_M,1).(q,1).e \overset{(2.1.2)}{=\!=} p.(a,1).(1\otimes q,1).(\rho_{M\otimes_A N},1).e \overset{1}{=}$$
$$p.(a,1).(1\otimes q,1).(1,\rho_P).H^A.e \overset{H^A}{=} p.(a,1).(1,\rho_P).H^A.(q,1).e \overset{p}{=}$$
$$(1,(1,\rho_P)).p.(a,1).H^A.(q,1).e \overset{2}{=}$$
$$(1,(1,\rho_P)).p.(a,1).H^A.(1,t).K^N.p.(q,1).e \overset{H^A}{=}$$
$$(1,(1,\rho_P)).p.(1,1\otimes t).(a,1).H^A.K^N.p.(q,1).e \overset{3}{=}$$
$$(1,(1,\rho_P)).p.(1,1\otimes t).(1,a).K^N.H^A.p.(q,1).e \overset{p}{=}$$
$$(1,(1,\rho_P)).(1,(1,1\otimes t)).(1,(1,a)).p.K^N.H^A.p.(q,1).e \overset{4}{=}$$
$$(1,(1,\rho_P)).(1,(1,1\otimes t)).(1,(1,a)).(1,u).H^A.p.(q,1).e \overset{(3.1)}{=\!=}$$
$$(1,1,\rho_P)).(1,\lambda).H^A.p.(q,1).e \overset{(3.3.6)}{=\!=}$$
$$(1,(1,\rho_P)).(1,\lambda).H^A.(1,e).p' \overset{H^A}{=} (1,(1,\rho_P)).(1,\lambda).(1,1\otimes e).H^A.p'$$
$$\overset{(3.3.2)}{=\!=} (1,e).(1,<\rho_N,\rho_P>).H^A.p'.$$

In this string of equations, 1 follows since e is an equalizer, 2 from Eilenberg and Kelly (1966), page 500, (7.2), 3 from Eilenberg and Kelly (1966) page 537, (6.7) and page 499, (7.1), and 4 from Eilenberg and Kelly (1966) page 499, (7.1). Since p' equalizes

$$(\rho_M, 1) \text{ and } (1, <\rho_N, \rho_P>).H^A ,$$

there is a unique morphism

$$\bar{p}: Hom_A(M \otimes_A N, P) \to Hom_A(M, Hom_A(N, P))$$

such that $e.\bar{p} = p'$, and hence we have

$$(3.3.7) \quad p_{MNP}.(q_{MN}, 1).e_{M\otimes_A N, P} = (1, e_{NP}).e_{M, Hom_A(N,P)}.\bar{p}_{MNP}.$$

Now since q is a coequalizer, $(1, q)$ is an equalizer, and also $\bar{p}$ is the unique morphism such that

$$p.(q,1).e = (1,e).e.\bar{p}.$$

Hence, since p is a natural isomorphism, it follows that $\bar{p}$ is a natural isomorphism as well. It remains to show that $\bar{p}$ is a morphism of $\underline{A}$-algebras, i.e., that we have

$$(3.3.8) \quad \bar{p}_{MNP}.<\rho_{M\otimes_A N}, \rho_P> = <\rho_M, <\rho_N, \rho_P>>. A\otimes\bar{p}_{MNP},$$

and as before it suffices to show that

$$(1,e).e.\bar{p}.<\rho_{M\otimes_A N}, \rho_P> = (1,e).e.<\rho_M, <\rho_N, \rho_P>>.1\otimes\bar{p}.$$

We have

$$\begin{aligned}
&(1,e).e.\bar{p}.<\rho_{M\otimes_A N}, \rho_P> \overset{(3.3.7)}{=\!=} p.(q,1).e.<\rho_{M\otimes_A N}, \rho_P> \overset{(3.3.2)}{=\!=} \\
&p.(q,1).(1,\rho_P),\lambda.1\otimes e \overset{\lambda}{=} p.(q,1).\lambda.1\otimes(q,1).1\otimes e \overset{p}{=} \\
&(1,(1,\rho_P)).p.\lambda.1\otimes(q,1).1\otimes e \overset{(3.2)}{=\!=} \\
&(1,(1,\rho_P)).(1,\lambda).\lambda.1\otimes p.1\otimes(q,1).1\otimes e \overset{(3.3.7)}{=\!=} \\
&(1,(1,\rho_P)).(1,\lambda).\lambda.1\otimes(1,e).1\otimes e.1\otimes\bar{p} \overset{\lambda}{=} \\
&(1,(1,\rho_P)).(1,\lambda).(1,1\otimes e).\lambda.1\otimes e.1\otimes\bar{p} \overset{(3.3.2)}{=\!=} \\
&(1,e).(1,<\rho_N, \rho_P>).\lambda.1\otimes e.1\otimes\bar{p} \overset{(3.3.2)}{=\!=} \\
&(1,e).e.<\rho_M, <\rho_N, \rho_P>>.1\otimes\bar{p}.
\end{aligned}$$

Hence we have that $\bar{p}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}$:

$$(Hom_A(M\otimes_A N,P),<\rho_{M\otimes_A N},\rho_P>) \to (Hom_A(M,Hom_A(N,P)),<\rho_M,<\rho_N,\rho_P>>)$$

is a natural isomorphism of $\underline{A}$-algebras.

We now claim that for any $(M,\rho_M),(N,\rho_N)\in V_o^{\underline{A}}$, $\bar{p}$ induces a $V^{\underline{A}}$-natural isomorphism of $V^{\underline{A}}$-functors

$$\hat{p}: \bar{L}^{M\otimes_A N} \to \bar{L}^M\bar{L}^N \quad \text{via} \quad \hat{p}_{(P,\rho_P)} = \bar{p}_{(M,\rho_M)(N,\rho_N)(P,\rho_P)}:$$

$$Hom_A(M\otimes_A N,P) \to Hom_A(M,Hom_A(N,P)).$$

To see this we need to verify that VN in Eilenberg and Kelly (1966), page 466, holds for $\hat{p}$, so that we must check that the following diagram commutes where $(P,\rho_P),(Q,\rho_Q)\in V_o^{\underline{A}}$:

$$(3.3.9)\qquad \begin{array}{ccc} Hom_A(P,Q) & \xrightarrow{\bar{L}^{M\otimes_A N}} & Hom_A(Hom_A(M\otimes_A N,P),Hom_A(M\otimes_A N,Q)) \\ \downarrow{\scriptstyle \bar{L}^N} & & \downarrow{\scriptstyle Hom_A(1,\bar{p})} \\ Hom_A(Hom_A(N,P),Hom_A(N,Q)) & & Hom_A(Hom_A(M\otimes_A N,P),Hom_A(M,Hom_A(N,Q))) \\ \downarrow{\scriptstyle \bar{L}^M} & \nearrow{\scriptstyle Hom_A(\bar{p},1)} & \\ Hom_A(Hom_A(M,Hom_A(N,P)),Hom_A(M,Hom_A(N,Q))) & & \end{array}$$

Now since e is a monomorphism in V_o, it suffices to check that

$$(1,(1,e)).(1,e),e.Hom_A(1,\bar{p}).\bar{L}^{M\otimes_A N} =$$
$$(1,(1,e)).(1,e).e.Hom_A(\bar{p},1).\bar{L}^M.\bar{L}^N.$$

We have that

$$(1,(1,e)).(1,e).e.Hom_A(1,\bar{p}).\bar{L}^{M\otimes_A N} \overset{e}{=}$$
$$(1,(1,e)).(1,e).(1,\bar{p}).e.\bar{L}^{M\otimes_A N} \overset{(3.3.7)}{=\!=}$$
$$(1,p).(1,(q,1)).(1,e).e.\bar{L}^{M\otimes_A N} \overset{(3.3.4)}{=\!=}$$
$$(1,p).(1,(q,1)).(e,1).L^{M\otimes_A N}.e =$$
$$(e,1).(1,p).(1,(q,1)).L^{M\otimes_A N}.e \overset{L}{=}$$

$$(e,1).(1,p).((q,1),1).L^{M\otimes N}.e =$$
$$(e,1).((q,1),1).(1,p).L^{M\otimes N}.e \overset{1}{=}$$
$$(e,1).((q,1),1).(p,1).L^{M}.L^{N}.e \overset{(3.3.7)}{=\!=\!=}$$
$$(\bar{p},1).(e,1).((1,e),1).L^{M}.L^{N}.e \overset{L}{=}$$
$$(\bar{p},1).(e,1).L^{M}.(e,1).L^{N}.e \overset{(3.3.4)}{=\!=\!=}$$
$$(\bar{p},1).(e,1).L^{M}.(1,e).e.\bar{L}^{N} \overset{L}{=}$$
$$(\bar{p},1).(e,1).(1,(1,e)).L^{M}.e.\bar{L}^{N} =$$
$$(1,(1,e)).(\bar{p},1).(e,1).L^{M}.e.\bar{L}^{N} \overset{(3.3.4)}{=\!=\!=}$$
$$(1,(1,e)).(\bar{p},1).(1,e).e.\bar{L}^{M}.\bar{L}^{N} \overset{e}{=}$$
$$(1,(1,e)).(1,e).e.Hom_A(\bar{p},1).\bar{L}^{M}.\bar{L}^{N}.$$

In this string of equations, 1 follows from MCC3' in Eilenberg and Kelly (1966), page 475.

It is now clear that the object $(M\otimes_A N, \rho_{M\otimes_A N})$ in $V^{\underline{A}}$ and the $V^{\underline{A}}$-natural isomorphism $\hat{p}: \bar{L}^{M\otimes_A N} \to \bar{L}^M \bar{L}^N$ form a representation of the $V^{\underline{A}}$-functor $\bar{L}^M \bar{L}^N: V^{\underline{A}} \to V^{\underline{A}}$ in the sense of Eilenberg and Kelly (1966), Remark 10.11, page 471, so that by Theorem 5.3 of Eilenberg and Kelly, the closed category $V^{\underline{A}}$ admits enrichment to a (symmetric) monoidal closed category $V^{\underline{A}}$. This completes the proof of Theorem 3.3.

We observe that Day (1970) has shown that for a symmetric monoidal closed category V such that V_0 has all small limits and colimits and for a commutative monoid M in V, the category of M-modules, viewed as a functor category, is also symmetric monoidal closed. We note that Day's proof depends heavily on the completeness and cocompleteness of V_0, while the proof given herein is motivated by the obvious examples and requires only the existence of equalizers and coequalizers in V_0.

4. COMMUTATIVE MONOIDS AND ALGEBRAS FOR A MONAD.

The following theorem describes a connection between the categories CM(V) of commutative monoids in V and V^T of T-algebras for a

commutative adjoint V-monad T on V. Briefly stated, it says that there is a monad T_1 on CM(V) with the property that T_1-algebras in CM(V) are (isomorphic to) commutative monoids in V^T. We denote by $U : CM(V) \to V_o$ the functor given on objects by $U(A, e, m) = A$.

THEOREM 4.1. *Let* V_o *have coequalizers and let* $T = (T, \eta, \mu)$ *be a commutative adjoint* V*-monad on* V. *Then there is a monad* $T_1 = (T_1, \eta_1, \mu_1)$ *on* CM(V) *such that*

$$U T_1 = T U, \quad U \eta_1 = \eta U, \quad U \mu_1 = \mu U,$$

and $CM(V^T)$ *is isomorphic to* $CM(V)^{T_1}$ *over* V.

PROOF. Again by Lemma 1.1 we may assume that

$$T = \underline{A} = (A \otimes (-), \eta^{\underline{A}}, \mu^{\underline{A}})$$

for a commutative monoid (A, e, m) in V. For any commutative monoid (A', e', m') in V, define

$$T_1(A', e', m') = (A \otimes A', e_1, m_1)$$

where

$$e_1 = e \otimes e' . l_I^{-1} \quad \text{and} \quad m_1 = m \otimes m' . \sigma_{A A'},$$

and where

$$\sigma_{A A'} : (A \otimes A') \otimes (A \otimes A') \to (A \otimes A) \otimes (A' \otimes A')$$

is the so-called «middle-four interchange» of Eilenberg and Kelly (1966), page 517, a coherently natural isomorphism. A simple calculation shows that $(A \otimes A', e_1, m_1)$ is a commutative monoid in V, so that

$$T_1 : CM(V) \to CM(V)$$

is defined on objects. For a morphism f in CM(V), let $T_1(f) = A \otimes f$, a morphism in CM(V) as well. Define natural transformations

$$\eta_1 : CM(V) \to T_1 \quad \text{and} \quad \mu_1 : T_1 T_1 \to T_1$$

by

$$\eta_{1\,(A',e',m')} = (\eta^{\underline{A}})_{A'} \quad \text{and} \quad \mu_{1\,(A',e',m')} = (\mu^{\underline{A}})_{A'}.$$

Again we see that $\eta_{1\,(A',e',m')}$ and $\mu_{1\,(A',e',m')}$ are morphisms in CM(V) and that η_1 and μ_1 are natural. It is immediate that

$$U T_1 = T U, \quad U \eta_1 = \eta U \quad \text{and} \quad U \mu_1 = \mu U,$$

and since U is faithful, it follows that $\mathrm{T}_1 = (T_1, \eta_1, \mu_1)$ is a monad on $\mathrm{CM}(\mathrm{V})$.

Now define $\Phi : \mathrm{CM}(\mathrm{V}^{\underline{A}}) \to \mathrm{CM}(\mathrm{V})^{\mathrm{T}_1}$ on objects by

$$\Phi((M, \rho_M), e', m') = ((M, e'.e, m'.q_{MM}), \rho_M),$$

where $q_{MM}: M \otimes M \to M \otimes_A M$ is the coequalizer of the two morphisms

$$r'_M, r''_M : A \otimes (M \otimes M) \to M \otimes M,$$

$$r'_M = \rho_M \otimes M . a^{-1}_{AMM} \quad \text{and} \quad r''_M = M \otimes \rho_M . a_{MAM} . c_{AM} \otimes M . a^{-1}_{AMM},$$

as in Theorem 2.1. The calculations showing $(M, e'.e, m'.q_{MM})$ to be a commutative monoid are straightforward and depend upon knowing that

$$m' : (M \otimes_A M, \rho_{M \otimes_A M}) \to (M, \rho_M) \quad \text{and} \quad e' : (A, m) \to (M, \rho_M)$$

are morphisms of $\underline{A}$-algebras making $((M, \rho_M), e', m')$ a commutative monoid in $\mathrm{V}^{\underline{A}}$, and upon recalling the definitions of the natural isomorphisms $\bar{a}$, $\bar{l}$ and $\bar{c}$ in $\mathrm{V}^{\underline{A}}$ from Theorem 2.1. Similarly one sees that $\rho_M : A \otimes M \to M$ is a morphism in $\mathrm{CM}(\mathrm{V})$ and also a T_1-structure on $(M, e'.e, m'.q_{MM})$. Defining Φ on morphisms by $\Phi(f) = f$, we have a functor

$$\Phi : \mathrm{CM}(\mathrm{V}^{\underline{A}}) \to \mathrm{CM}(\mathrm{V})^{\mathrm{T}_1}.$$

To see that Φ is an isomorphism, define $\Phi' : \mathrm{CM}(\mathrm{V})^{\mathrm{T}_1} \to \mathrm{CM}(\mathrm{V}^{\underline{A}})$ on objects by

$$\Phi'((M, e', m'), \rho_M) = ((M, \rho_M), e'', m''),$$

where

$$e'' = \rho_M . A \otimes e' . r^{-1}_A : A \to M$$

and $m'' : M \otimes_A M \to M$ is the unique morphism such that $m' = m''.q_{MM}$. Note that the existence of m'' follows since m' coequalizes r'_M and r''_M. It is immediate that (M, ρ_M) is an $\underline{A}$-algebra, and one can easily show that

$$e'' : (A, m) \to (M, \rho_M) \quad \text{and} \quad m'' : (M \otimes_A M, \rho_{M \otimes_A M}) \to (M, \rho_M)$$

are $\underline{A}$-algebra morphisms making $((M, \rho_M), e'', m'')$ a commutative monoid in $\mathrm{V}^{\underline{A}}$. Again defining Φ' on morphisms by $\Phi'(f) = f$, we have a functor $\Phi' : \mathrm{CM}(\mathrm{V})^{\mathrm{T}_1} \to \mathrm{CM}(\mathrm{V}^{\underline{A}})$, and a direct calculation shows that $\Phi' = \Phi^{-1}$, so

that Φ is an isomorphism (over V) as required.

5. EXAMPLES.

1. The following example provided part of the motivation (and the notation) for this paper. The category *Ab* of abelian groups and group homomorphisms is known to be a symmetric monoidal closed category. Moreover, a commutative monoid in *Ab* is a commutative ring R with identity, and if $\underline{R}$ denotes the corresponding commutative adjoint (*Ab*-)monad on *Ab*, it is clear that $Ab^{\underline{R}} \cong R\text{-}Mod$, the category of R-modules and R-linear homomorphisms. Since *Ab* has equalizers and coequalizers, it follows from Theorem 3.3 that *R-Mod* is also a symmetric monoidal closed category, which is indeed a well known result. Furthermore, one sees that $\mathrm{CM}(R\text{-}Mod) \cong R\text{-}Alg$, the category of commutative R-algebras and R-homomorphisms, and since $\mathrm{CM}(Ab) \cong Comm$, the category of commutative rings with identity, we have $R\text{-}Alg \cong Comm^{T_1}$. Hence any R-algebra S can be viewed as either an R-module S with a multiplicative structure, or as a commutative ring S with an action of R on S via a ring homomorphism $R \otimes S \to S$ (i.e., a ring homomorphism $R \to S$, since $R \otimes S \cong R \amalg S$), again a well known fact; e.g., Mac Lane (1967), page 173.

2. The category *Sets* of sets and functions is a cartesian closed category, and a commutative monoid in *Sets* is just an abelian monoid M. Hence, $Sets^{\underline{M}} \cong M\text{-}sets$, the category of M-sets and M-functions, is a symmetric monoidal closed category by Theorem 3.3.

3. The category $\mathrm{E}(Ab)$, whose objects are pairs (A, f) with A an abelian group and f an endomorphism on A, is a symmetric monoidal closed category, where $(A, f) \otimes (B, g) = (A \otimes B, f \# g)$, with

$$(f \# g)(a \otimes b) = f(a) \otimes b + a \otimes g(b), \quad \text{for any } a \in A,\ b \in B,$$

and where

$$((A,f),(B,g)) = ((AB), \{f, g\}), \quad \text{with } \{f, g\}(h) = gh - hf$$

for any $h: A \to B$ as in Keigher (a) and (b). Also, a commutative monoid

in $E(Ab)$ is a differential ring (A, d), and

$$E(Ab)^{\underline{(A,d)}} \cong (A, d)\text{-}Mod,$$

the category of differential modules over (A, d) by Keigher (b). It follows from Theorem 3.3 that $(A, d)\text{-}Mod$ is a symmetric monoidal closed category. Clearly we have

$$CM((A, d)\text{-}Mod) \cong (A, d)\text{-}Alg,$$

the category of differential algebras over (A, d), and since

$$CM(E(Ab)) \cong Diff,$$

the category of differential rings, Theorem 4.1 tells us that

$$(A, d)\text{-}Alg \cong Diff^{T_1}$$

as well.

4. The category Ban of real Banach spaces and continuous linear transformations of norm not exceeding one is a symmetric monoidal closed category as in Wick-Negrepontis (1973), and a commutative monoid in Ban is a Banach algebra A. Moreover, Ban has equalizers and coequalizers, so that the category $Ban^{\underline{A}}$ is also symmetric monoidal closed by Theorem 3.3. The category $Ban^{\underline{A}}$ is called the category of Banach A-modules for the Banach algebra A, and is of some interest to functional analysts.

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